# STRICT $\mathcal{C}^p$-TRIANGULATIONS - A NEW APPROACH TO DESINGULARIZATION

Wiesław Pawłucki

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Abstract. Let $R$ be any real closed field expanded by some o-minimal structure. Let $f : A \longrightarrow R^d$ be a definable and continuous mapping defined on a definable, closed, bounded subset $A$ of $R^n$. Let $\mathcal{E}$ be a finite family of definable subsets of $R^n$ contained in $A$. Let $p$ be any positive integer. We prove that then there exists a finite simplicial complex $\mathcal{T}$ in $R^n$ and a definable homeomorphism $h : |\mathcal{T}| \longrightarrow A$, where $|\mathcal{T}| := \cup\mathcal{T}$, such that for each simplex $\Delta \in \mathcal{T}$, the restriction of $h$ to its relative interior $\overset{\circ}{\Delta}$ is a $\mathcal{C}^p$-embedding of $\overset{\circ}{\Delta}$ into $R^n$ and moreover both $h$ and $f \circ h$ are of class $\mathcal{C}^p$ in the sense that they have definable $\mathcal{C}^p$-extensions defined on an open definable neighborhood of $|\mathcal{T}|$ in $R^n$. Then we call a pair $(\mathcal{T}, h)$ a *strict $\mathcal{C}^p$-triangulation* of $A$. In addition this triangulation can be made compatible with $\mathcal{E}$ in the sense that for each $E \in \mathcal{E}$, $h^{-1}(E)$ is a union of some $\overset{\circ}{\Delta}$, where $\Delta \in \mathcal{T}$. We also give an application to approximation theory.

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**1.Introduction and Main Theorem.**

We will work with an arbitrary fixed o-minimal expansion of any real closed field $R$; e.g. the field of real numbers $\mathbb{R}$ with semialgebraic subsets of spaces $\mathbb{R}^n$, where $n \in \mathbb{N}$. O-minimal geometry (see [C] or [vdD] for fundamental notions and results) is a far-going generalization of semialgebraic and subanalytic geometries (presented in [BCR], [Ł], [Ga], [H], [BM], [S]). We will deal only with subsets of $R^n$ and mappings $f : A \longrightarrow R^m$, where $A \subset R^n$, which are *definable* in this structure (mapping $f$ is called definable if the graph of $f$ is a definable subset of $R^{n+m}$). Therefore we will principally skip the adjective definable.

We adopt the following general definition. If $\mathcal{K}$ is any family of subsets of a set $X$, then by *a refinement of* $\mathcal{K}$ we understand any family $\mathcal{L}$ of subsets of $X$ such that each $L \in \mathcal{L}$ is contained in some $K \in \mathcal{K}$ and each $K \in \mathcal{K}$ is the union $\cup\mathcal{L}'$ of some subfamily $\mathcal{L}' \subset \mathcal{L}$. The term refinement will be also used in a different sense; namely, if $\mathcal{F}$ is a family of functions defined on a set $X$ we will say that a family $\mathcal{G}$ of functions defined on $X$ is *a refinement of* $\mathcal{F}$ if simply $\mathcal{F} \subset \mathcal{G}$.

If $\mathcal{K}$ is any family of subsets of a set $X$, then we will denote by $|\mathcal{K}|$ the union of all subsets $K$ belonging to $\mathcal{K}$.

The interior of a subset $A$ of a topological space will be in general denoted int$A$, but sometimes we find the Bourbaki notation $\overset{\circ}{A}$ more handy, while for the closure of $A$ we will use either $\overline{A}$ or cl$A$.

We adopt a standard definition of a *simplex* of dimension $k$ in $R^n$ as the convex hull of $k+1$ points $a_0, \dots, a_k$ affinely independent in $R^n$; i.e.

$$\Delta = [a_0, \dots, a_k] := \{\sum_{i=0}^{k} \alpha_i a_i : \alpha_i \geqslant 0 \ (i \in \{0, \dots, k\}), \sum_{i=0}^{k} \alpha_i = 1\}.$$

If $0 \leqslant i_0 < i_1 < \dots < i_l \leqslant k$, then the simplex $[a_{i_0}, \dots, a_{i_l}]$ is called a *face* of $\Delta$ of dimension $l$. The points $a_0, \dots, a_k$ are called *vertices* of $\Delta$. The *boundary* $\partial\Delta$ of a simplex $\Delta$ is the union of all faces of $\Delta$ of dimension $< k$. Its *relative interior* is by definition

$$\overset{\circ}{\Delta} := \Delta \setminus \partial\Delta = (a_0, \dots, a_k) := \{\sum_{i=0}^{k} \alpha_i a_i : \alpha_i > 0 \ (i \in \{0, \dots, k\}), \sum_{i=0}^{k} \alpha_i = 1\}.$$

It will be convenient for us to use a more general notion of a *convex polyhedron*, or simply *polyhedron*, in $R^n$ which is defined as the convex hull of any finite subset of $R^n$. It is clear that the notions of dimension, faces, boundary, vertices and relative interior generalize to all polyhedra and that polyhedra are subsets definable in PL-geometry.

By a *polyhedral complex* in $R^n$ we will understand a finite family $\mathcal{P}$ of polyhedra in $R^n$ such that for each $P \in \mathcal{P}$ all faces of $P$ belong to $\mathcal{P}$ and for each pair $P_1, P_2 \in \mathcal{P}$, $P_1 \cap P_2$ is a common face of both $P_1$ and $P_2$. A polyhedral complex consisting of simplexes is called a *simplicial complex*. Observe that if we restrict our consideration to polyhedral complexes $\mathcal{P}$ such that $|\mathcal{P}|$ is of constant dimension $n$, then a polyhedral complex can be defined as a finite family of polyhedra of dimension $n$ such that the intersection any two of them is their common face. We will use this identification concerning simplicial complexes as well.

Let $p$ be any positive integer and let $A$ be any definable, bounded and closed subset of $R^n$. A $\mathcal{C}^p$*-triangulation* of $A$ is a pair $(\mathcal{T}, h)$, where $\mathcal{T}$ is a simplicial complex in $R^n$ and $h$ is a definable homeomorphism of $|\mathcal{T}|$ onto $A$ such that for each simplex $\Delta \in \mathcal{T}$ the restriction $h|\mathring{\Delta}$ is a $\mathcal{C}^p$-embedding of $\mathring{\Delta}$ into $R^n$. If $\mathcal{E}$ is any finite family of definable subsets of $A$ we say that a *triangulation* $(\mathcal{T}, h)$ *is compatible with* $\mathcal{E}$ if for each $E \in \mathcal{E}$ the inverse image $h^{-1}(E)$ is a union of some $\mathring{\Delta}$, where $\Delta \in \mathcal{T}$. A $\mathcal{C}^p$-triangulation of $A$ will be called a *strict* $\mathcal{C}^p$*-triangulation* of $A$ if the mapping $h : |\mathcal{T}| \longrightarrow R^n$ is of class $\mathcal{C}^p$ in the sense that it admits a definable extension $\tilde{h} : \Omega \longrightarrow R^n$ of class $\mathcal{C}^p$ defined on an open definable neighborhood $\Omega$ of $|\mathcal{T}|$ in $R^n$.

**Main Theorem.** *Let $R$ be any real closed field expanded by some o-minimal structure. Let $f : A \longrightarrow R^d$ be a definable and continuous mapping defined on a definable, closed, bounded subset $A$ of $R^n$. Let $\mathcal{E}$ be a finite family of definable subsets of $R^n$ contained in $A$. Let $p$ be any positive integer.*

*Then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $A$ compatible with the family $\mathcal{E}$ and such that $f \circ h$ is of class $\mathcal{C}^p$.*

The proof of the Main Theorem is an interplay between PL- and o-minimal geometries. The general idea comes from our earlier paper about $\mathcal{C}^p$-parametrizations of sets definable in o-minimal structures [K-CPV]. In that paper we parametrized definable sets by ($\mathcal{C}^p$-mappings defined on) cubes (similarly as in the classical analytic rectilinearization theorem for subanalytic sets [H],[BM]), which inevitably spoils injectivity of the parametrization. Similarly, blowing-up operations evidently spoil injectivity. Instead of cubes or blowings-up we use simplexes as in the classical triangulation theorem [vdD, Chapter 8], which gives existence of $\mathcal{C}^p$-triangulations. All the problem is to make a triangulating homeomorphism $\mathcal{C}^p$-smooth. Our procedure of smoothing is based on the case of dimension one; it means on the Main Theorem for $n = 1$, the proof of which we will shortly explain now, assuming for simplicity that $d = 1$. Without any loss of generality we can assume that $f : [a, b] \longrightarrow R$ is a continuous definable function defined on a bounded, closed interval. There exists a finite sequence $c_0 = a < c_1 < \cdots < c_{s+1} = b$ such that for each $i \in \{0, \ldots, s\}$, the restriction $f|(c_i, c_{i+1})$ is of class $\mathcal{C}^{p+1}$ and either $|f'| \leqslant 1$ on $(c_i, c_{i+1})$ or $|f'(x)| > 1$ on $(c_i, c_{i+1})$. Now we use a simple but beautiful trick of Coste-Reguiat [CR] reducing the problem to that where $|f'| \leqslant 1$ on $[a, b] \setminus \{c_0, \ldots, c_{s+1}\}$. Namely, define $g : [a, b] \longrightarrow R$ by an inductive formula. First, put $g(a) = g(c_0) = f(a)$. Then we define $g$ on $[c_i, c_{i+1}]$ depending on two following cases:

*Case I:* if $|f'| \leqslant 1$ on $[c_i, c_{i+1}]$, then we put $g(x) := g(c_i) + x - c_i$, for each $x \in [c_i, c_{i+1}]$, and

*Case II:* if $|f'| > 1$ on $[c_i, c_{i+1}]$, then we put $g(x) := g(c_i) + |f(x) - f(c_i)|$, for each $x \in [c_i, c_{i+1}]$.

Put $d_i = g(c_i)$ for $i \in \{0, \dots, s+1\}$. Observe that $g : [c_0, c_{s+1}] \longrightarrow [d_0, d_{s+1}]$ is a strictly increasing homeomorphism such that $g'(x) \geqslant 1$ for $x \in [c_0, c_{s+1}] \setminus \{c_0, \dots, c_{s+1}\}$. Take now the inverse $h := g^{-1} : [d_0, d_{s+1}] \longrightarrow [c_0, c_{s+1}]$. Then $0 < h'(y) \leqslant 1$ and $|(f \circ h)'(y)| \leqslant 1$, for each $y \in (d_i, d_{i+1})$, where $i \in \{0, \dots, s\}$. Now we use a trick of Yomdin-Gromov (see Lemma 4.1 and Corollary 4.2 below and compare with [Y1], [Y2] and [G]). Passing perhaps to a finer subdivision one can assume that on each of the intervals $(d_i, d_{i+1})$ each of the derivatives $h^{(\nu)}$ and $(f \circ h)^{(\nu)}$, where $\nu \in \{2, \dots, p+1\}$, exists and has a constant sign. It follows that substituting $y = \varphi(u) = (u - d_0)^q + d_0$, where $q$ is any fixed odd integer grater than $p$, we get two functions $h \circ \varphi$ and $f \circ h \circ \varphi$ defined on an interval $[d_0, d'_{s+1}]$, which are of class $\mathcal{C}^p$ at $d_0$ and $p$-flat at $d_0$. Let $d'_1 := \varphi^{-1}(d_1)$. Now substituting $u = \psi(w) := (w - d'_1)^q + d'_1$ we get two functions $h \circ \varphi \circ \psi$ and $f \circ h \circ \varphi \circ \psi$ defined on an interval $[d''_0, d''_{s+1}]$ which are of class $\mathcal{C}^p$ both at $d''_0$ and at $d''_1 = d'_1$ and $p$-flat at these points. Continuing this process we finally get a homeomorphism $H : [\tilde{a}, \tilde{b}] \longrightarrow [a, b]$ of class $\mathcal{C}^p$ such that $f \circ H$ is of class $\mathcal{C}^p$. In the case $n > 1$ we use the same smoothing procedure but with parameters. In order to make it possible we introduce two devices: *capsules* which are cells without vertical line segments in the boundary (see Section 2) and *detectors* which are special differentiable functions of choice (see Section 3).

The advantage of our method of desingularization is that it works for arbitrary o-minimal structure, including in particular the following two examples:

(1) the o-minimal structure of $\mathbb{R}$-subanalytic sets and mapping; i.e. the structure generated on the ordered field of real numbers $\mathbb{R}$ by real analytic bounded subsets of $\mathbb{R}^n$ $(n \in \mathbb{N})$ and all power functions $(0, \infty) \ni t \longmapsto t^\alpha \in (0, \infty)$ with real irrational $\alpha$ (for a $\mathcal{C}^p$-rectilinearization and uniformization theorems in this structure see [Pi]),

(2) an o-minimal structure of Le Gal and Rolin [LR] which does not admit $\mathcal{C}^\infty$ cell decompositions.

These examples explain why in our Main Theorem we deal with finite classes of differentiability rather than with $\mathcal{C}^\infty$. Besides, the $\mathcal{C}^\infty$-analogue of the theorem is not true even in the semialgebraic case as can be easily checked; consider for example the continuous semialgebraic function

$$f(x_1, x_2) := \begin{cases} \dfrac{x_2^3}{x_1^2 + x_2^2}, & \text{for } (x_1, x_2) \in [-1, 1]^2 \setminus \{(0, 0)\} \\ 0, & \text{for } (x_1, x_2) = (0, 0). \end{cases}$$

The case $p = 1$ has already been proved in a slightly weaker form for semialgebraic category by Ohmoto and Shiota [OS], who used strict $\mathcal{C}^1$-triangulations to develop the theory of integration on sets with sinularities. Our Main Theorem for $p = 1$ in full extent has been proved by Czapla and Pawłucki [CP].

Throughout the paper we use the following notation for linear projections

$$\pi^n_m : R^n \ni (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_m) \in R^m$$

where $m \leqslant n$.

We end this introduction by a useful observation that without any loss of generality we can assume in the Main Theorem that instead of $A$ we have to triangulate a big polyhedron $P$ containing $A$, because by the Tietze Theorem (cf. [vdD, Chapter 8, (3.10)]) the mapping $f$ can be extended to a continuous mapping defined on $P$.

## 2. Capsules.

We define two special notions which will play essential role in the proof of the Main Theorem. These are *capsules* studied in the present section and *detectors* to which the next section is devoted.

A *capsule* in $R^{n+1}$ is a subset $K$ of $R^{n+1}$ of the form

$$K = \{(x,t) \in D \times R : \alpha(x) \leqslant t \leqslant \beta(x)\},$$

where $D$ is a subset of $R^n$ such that $D = \overline{\mathrm{int}D}$, $\mathrm{int}D$ is bounded, connected and $\alpha, \beta : D \longrightarrow R$ are continuous functions such that $\alpha < \beta$ on $\mathrm{int}D$ and $\alpha = \beta$ on $\partial D$. The subset $\{(x,t) \in K : \ x \in \partial D\}$ of $K$ will be called the *rim* of the capsule $K$.

**Proposition 2.1.** *For any subset $E$ of $R^{n+1}$ the following conditions are equivalent*

(2.1.1) *$E$ is a finite union of capsules in $R^{n+1}$.*

(2.1.2) *$E = \overline{\mathrm{int}E}$ is bounded and $\partial E$ does not contain any nontrivial line segment parallel to the $t$-axis.*

(2.1.3) *$E$ is a finite union of capsules in $R^{n+1}$ whose interiors are pairwise disjoint.*

*Proof.* Obviously (2.1.1) implies (2.1.2). Assume now (2.1.2) satisfied. Let $\pi : R^{n+1} \ni (x,t) \longmapsto x \in R^n$. Since $\mathrm{int}E$ is bounded and $\pi(E)$ is closed,

$$\pi(E) = \pi(\overline{\mathrm{int}E}) = \overline{\pi(\mathrm{int}E)} \subset \overline{\mathrm{int}\pi(E)} \subset \pi(E),$$

hence $\pi(E) = \overline{\mathrm{int}\pi(E)}$. Take a cell decomposition of $R^{n+1}$ compatible with $\mathrm{int}E$ and with $\partial E$ (cf. [vdD, Chapter 3, (2.11)]). This allows us to represent $\mathrm{int}E$ as a finite union of pairwise disjoint cells of the form

$$(\varphi, \psi) = \{(x,t) : \ x \in S, \ \varphi(x) < x < \psi(x)\},$$

where $S \subset \pi(\mathrm{int}E)$, $\varphi, \psi : S \longrightarrow R$ are continuous, $\varphi < \psi$ on $S$ and the graphs[1] of $\varphi$ and $\psi$ are contained in $\partial E$. Using classical triangulation applied to $\pi(\mathrm{int}E)$ and all $S$ (cf. [vdD, Chapter 8, (1.7)) we can additionally assume that $S = \pi(\varphi, \psi)$ satisfies the following *Łojasiewicz's* ($s$)*-condition* (cf. [Ł, Section 25]): each point $a \in \overline{S} \setminus S$ admits a neighborhood basis $\mathcal{U}$ in $R^n$ such that the trace $U \cap S$ of each $U \in \mathcal{U}$ on $S$ is connected. Then the set of all limit values of $\varphi$ at each point $a \in \overline{S} \setminus S$ can be identified with

$$\overline{\varphi} \cap (\{a\} \times R) = \{a\} \times \bigcap \{\overline{\varphi(U)} : \ U \in \mathcal{U}\},$$

[1] We identify mappings with their graphs denoting both by the same letter.

which is a nonempty, connected subset of the vertical line $\{a\} \times R$ and of $\partial E$ at the same time; hence, a singleton. Consequently, both $\varphi$ and $\psi$ have continuous extensions $\overline{\varphi}, \overline{\psi} : \overline{S} \longrightarrow R$ to $\overline{S}$ and next, by the Tietze Theorem (cf. [vdD, Chapter 8, (3.10)]), to the whole $\pi(E)$. Using all these extensions and functions min and max we can find a sequence of continuous functions

$$\alpha_1 \leqslant \cdots \leqslant \alpha_p : \pi(E) \longrightarrow R,$$

such that

(2.1.4) $$\text{for each } x \in \pi(\text{int}E) \quad \text{the fiber } (\text{int}E)_x$$
$$\text{is a union of some intervals } (\alpha_i(x), \alpha_j(x)), \text{ where } i < j,$$

and

(2.1.5) $$\pi^{-1}(\pi(\text{int}E)) \cap \partial E \subset \bigcup_i \alpha_i.$$

Refining the sequence $\alpha_1, \dots, \alpha_p$ by some extra functions we can assume that all the sets

$$(\alpha_i, \alpha_{i+1}) := \{(x,t) : \, x \in \pi(E), \, \alpha_i(x) < t < \alpha_{i+1}(x)\}$$

are connected and nonempty. It follows from (2.1.5) that if $(\alpha_i, \alpha_{i+1}) \cap \text{int}E \neq \emptyset$, then $(\alpha_i, \alpha_{i+1}) \subset \text{int}E$. Let $\{i_1 < \cdots < i_s\} = \{i : \, (\alpha_i, \alpha_{i+1}) \subset \text{int}E\}$. Then by (2.1.4)

$$(\alpha_{i_1}, \alpha_{i_1+1}) \cup \cdots \cup (\alpha_{i_s}, \alpha_{i_s+1})$$

is dense in int$E$; hence in $E$. Let $P_\nu := \pi(\alpha_{i_\nu}, \alpha_{i_\nu+1})$. Now if $x \in \overline{P_\nu} \setminus P_\nu$ and $x \in \pi(\text{int}E)$, then of course $\alpha_{i_\nu}(x) = \alpha_{i_\nu+1}(x)$ and if $x \in \overline{P_\nu} \setminus P_\nu$ and $x \notin \pi(\text{int}E)$, then $\{x\} \times [\alpha_{i_\nu}(x), \alpha_{i_\nu+1}(x)] \subset \partial E$, hence again $\alpha_{i_\nu}(x) = \alpha_{i_\nu+1}(x)$. However $\overline{(\alpha_{i_\nu}, \alpha_{i_\nu+1})}$ may not be a capsule yet because the condition int$\overline{P_\nu} = P_\nu$ may not be a priori satisfied. To solve this problem we prove the following lemma.

**Lemma.** *Let $P$ be a bounded open subset of $R^n$ and let $\alpha, \beta : \overline{P} \longrightarrow R$ be two continuous functions such that $\alpha < \beta$ on $P$ and $\alpha = \beta$ on $\partial P$. Then $\overline{(\alpha, \beta)}$ can be represented as a finite union of capsules with pairwise disjoint interiors.*

*Proof of Lemma.* Without any loss of generality we can assume that $\alpha \equiv 0$. Next, using classical triangulation we reduce the problem to PL-geometry. Then the subset $A := (\text{int}\overline{P}) \setminus P$ is contained in a finite number $H_1, \dots, H_q$ of affine hyperplanes, $q$ minimal. We argue by induction on $q$. By affine change of coordinates in $R^n$, we can assume that $H_q = \{(x_1, \dots, x_n) : \, x_n = 0\}$. Then the function $\gamma(x) := Mx_n$, with $|M|$ big enough, cuts the cell $\overline{(0, \beta)}$ into two $\overline{(0, \gamma)}$ and $\overline{(\gamma, \beta)}$ for each of which $q' < q$.

This ends the proof of Proposition 1.

**Remark 2.2.** *If $E$ fulfills the conditions of Proposition 1 and $\lambda_j : \pi(E) \longrightarrow R$ ($j \in \{1, \dots, r\}$) is a given finite family of continuous functions, then there exists a finite family of continuous functions*

$$\alpha_1 \leqslant \cdots \leqslant \alpha_s : R^n \longrightarrow R$$

*such that $E$ is a union of some capsules of the form $\overline{(\alpha_i, \alpha_{i+1})}$ which are compatible with every $\lambda_j$ in the sense that either $\lambda_j(x) \leqslant t$, for each $(x,t) \in \overline{(\alpha_i, \alpha_{i+1})}$, or $\lambda_j(x) \geqslant t$, for each $(x,t) \in \overline{(\alpha_i, \alpha_{i+1})}$.*

**Remark 2.3.** *If $K_0, K_1, \dots, K_p$ are capsules in $R^{n+1}$ and $K_\nu \subset K_0$ when $1 \leqslant \nu \leqslant p$, then there exists a finite family of continuous functions*

$$\alpha_1 \leqslant \cdots \leqslant \alpha_s : R^n \longrightarrow R$$

*such that $\overline{(\alpha_i, \alpha_{i+1})}$, $(i \in \{0, \dots, s-1\})$ is a family of capsules which is a refinement of $K_0, \dots, K_p$.*

**Corollary 2.4.** *For any finite family $\mathcal{K}$ of capsules in $R^{n+1}$ there exists a finite family $\mathcal{L}$ of capsules in $R^{n+1}$ which is a refinement of $\mathcal{K}$ and the interiors of capsules from $\mathcal{L}$ are pairwise disjoint.*

**Proposition 2.5.** *Let $K$ be any capsule in $R^{n+1}$ and let $\mathcal{V}$ be a finite family of open subsets of* $\mathrm{int}K$ *covering the whole* $\mathrm{int}K$*. Then there exists a finite family $\mathcal{L}$ of capsules in $R^{n+1}$ whose interiors are pairwise disjoint, $\cup\mathcal{L} = K$ and for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ such that* $\mathrm{int}L \subset V$.

*Proof.* Put $K = \{(x,t) \in D \times R : \alpha(x) \leqslant t \leqslant \beta(x)\}$. There are two parts of the proof.

*Part I. We first prove by induction on $k$ that if $A$ is any subset of* $\mathrm{int}D$ *of dimension $k$, then there exists a finite family of capsules in $R^{n+1}$ such that for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ containing* $\mathrm{int}L$ *and for each $a \in A$ there exists $L \in \mathcal{L}$ and $\varepsilon > 0$ such that $\{a\} \times (\alpha(a), \alpha(a) + \varepsilon) \subset \mathrm{int}L$.*

Applying triangulation to $D$ compatible with $A$, we can assume that $A$ is an open subset of $R^k = \{(x_1, \dots, x_n) : x_{k+1} = \cdots = x_n = 0\}$. Partitioning $A$, using induction hypothesis and cell decomposition, we can assume that $A$ is connected, there exists one $V \in \mathcal{V}$ and a function

$$\eta : A \longrightarrow (0, \infty)$$

such that $\{a\} \times (\alpha(a), \alpha(a) + \eta(a)] \subset V$, for each $a \in A$. Replacing $\eta$ by $\tilde{\eta}(a) := \min\{\eta(a), d(a, \overline{A} \setminus A)\}$, we can assume that $\eta(a) \to 0$, when $d(a, \overline{A} \setminus A) \to 0$. For each $t \in [\alpha(a), \alpha(a) + \eta(a)]$ put $\rho(a,t) := \frac{1}{2} d((a,t), K \setminus V)$. Since for each $a \in A$, $\rho(a, \alpha(a)) = 0$ and $\rho(a,t) > 0$, when $t > \alpha(a)$, we can modify $\eta$ in such a way that

$$(\alpha(a), \alpha(a) + \eta(a)] \ni t \longmapsto \rho(a,t) \in (0, \infty)$$

is strictly increasing. Again by partitioning $A$ and using induction hypothesis we can assume that $\eta$ is continuous and replacing $\eta$ by $\tilde{\eta}(a) := \min\{\eta(a), d(a, \overline{A} \setminus A)\}$, we can assume that $\eta(a) \to 0$, when $d(a, \overline{A} \setminus A) \to 0$. It follows from the definition of $\rho$ that for each $a \in A$ and $t \in (\alpha(a), \alpha(a) + \eta(a)]$

$$\{(x_1, \dots, x_n, t) : a = (x_1, \dots, x_k), \ (x_{k+1}^2 + \cdots + x_n^2)^{\frac{1}{2}} \leqslant \rho(a,t)\} \subset V.$$

Now we define the wanted capsule. Put

$$E := \{(x_1, \dots, x_n) : a = (x_1, \dots, x_k) \in \overline{A}, \ (x_{k+1}^2 + \cdots + x_n^2)^{\frac{1}{2}} \leqslant \rho(a, \alpha(a) + \eta(a))\}$$

and $L := \{(x_1, \ldots, x_n, t) : (x_1, \ldots, x_n) \in E,$

$$\rho^{-1}(x_1, \ldots, x_k, (x_{k+1}^2 + \cdots + x_n^2)^{\frac{1}{2}}) \leqslant t \leqslant \alpha(x_1, \ldots, x_k) + \eta(x_1, \ldots, x_k)\},$$

where $\rho^{-1}$ denotes the inverse of $\rho$ with respect to the last variable.

*Part II.* According to Part I, there exists a finite family $\mathcal{L}$ of capsules in $R^{n+1}$ such that for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ containing $\mathrm{int}L$ and for each $a \in D$ there exists $L \in \mathcal{L}$ and $\varepsilon > 0$ such that $\{a\} \times (\alpha(a), \alpha(a) + \varepsilon) \subset \mathrm{int}L$ and there exists $M \in \mathcal{L}$ and $\theta > 0$ such that $\{a\} \times (\beta(a), \beta(a) - \theta) \subset \mathrm{int}M$.

By Corollary 2.3 there exists a finite family $\mathcal{L}'$ of capsules in $R^{n+1}$ which is refinement of the family $\mathcal{L} \cup \{K\}$ and the interiors of which are pairwise disjoint. It follows that if $L' \in \mathcal{L}'$ and $L'$ is not contained in any of the capsules from $\mathcal{L}$, then $L'$ is of the form

$$L' = \{(x, t) : x \in Q,\ \gamma(x) \leqslant t \leqslant \delta(x)\},$$

where $\mathcal{V}$ is an open covering of $L'|\mathrm{int}Q = \{(x, t) : x \in \mathrm{int}Q,\ \gamma(x) \leqslant t \leqslant \delta(x)\}$. Thus to finish the proof it suffices to prove the following.

*If $K = \{(x, t) \in D \times R : \alpha(x) \leqslant t \leqslant \beta(x)\}$ is a capsule in $R^{n+1}, K^* := K \cap (\partial D \times R)$, $\mathcal{V}$ is a finite family of open subsets of $R^{n+1}$ such that $K \setminus K^* \subset \cup\mathcal{V}$ and $A$ is a subset of* $\mathrm{int}D$ *of dimension $k$, then there exists a finite family $\mathcal{L}$ of capsules in $R^{n+1}$ contained in $K$ such that $\cup\mathcal{L} \setminus K^*$ is a neighborhood of $K|A$ in $K \setminus K^*$ and for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ such that $L \setminus K^* \subset V$.*

We proceed again by induction on $k$. Take a cell decomposition $\mathcal{C}$ of the set $\cup\mathcal{V}$ compatible with each of $V \in \mathcal{V}$ and with $K|A$. Let

$$\{B_1, \ldots, B_s\} = \{\pi(C) :\ C \in \mathcal{C},\ C \subset K|A,\ \dim \pi(C) = k\}.$$

Now we apply the induction hypothesis to $E := A \setminus (B_1 \cup \cdots \cup B_s)$. There exists a finite family $\mathcal{L}$ of capsules in $R^{n+1}$ contained in $K$ such that $\cup\mathcal{L} \setminus K^*$ is a neighborhood of $K|E$ in $K \setminus K^*$ and for each $L \in \mathcal{L}$ there exists $V \in \mathcal{V}$ such that $L \setminus K^* \subset V$. Fix one $B_\mu = B$. Then

$$K|B = [\gamma_0, \gamma_1] \cup \cdots \cup [\gamma_{m-1}, \gamma_m],$$

where $\gamma_\nu : B \longrightarrow R \quad (\nu \in \{0, \ldots, m\})$ are continuous, $\gamma_0 < \cdots < \gamma_m, \gamma_0 = \alpha|B, \gamma_m = \beta|B$ and each of $[\gamma_\nu, \gamma_{\nu+1}]$ is contained in some $V \in \mathcal{V}$. There is an open subset $T_0$ of $B$ such that $\overline{T}_0 \cap \mathrm{int}D \subset B$ and $\cup\mathcal{L} \setminus K^*$ is a neighborhood of $K|(B \setminus T_0)$. Take also open subsets $T_1, T_2$ of $B$ such that $\overline{T}_i \cap \mathrm{int}D \subset T_j \subset \overline{T}_j \cap \mathrm{int}D \subset B$ if $0 \leqslant i < j \leqslant 2$. By Tietze Theorem for each $\nu \in \{1, \ldots, m\}$ there exists a continuous function

$$\tilde{\gamma_\nu} : \overline{T}_2 \longrightarrow R$$

such that $\tilde{\gamma}_\nu|\overline{T}_1 = \gamma_\nu|\overline{T}_1,\ \tilde{\gamma}_\nu|\partial T_2 = \gamma_{\nu-1}|\partial T_2$ and $\gamma_{\nu-1} \leqslant \tilde{\gamma}_\nu \leqslant \gamma_\nu$ on $\overline{T}_2$. Then

$$\bigcup_{\nu=1}^{m} [\gamma_{\nu-1}|\overline{T}_2, \tilde{\gamma}_\nu] \setminus K^*$$

is a neighborhood of $K|\overline{T}_0 \cap \mathrm{int}D$ in $K \setminus K^*$. A similar neighborhood we built over every $B_\mu$. Applying Proposition 2.1 we finish the proof.

In the proof of Proposition 8.2 in Section 8 we will need the following lemma.

**Lemma 2.6.** *Every* PL*-capsule in* $R^{n+1}$ *is a finite union of convex* PL*-capsules, whose interiors are pairwise disjoint.*

*Proof.* The boundary $\partial S$ of any PL-capsule $S$ is contained in a finite number of graphs of affine functions

$$\partial S \subset \varphi_1 \cup \cdots \cup \varphi_s,$$

where $s$ is the smallest possible. We argue by induction on the number $q$ of $\varphi_\nu$ such that $S$ is not contained in just one closed half-space cut by $\varphi_\nu$. If $q = 0$, clearly $S$ is convex. Otherwise there is $\nu$ such that

$$T_1 := \mathrm{cl}\{(x, y) \in \mathrm{int}S : \ y < \varphi_\nu(x)\} \quad \text{and} \quad T_2 := \mathrm{cl}\{(x, y) \in \mathrm{int}S : \ y > \varphi_\nu(x)\},$$

are finite unions of PL-capsules, for which the number $q$ is smaller. The lemma follows.

## 3. Detectors.

In this section we will need $\mathcal{C}^p$-partitions of unity. Although it is well-known that $\mathcal{C}^p$-partitions of unity exist in any o-minimal structure, however for the reader's convenience and making the paper self-contained, we give a short proof in the first two lemmas.

**Lemma 3.1.** *Let* $\Omega$ *be an open subset of* $R^n$ *and let* $A$ *and* $B$ *be two closed, disjoint subsets of* $\Omega$*. Then there exists a* $\mathcal{C}^p$*-function* $\varphi : \Omega \longrightarrow [0, 1]$ *such that* $\varphi = 1$ *on* $A$ *and* $\varphi = 0$ *on* $B$*.*

*Proof.* By the Whitney extension theorem in the version from [KP], there exists a $\mathcal{C}^p$-function $\psi : \Omega \longrightarrow R$ such that $\psi = 1$ on $A$ and $\psi = 0$ on $B$. Now it suffices to put $\varphi := \lambda \circ \psi$, where $\lambda : R \longrightarrow [0, 1]$ is a $\mathcal{C}^p$-function such that $\lambda(0) = 0$ and $\lambda(1) = 1$.

**Lemma 3.2.** *Let* $\Omega$ *be an open subset of* $R^n$ *and let* $A_1, \ldots, A_m$ *be a finite family of closed and pairwise disjoint subsets of* $\Omega$*. Then there exist* $\mathcal{C}^p$*-functions* $\varphi_j : \Omega \longrightarrow [0, 1] \quad (j \in \{1, \ldots, m\})$ *such that*

$$\sum_{j=1}^{m} \varphi_j(x) = 1, \qquad \textit{for each } x \in \Omega$$

*and for each* $j \in \{1, \ldots, m\} \quad \varphi_j = 1 \ \ on \ A_j$*.*

*Proof.* Induction on $m$. Let $m > 1$. By the induction hypothesis there are $\psi_1, \ldots, \psi_{m-1} : \Omega \longrightarrow [0, 1]$ of class $\mathcal{C}^p$ such that

$$\sum_{i=1}^{m-1} \psi_i(x) = 1, \qquad \text{for each } x \in \Omega$$

and $\psi_i = 1$ on $A_i$. By Lemma 3.1 there exists a $\mathcal{C}^p$-function $\sigma_1 : \Omega \longrightarrow [0, 1]$ such that $\sigma_1 = 1$ on $A_m$ and $\sigma_1 = 0$ on $A_1 \cup \cdots \cup A_{m-1}$. There exists an open neighborhood $U$ of $A_1$ in $\Omega$ such that $\sigma_1 > 0$ on $U$ and $U \subset \Omega \setminus (A_1 \cup \cdots \cup A_{m-1})$. By Lemma 3.1 there exists a $\mathcal{C}^p$-function $\sigma_2 : \Omega \longrightarrow [0,1]$ such that $\sigma_2 = 1$ on $\Omega \setminus U$ and $\sigma_2 = 0$ on $A_m$. Then the $\mathcal{C}^p$-function

$$\sigma_1 + \sigma_2 : \Omega \longrightarrow [0,2]$$

is positive on $\Omega$, so we can built the following $\mathcal{C}^p$-function on $\Omega$

$$\rho_1(x) := \frac{\sigma_1(x)}{\sigma_1(x) + \sigma_2(x)} \quad \text{and} \quad \rho_2(x) := \frac{\sigma_2(x)}{\sigma_1(x) + \sigma_2(x)}.$$

Of course, $\rho_1(x) + \rho_2(x) \equiv 1$, $\rho_1 = 0$ on $A_1 \cup \cdots \cup A_{m-1}$, while $\rho_2 = 0$ on $A_m$; hence $\rho_1 = 1$ on $A_m$ and $\rho_2 = 1$ on $A_1 \cup \cdots \cup A_{m-1}$. Finally we put $\varphi_1 := \psi_1 \rho_2, \ldots, \varphi_{m-1} := \psi_{m-1}\rho_2$ and $\varphi_m := \rho_1$.

**Proposition 3.3.** *Let $\Omega$ be an open subset of $R^n$, $E$ a closed subset of $\Omega$ of dimension $k$ and $C$ a convex, closed bounded subset of $R^m$. Let $f : E \times C \longrightarrow [0,\infty)$ be a continuous function and define*

$$g(x) := \sup_{y \in C} f(x,y), \quad \textit{for each} \quad x \in E.$$

*Assume that $g(x) > 0$, for each $x \in E$. Let $p \in \mathbb{N}$.*

*Then there exists a family $\omega_j : \Omega \longrightarrow \operatorname{int} C$ $(j \in \{0,\ldots,k\})$ of $\mathcal{C}^p$-mappings such that*

$$\frac{1}{2} g(x) < \sup_j f(x, \omega_j(x)), \qquad \textit{for each} \quad x \in E.$$

*The mappings $\omega_j$ will be called* detectors of class $\mathcal{C}^p$ for $f$ over $E$.

*Proof.* Induction on $k$. If $k = 0$ it suffices to know that there exists a $\mathcal{C}^p$-mapping $\omega : \Omega \longrightarrow C$ which has prescribed values at a finite number of points; an immediate consequence of existence of definable $\mathcal{C}^p$-partitions of unity (Lemma 3.2).

Suppose now that $k > 0$. By the definable choice there exists a mapping $\omega_k : E \longrightarrow \operatorname{int} C$ such that

$$\frac{1}{2} g(x) < f(x, \omega_k(x)), \quad \text{for each} \quad x \in D. \tag{3.3.1}$$

There exists a closed subset $E_1$ of $E$ of dimension $l < k$ such that $E \setminus E_1$ is a $\mathcal{C}^p$-submanifold of $R^n$ of dimension $k$ and $\omega_k | E \setminus E_1$ is a $\mathcal{C}^p$-mapping. Moreover, by [KP] we can assume that $E \setminus E_1$ can be represented as a finite union

$$E \setminus E_1 = \bigcup_{\nu} \Gamma_\nu \tag{3.3.2}$$

of pairwise disjoint $k$-dimensional $\mathcal{C}^p$-submanifolds each of which, in some linear coordinate system is the graph of a $\mathcal{C}^p$-mapping

$$\Gamma_\nu = \{(x_1, \ldots, x_k, \gamma^\nu_{k+1}(x_1,\ldots,x_k), \ldots, \gamma^\nu_n(x_1,\ldots,x_k)) : (x_1,\ldots,x_k) \in D_\nu\},$$

of a $\mathcal{C}^p$-mapping $\gamma^\nu = (\gamma^\nu_{k+1}, \dots, \gamma^\nu_n) : D_\nu \longrightarrow R^{n-k}$ defined on some open subset $D_\nu \subset R^k$.

By natural projection

$$D_\nu \times R^{n-k} \ni (x_1, \dots, x_n) \longmapsto (x_1, \dots, x_k, \gamma^\nu(x_1, \dots, x_k)) \in \Gamma_n u$$

$\omega_k | \Gamma_\nu$ can be extended to a $\mathcal{C}^p$-mapping to a neighborhood of $\Gamma_\nu$; hence $\omega_k|E \backslash E_1$ can be extended to a $\mathcal{C}^p$-mapping defined on a neighborhood of $E \setminus E_1$. Consequently, $\omega_k|E \setminus E_1$ extends to a $\mathcal{C}^p$-Whitney field defined on $E \setminus E_1$. By the induction hypothesis, there exist $\mathcal{C}^p$-mappings $\omega_j : \Omega \longrightarrow \mathrm{int}C$ $(j \in \{0, \dots, k_1\})$ such that

$$\frac{1}{2} g(x) < \sup_j f(x, \omega_j(x)), \qquad \text{for each } x \in E_1 \tag{3.3.3.}$$

There exists an open neighborhood $W$ of $E_1$ in $\Omega$ such that (3.3.3) holds true for each $x \in W \cap E$. Then $E \setminus W$ is a closed subset of $\Omega$ contained in $E \setminus E_1$. By the Whitney Extension Theorem, there exists a $\mathcal{C}^p$-mapping $F : \Omega \longrightarrow R^m$ which extends $\omega_k|E \setminus W$. Then $U := F^{-1}(\mathrm{int}c)$ is an open neighborhood of $E \setminus W$ in $\Omega$. By Lemma 3.2, there exists $\mathcal{C}^p$-functions $\varphi_1, \varphi_2 : \Omega \longrightarrow [0,1]$ such that $\varphi_1 + \varphi_2 \equiv 1$, $\varphi_1 = 1$ on $E \setminus W$ and $\varphi_2 = 1$ on $\Omega \setminus U$. Choose any $c_0 \in \mathrm{int}C$ and put $\tilde{\omega}_k := \varphi_1 F + \varphi_2 c_0$. Then $\omega_0, \dots, \omega_{k-1}, \tilde{\omega}_k$ is the desired sequence for $E$.

**Example 3.2** The following example shows the assumption $g(x) > 0$, for each $x \in E$, in Proposition 3.1 cannot be omitted. Put

$$E := \{(x_1, x_2) \in R^2 : \, x_1^2 + x_2^2 \leqslant \frac{1}{4}\} \qquad \text{and} \qquad C = [0,1].$$

Consider $f : E \times C \longrightarrow [0, \infty)$ defined in the following way:

$$f(x_1, x_2, y) = 0, \qquad \text{when } x_1^2 + x_2^2 > 0 \text{ and } y \leqslant \frac{|x_1||x_2|}{2(x_1^2 + x_2^2)};$$

$$f(x_1, x_2, y) = y - \frac{|x_1||x_2|}{2(x_1^2 + x_2^2)}, \qquad \text{when } x_1^2 + x_2^2 > 0 \text{ and}$$
$$\frac{|x_1||x_2|}{2(x_1^2 + x_2^2)} \leqslant y \leqslant \frac{|x_1||x_2|}{2(x_1^2 + x_2^2)} + x_1^2 + x_2^2;$$

$$f(x_1, x_2, y) = 2(x_1^2 + x_2^2) - \Big(y - \frac{|x_1||x_2|}{2(x_1^2 + x_2^2)}\Big), \qquad \text{when } x_1^2 + x_2^2 > 0 \text{ and}$$
$$\frac{|x_1||x_2|}{2(x_1^2 + x_2^2)} + x_1^2 + x_2^2 \leqslant y \leqslant \frac{|x_1||x_2|}{2(x_1^2 + x_2^2)} + 2(x_1^2 + x_2^2);$$

$$f(x_1, x_2, y) = 0, \qquad \text{when } x_1^2 + x_2^2 > 0 \text{ and } \frac{|x_1||x_2|}{2(x_1^2 + x_2^2)} + 2(x_1^2 + x_2^2) \leqslant y \leqslant 1;$$

$$f(x_1, x_2, y) = 0, \qquad \text{when } x_1^2 + x_2^2 = 0.$$

Clearly, $g(x_1, x_2) = x_1^2 + x_2^2$ and $f$ does not admit event continuous detectors over $E$.

## 4. Yomdin-Gromov trick and a smoothing homeomorphism $\omega$.

This paragraph concerns a method of smoothing functions of one variable mimicking Yomdin and Gromov (cf. $[Y_1, Y_2]$ and $[G;$ Section 4.1$]$) which appeared useful to get smooth parametrizations of subsets definable in o-minimal structures (cf. [K-CPV]). It is crucial in the proof of our basic Lemma 5.1.

**Lemma 4.1.** *Let $\lambda : (a,b) \longrightarrow R$ be a definable $\mathcal{C}^{p+1}$-function, where $p \in \mathbb{N}$, $p \geqslant 1$, defined on an open interval $(a,b) \subset R$ such that, for each $\nu \in \{2, \dots, p+1\}$, $\lambda^{(\nu)} \geqslant 0$ on $(a,b)$ or $\lambda^{(\nu)} \leqslant 0$ on $(a,b)$. Then, for any closed interval $[t-r, t+r] \subset (a,b)$, where $r \in R$ and $r > 0$,*

$$|\lambda^{(p)}(t)| \leqslant 2^{\binom{p+2}{2}-2} \sup_{[t-r,t+r]} |\lambda| \frac{1}{r^p}.$$

*Proof.* Induction on $p$ (see [K-CPV; Lemma 2.1] for details).

Applying Lemma 4.1 to $\lambda'$ in the place of $\lambda$ and $\mu - 1$ in the place of $p$, we have the following

**Corollary 4.2.** *Under the assumptions of Lemma 4.1,*

$$|\lambda^{(\mu)}(t)| \leqslant C_p \sup_{(a,b)} |\lambda'| \frac{1}{|t-a|^{\mu-1}},$$

*for each $t \in \left(a, \frac{a+b}{2}\right]$ and $\mu \in \{2, \dots, p\}$, where $C_p := 2^{\binom{p+1}{2}-2}$. In particular, if $\lambda'$ is bounded; i. e. $|\lambda'| \leqslant M$, where $M \in R$ and $M > 0$, then*

$$\text{(4.2.1)} \qquad |\lambda^{(\mu)}(t)| \leqslant C_p M \frac{1}{|t-a|^{\mu-1}}, \quad \textit{for each} \quad t \in \left(a, \frac{a+b}{2}\right], \ \mu \in \{2, \dots, p\}.$$

**Lemma 4.3.** *Let $\lambda : (a,c] \longrightarrow R$ be a definable $\mathcal{C}^p$-function, where $a, c \in R$, $a < c$ such that*

$$\text{(4.3.1)} \qquad |\lambda^{(\mu)}(t)| \leqslant L \frac{1}{|t-a|^{\mu-1}}, \quad \textit{for each} \quad t \in (a,c], \ \mu \in \{1, \dots, p\}$$

*where $L \in R$ is a positive constant. Fix $m \in \mathbb{N}$, $m \geqslant p+1$. Fix any $\alpha \in R$. Put $\varphi(\tau) := \lambda(a + (\tau - \alpha)^m)$, for each $\tau \in (\alpha, \beta]$, where $\beta = \alpha + \sqrt[m]{c-a}$.*

*Then there exists a positive constant $M$ depending only on $L$ and $m$ such that $|\varphi^{(\mu)}(\tau)| \leqslant L|\tau - \alpha|^{m-\mu}$, for each $\tau \in (\alpha, \beta]$ and $\mu \in \{1, \dots, p\}$. Consequently, $\varphi$ has a unique extension to a $\mathcal{C}^p$-function $\varphi : [\alpha, \beta] \longrightarrow R$ $p$-flat at $\alpha$.*

*Proof.* Without any loss of generality we can assume that $a = 0 = \alpha$. Then $\varphi(\tau) = \lambda(\tau^m)$. For each $\mu \in \{1, \dots, p\}$, $\varphi^{(\mu)}(\tau) =$

$$a_{1\mu}\tau^{m-\mu}\lambda'(\tau^m) + a_{2\mu}\tau^{2m-\mu}\lambda''(\tau^m) + a_{3\mu}\tau^{3m-\mu}\lambda^{(3)}(\tau^m) + \dots + a_{\mu\mu}\tau^{\mu m-\mu}\lambda^{(\mu)}(\tau^m),$$

where $a_{i\mu}$ are positive integers defined inductively by the following formulae

$$a_{1\mu} = \frac{m!}{(m-\mu)!}, \quad a_{i\mu} = m a_{(i-1)(\mu-1)} + (im - \mu + 1) a_{i(\mu-1)}, \quad a_{\mu\mu} = m^\mu.$$

By (4.3.1), it follows that $|\varphi^{(\mu)}(\tau)| \leqslant$

$$a_{1\mu}\tau^{m-\mu}L + a_{2\mu}\tau^{2m-\mu}\frac{L}{\tau^m} + a_{3\mu}\tau^{3m-\mu}\frac{L}{\tau^{2m}} + \cdots + a_{\mu\mu}\tau^{\mu m-\mu}\frac{L}{\tau^{(\mu-1)m}} =$$

$$L(a_{1\mu} + \cdots + a_{\mu\mu})\tau^{m-\mu}.$$

It will be convenient to have the $p$-flatness of a parametrization of the segment $[a, c]$ at the right end as well. It is why we use the following increasing parametrization of the segment $[\alpha, \beta]$ $p$-flat at right end:

$$\tau := \alpha + \sqrt[m]{c-a} - (\gamma - s)^m,$$

where $\gamma \in R$ is arbitrary, $s \in [\gamma, \delta]$ and $\delta = \gamma + \sqrt[2m]{c-a}$. This leads us to the following.

**Corollary 4.4.** *Let $\lambda : (a, b) \longrightarrow R$ be a $\mathcal{C}^{p+1}$-function, where $p \in \mathbb{N}$, $p \geqslant 1$, defined on an open interval $(a, b) \subset R$ such that $\lambda'$ is bounded and, for each $\nu \in \{2, \ldots, p+1\}$, $\lambda^{(\nu)} \geqslant 0$ on $(a, b)$ or $\lambda^{(\nu)} \leqslant 0$ on $(a, b)$. Let $m \in \mathbb{N}$, $m \geqslant p+1$. Let $\gamma_0 \in R$ be fixed arbitrarily, $\gamma_1 := \gamma_0 + \sqrt[2m]{(b-a)/2}$, $\gamma_2 := \gamma_1 + \sqrt[2m]{(b-a)/2} = \gamma_0 + 2\sqrt[2m]{(b-a)/2}$. Put*

$$\omega(a, b; s) := \begin{cases} a + \left[\sqrt[m]{(b-a)/2} - (\gamma_1 - s)^m\right]^m, & \text{if } s \in [\gamma_0, \gamma_1] \\ b - \left[\sqrt[m]{(b-a)/2} - (s - \gamma_1)^m\right]^m, & \text{if } s \in [\gamma_1, \gamma_2]. \end{cases}$$

*Then $\omega : [\gamma_0, \gamma_2] \longrightarrow [a, b]$ is an increasing homeomorphism such that $\omega(\gamma_0) = a$, $\omega(\gamma_1) = \frac{a+b}{2}$, $\omega(\gamma_2) = b$ and $\lambda \circ \omega$ extends uniquely to a $\mathcal{C}^p$ function $\lambda \circ \omega : [\gamma_0, \gamma_2] \longrightarrow R$ $p$-flat at points $\gamma_0, \gamma_1$ and $\gamma_2$.*

**Corollary 4.5.** *Let $y_0 \leqslant y_1 \leqslant \cdots \leqslant y_r$ be (at most) $r+1$ points in $R$. Let $\lambda : [y_0, y_r] \longrightarrow R$ be a continuous function such that, for each $i \in \{0, \ldots, r-1\}$, if $y_i < y_{i+1}$, then $\lambda|(y_i, y_{i+1})$ satisfies the assumptions of Corollary 4.4. Let $m \in \mathbb{N}$, $m \geqslant p+1$. Let the sequence of points in $R$*

$$\gamma_0 \leqslant \gamma_1 \leqslant \gamma_2 \leqslant \cdots \leqslant \gamma_{2r}$$

*be defined inductively by: $\gamma_0 \in R$ fixed arbitrarily, $\gamma_{2i+1} := \gamma_{2i} + \sqrt[2m]{(y_{i+1} - y_i)/2}$, $\gamma_{2i+2} := \gamma_{2i+1} + \sqrt[2m]{(y_{i+1} - y_i)/2}$ $(i \in \{0, \ldots, r-1\})$. Put $\omega(y_0, \ldots, y_r; s) :=$*

$$\begin{cases} y_i + \left[\sqrt[m]{(y_{i+1} - y_i)/2} - (\gamma_{2i+1} - s)^m\right]^m, & \text{if } s \in [\gamma_{2i}, \gamma_{2i+1}] \\ y_{i+1} - \left[\sqrt[m]{(y_{i+1} - y_i)/2} - (s - \gamma_{2i+1})^m\right]^m, & \text{if } s \in [\gamma_{2i+1}, \gamma_{2i}], \end{cases}$$

*for $i \in \{0, \ldots, r-1\}$ and*

$$\omega(y_0, \ldots, y_r; s) := \begin{cases} y_0 - (\gamma_0 - s)^m, & \text{if } s \in (-\infty, \gamma_0], \\ y_r + (s - \gamma_{2r})^m, & \text{if } s \in [\gamma_{2r}, \infty). \end{cases}$$

*Then $\omega : R \longrightarrow R$ is an increasing homeomorphism of class $\mathcal{C}^p$ such that $\omega(\gamma_{2i}) = y_i$ and $\omega(\gamma_{2i+1}) = \frac{y_i + y_{i+1}}{2}$ $(i \in \{0, \ldots, r-1\})$, and $\lambda \circ \omega : [\gamma_0, \gamma_{2r}] \longrightarrow R$ is of class $\mathcal{C}^p$, $p$-flat at points $\gamma_0, \ldots, \gamma_{2r}$.*

## 5. Basic lemmata.

**Lemma 5.1.** *Let $D$ be a bounded subset of $R^{n-1}$ such that $D = \overline{\mathrm{int} D}$, let $m$, $p$ be positive integers such that $m \geqslant p+1$. Let*

$$\alpha_0 \leqslant \alpha_1 \leqslant \cdots \leqslant \alpha_r : D \longrightarrow R$$

*be a finite sequence of continuous functions such that $\mathcal{K} := \{\overline{(\alpha_i, \alpha_{i+1})} : i \in \{0, \ldots, r-1\}\}$ is a family of capsules in $R^n$. Let $\mathcal{K}_1 \subset \mathcal{K}$ and put $A := |\mathcal{K}|$ and $A_1 := |\mathcal{K}_1|$.*

*Let $f = (f_1, \ldots, f_d) : A_1 \longrightarrow R^d$ be a continuous mapping such that for each $K \in \mathcal{K}_1$ there exists continuous partial derivatives*

$$\frac{\partial^\sigma (f|\overset{\circ}{K})}{\partial x_n^\sigma} \qquad (\sigma \in \{1, \ldots, p+1\}).$$

*Then there exists a finite sequence of continuous functions*

$$\delta_0 \leqslant \delta_1 \leqslant \cdots \leqslant \delta_k : D \longrightarrow R$$

*and a homeomorphism*

$$\Phi : [\delta_0, \delta_k] \longrightarrow [\alpha_0, \alpha_r]$$

*such that:*

(5.1.1) *$\Phi$ is of the form $\Phi(x', \xi_n) = (x', \varphi(x', \xi_n))$, where $x' = (x_1, \ldots, x_{n-1})$.*

(5.1.2) *For each $j \in \{0, \ldots, k-1\}$ the derivatives*

$$\frac{\partial^\sigma \varphi}{\partial \xi_n^\sigma} \qquad (\sigma \in \{1, \ldots, p+1\})$$

*exist continuous in $(\delta_j, \delta_{j+1})$ and have continuous extensions by zero to $\overline{(\delta_j, \delta_{j+1})}$; moreover*

$$\frac{\partial \varphi}{\partial \xi_n} > 0 \qquad \text{on } (\delta_j, \delta_{j+1}).$$

(5.1.3) *The sequence $\theta_j(x') := \varphi(x', \delta_j(x'))$, where $x' \in D$ and $j \in \{0, \ldots, k\}$, is a refinement of $\alpha_0, \ldots, \alpha_r$; in particular, $\alpha_0 = \theta_0$ and $\alpha_r = \theta_k$.*

(5.1.4) *$\mathcal{L} := \{\overline{(\delta_j, \delta_{j+1})} : j \in \{0, \ldots, k-1\}\}$ is a family of capsules in $R^n$ such that $\{\Phi(L) : L \in \mathcal{L}\}$ is a family of capsules which is a refinement of $\mathcal{K}$.*

(5.1.5) *Put $\mathcal{L}_1 := \{L \in \mathcal{L} : \Phi(L) \subset K, \text{ for some } K \in \mathcal{K}_1\}$. For each $L \in \mathcal{L}_1$, there exist continuous partial derivatives*

$$\frac{\partial^\sigma (f \circ \Phi|\overset{\circ}{L})}{\partial \xi_n^\sigma} \qquad (\sigma \in \{1, \ldots, p+1\})$$

*and these for $\sigma \in \{1, \dots, p\}$ extend continuously by zero to $L$.*

(5.1.6) *On each capsule $L \in \mathcal{L}$ the function $\varphi$ is either of the form*

$$\xi_n^{2m} + a_1(x')\xi_n^{2m-1} + \cdots + a_{2m}(x'), \textit{ where } a_1, \dots, a_{2m} : D \to R \textit{ are continuous}$$

*(it is so in particular, when $L \notin \mathcal{L}_1$)*
*or of the form*

$$\pm f_{\varkappa}^{-1}(x', \pm\xi_n^{2m} + a_1(x')\xi_n^{2m-1} + \cdots + a_{2m}(x')), \quad \textit{where } a_1, \dots, a_{2m} : D \to R$$

*are continuous and where $\varkappa \in \{1, \dots, d\}$ and $f_{\varkappa}^{-1}$ denotes the inverse of $f_{\varkappa}$ with respect to the variable $x_n$ on the capsule $\Phi(L)$ on which*

$$\left|\frac{\partial f_{\varkappa}}{\partial x_n}\right| \geqslant c^{-1}, \quad \textit{with some constant } c > 1.$$

*Proof.* Fix any $c > 1$. By Proposition 2.5, passing perhaps to a refinement of $\mathcal{K}$ one can assume that for each $K \in \mathcal{K}$ we have either

$$\left|\frac{\partial f_{\varkappa}}{\partial x_n}\right| \leqslant c, \quad \text{in } \overset{\circ}{K} \quad \text{for each} \quad \varkappa \in \{1, \dots, d\}, \tag{5.1.7}$$

or

$$\left|\frac{\partial f_{\varkappa}}{\partial x_n}\right| \geqslant c^{-1}, \quad \text{in } \overset{\circ}{K} \quad \text{for some } \varkappa \in \{1, \dots, d\}, \tag{5.1.8}$$

and in the second case among $f_{\varkappa}$ satisfying (5.1.8) there is one, denote it by $f_K$, such that

$$\left|\frac{\partial f_{\varkappa}}{\partial x_n}\right| \Big/ \left|\frac{\partial f_K}{\partial x_n}\right| \leqslant c^d, \quad \text{in } \overset{\circ}{K} \quad \text{for each} \quad \varkappa \in \{1, \dots, d\}. \tag{5.1.9}$$

Now we define a function $\lambda : [\alpha_0, \alpha_r] \longrightarrow R$ inductively as follows. Put first

$$\lambda(x', \alpha_0(x')) := \alpha_0(x'), \qquad \text{for each } x' \in D.$$

We define $\lambda$ on $[\alpha_i, \alpha_{i+1}]$ according to the following two cases.

*Case I.* If $\overline{(\alpha_i, \alpha_{i+1})} \notin \mathcal{K}_1$ or if $\overline{(\alpha_i, \alpha_{i+1})} \in \mathcal{K}_1$ and (5.1.7) is satisfied on $(\alpha_i, \alpha_{i+1})$, then put

$$\lambda(x', x_n) := \lambda(x', \alpha_i(x')) + x_n - \alpha_i(x'), \qquad \text{for each } (x', x_n) \in [\alpha_i, \alpha_{i+1}].$$

*Case II.* If $K = \overline{(\alpha_i, \alpha_{i+1})} \in \mathcal{K}_1$ and (5.1.8) is satisfied on $(\alpha_i, \alpha_{i+1})$, then put

$$\lambda(x', x_n) := \lambda(x', \alpha_i(x')) + |f_K(x', x_n) - f_K(x', \alpha_i(x'))|, \qquad \text{for each } (x', x_n) \in [\alpha_i, \alpha_{i+1}].$$

Put $\Lambda(x', x_n) := (x', \lambda(x', x_n)))$. Then $\Lambda$ is a homeomorphism of $[\alpha_0, \alpha_r]$ onto $[\beta_0, \beta_r]$, where $\beta_i(x') := \lambda(x', \alpha_i(x'))$ $(x' \in D,\ i \in \{0, \dots, r\})$ and $\overline{(\beta_i, \beta_{i+1})}$ $(i \in \{0, \dots, r-1\})$ are capsules in $R^n$.

The partial derivatives

$$\frac{\partial^\sigma \lambda}{\partial x_n^\sigma} \qquad (\sigma \in \{1, \dots, p+1\})$$

exist and are continuous in every $(\alpha_i, \alpha_{i+1})$ and $\frac{\partial \lambda}{\partial x_n} \equiv 1$ or $\frac{\partial \lambda}{\partial x_n} \geqslant c^{-1}$ on $(\alpha_i, \alpha_{i+1})$; hence $\lambda : [\alpha_0, \alpha_r] \longrightarrow R$ is continuous, strictly increasing with respect to $x_n$. Let

$$\Psi : [\beta_0, \beta_r] \ni (x', \zeta_n) \longmapsto (x', \psi(x', \zeta_n)) \in [\alpha_0, \alpha_r]$$

denote the inverse homeomorphism to $\Lambda$. Then

$$0 < \frac{\partial \psi}{\partial \zeta_n}(x', \zeta_n) = \frac{1}{\frac{\partial \lambda}{\partial x_n}(x', \psi(x', \zeta_n))} \leqslant \max\{1, c\} = c$$

on every $(\beta_i, \beta_{i+1})$. Fix now any $K = \overline{(\alpha_i, \alpha_{i+1})} \in \mathcal{K}$.

If $K$ is of type as in Case I, then for each $(x', \zeta_n) \in (\beta_i, \beta_{i+1})$

$$\beta_i(x') + \psi(x', \zeta_n) - \alpha_i(x') \equiv \zeta_n; \quad \text{hence} \quad \psi(x', \zeta_n) = \zeta_n - \beta_i(x') + \alpha_i(x');$$

consequently, if $K \in \mathcal{K}_1$, then for each $\varkappa \in \{1, \dots, d\}$

$$\Big|\frac{\partial (f_\varkappa \circ \Psi)}{\partial \zeta_n}(x', \zeta_n)\Big| = \Big|\frac{\partial f_\varkappa}{\partial x_n}(x', \psi(x', \zeta_n)\Big| \leqslant c.$$

If $K \in \mathcal{K}_1$ is of type as in Case II, then for each $(x', \zeta_n) \in (\beta_i, \beta_{i+1})$

$$\beta_i(x') + |f_K(x', \psi(x', \zeta_n)) - f_K(x', \alpha_i(x'))| \equiv \zeta_n,$$

$$\text{hence} \quad \psi(x', \zeta_n) = f_K^{-1}\big(x', \pm(\zeta_n - \beta_i(x')) + f_K(x', \alpha_i(x')\big);$$

consequently, for each $\varkappa \in \{1, \dots, d\}$

$$\Big|\frac{\partial (f_\varkappa \circ \Psi)}{\partial \zeta_n}(x', \zeta_n)\Big| = \Big|\frac{\partial f_\varkappa}{\partial x_n}(x', \psi(x', \zeta_n))\Big| \Big/ \Big|\frac{\partial f_K}{\partial x_n}(x', \psi(x', \zeta_n))\Big| \leqslant c^d.$$

By Proposition 2.4, passing to a refinement $\overline{(\gamma_j, \gamma_{j+1})}$ $(j \in \{0, \dots, s-1\})$ of capsules $\overline{(\beta_i, \beta_{i+1})}$ , where the sequence $\gamma_0 \leqslant \gamma_1 \leqslant \dots \leqslant \gamma_s$ is a refinement of the sequence $\beta_0 \leqslant \dots \beta_r$, we can additionally assume that for each $j \in \{0, \dots, s-1\}$ and each $\sigma \in \{2, \dots, p+1\}$ we have either

$$\Big|\frac{\partial^\sigma \psi}{\partial \zeta_n^\sigma}(x', \zeta_n)\Big| \leqslant c, \qquad \text{on} \qquad (\gamma_j, \gamma_{j+1}) \tag{5.1.10}$$

or

$$\Big|\frac{\partial^\sigma \psi}{\partial \zeta_n^\sigma}(x', \zeta_n)\Big| \geqslant c^{-1}, \qquad \text{on} \qquad (\gamma_j, \gamma_{j+1}) \tag{5.1.11}$$

and, similarly, for each $\varkappa \in \{1, \dots, d\}$, either

$$\Big|\frac{\partial^\sigma (f_\varkappa \circ \Psi)}{\partial \zeta_n^\sigma}(x', \zeta_n)\Big| \leqslant c, \qquad \text{on} \qquad (\gamma_j, \gamma_{j+1}) \tag{5.1.12}$$

or

$$\Big|\frac{\partial^\sigma (f_\varkappa \circ \Psi)}{\partial \zeta_n^\sigma}(x', \zeta_n)\Big| \geqslant c^{-1}, \qquad \text{on} \qquad (\gamma_j, \gamma_{j+1}). \tag{5.1.13}$$

Notice that the condition (5.1.13) implies a constant sign of the partial derivative involved on $(\gamma_j, \gamma_{j+1})$.

Finally, we modify the homeomorphism $\Psi$ with respect to the variable $\zeta_n$ by means of the smoothing homeomorphism $\omega$ with a parameter (Corollary 4.5):

$$\Phi(x', \xi_n) := \Psi\big(x', \omega(\gamma_0(x'), \dots, \gamma_s(x'); \xi_n)\big),$$

where $(x', \xi_n) \in [\delta_0, \delta_{2s}]$ and where $\delta_0 \leqslant \dots \leqslant \delta_{2s} : D \longrightarrow R$ is a sequence of continuous functions.

**Lemma 5.2.** *Let $\Delta \subset R^n$ be a simplex of dimension $n$, $p$ a positive integer and let*

$$\beta_0 \leqslant \beta_1 \leqslant \dots \beta_k : \Delta \longrightarrow R$$

*be $\mathcal{C}^p$-functions such that for every face $S$ of $\Delta$ and each $j \in \{0, \dots, k-1\}$ either $\beta_{j+1} - \beta_j \neq 0$ on $\mathring{S}$ or $\beta_{j+1} - \beta_j \equiv 0$ on $S$ and let in the latter $\beta_{j+1} - \beta_j$ be $p$-flat on $S$.*

*Let*

$$\lambda_0 \leqslant \lambda_1 \leqslant \dots \leqslant \lambda_k : \Delta \longrightarrow R$$

*be continuous* PL*-functions such that for every face $S$ of $\Delta$ and $j \in \{0, \dots, k\}$ $\lambda_j | S$ is affine and*

$$\beta_j \equiv \beta_{j+1} \quad \text{on } S \quad \Longleftrightarrow \quad \lambda_j \equiv \lambda_{j+1} \quad \text{on } S \quad (j \in \{0, \dots, k-1\}) \tag{5.2.1}$$

*Then the formula*

$$\Psi(u, \zeta) = \begin{cases} \Big(u, \frac{\zeta - \lambda_j(u)}{\lambda_{j+1} - \lambda_j(u)}(\beta_{j+1}(u) - \beta_j(u)) + \beta_j(u)\Big), & \text{if } \lambda_j(u) < \lambda_{j+1}(u), \\ (u, \beta_j(u)), & \text{if } \lambda_j(u) = \lambda_{j+1}(u), \end{cases}$$

*for $(u, \zeta) \in [\lambda_j, \lambda_{j+1}]$, defines a homeomorphism $[\lambda_0, \lambda_k]$ onto $[\beta_0, \beta_k]$, such that $\Psi(u, \lambda_j(u)) = (u, \beta_j(u))$, for $u \in \Delta$, $j \in \{0, \dots, k\}$ and for each $j \in \{0, \dots, k-1\}$ $\Psi | [\lambda_j, \lambda_{j+1}]$ is of class $\mathcal{C}^p$.*

*Proof.* Assume that $\lambda_j < \lambda_{j+1}$ on $\mathring{\Delta}$. By a linear change of coordinates we can assume that

$$\Delta = \{u \in R^n : u_\nu \leqslant 0 \quad (\nu \in \{1, \dots, n\}) \quad \sum_{\nu=1}^n u_\nu \leqslant 1\}$$

and $\quad S := \{u \in \varDelta :\ \lambda_j(u) = \lambda_{j+1}(u)\}$

$$= \{u \in \varDelta :\ \beta_j(u) = \beta_{j+1}(u)\} = \{u \in \varDelta : u_{l+1} = \cdots = u_n = 0\}.$$

Then for each $u \in \varDelta$

$$\lambda_{j+1}(u) - \lambda_j(u) = \sum_{\nu=l+1}^{n} c_\nu u_\nu, \quad \text{where } \ c_\nu > 0 \ \ (\nu \in \{l+1, \ldots, n\}).$$

We want to check that

$$\frac{\partial^{|\sigma|+\rho}}{\partial u^\sigma \partial \zeta^\rho}\Big[\frac{\zeta - \lambda_j(u)}{\lambda_{j+1}(u) - \lambda_j(u)}(\beta_{j+1}(u) - \beta_j(u))\Big] \longrightarrow 0,$$

when $(\lambda_j, \lambda_{j+1}) \ni (u, \zeta) \to (u_0, \lambda_j(u_0)) \in S \times R,\ \sigma \in \mathbb{N}^n,\ \rho \in \mathbb{N}$ and $|\sigma| + \rho \leqslant p$.

In view of the Leibnitz formula, it suffices to check that

$$(\zeta - \lambda_j(u)) D^\sigma \Big[\frac{1}{\lambda_{j+1} - \lambda_j}\Big](u) D^\rho(\beta_{j+1} - \beta_j)(u) \longrightarrow 0,$$

when $\sigma, \rho \in \mathbb{N}^n,\ |\sigma| + |\rho| \leqslant p$ and $(u, \zeta) \to (u_0, \lambda_j(u_0))$, and

$$D^\sigma \Big[\frac{1}{\lambda_{j+1} - \lambda_j}\Big](u) D^\rho(\beta_{j+1} - \beta_j)(u) \longrightarrow 0,$$

when $\sigma, \rho \in \mathbb{N}^n,\ |\sigma| + |\rho| \leqslant p - 1$ and $(u, \zeta) \to (u_0, \lambda_j(u_0))$.

In the first case, by the Taylor formula

$$(\zeta - \lambda_j(u)) D^\sigma \Big[\frac{1}{\lambda_{j+1} - \lambda_j}\Big](u) D^\rho(\beta_{j+1} - \beta_j)(u) = (\zeta - \lambda_j(u)) \frac{C}{(\lambda_{j+1}(u) - \lambda_j(u))^{|\sigma|+1}}$$

$$\sum_{|\delta| = p - |\rho|} \frac{1}{\delta!}(u - \pi(u))^\delta D^{\rho + (0,\delta)}(\beta_{j+1} - \beta_j)(\pi(u) + \theta(u - \pi(u))),$$

where $C > 0$, $\pi(u) = (u_1, \ldots, u_l, 0, \ldots, 0)$ and $\theta \in (0, 1)$. Consequently, with some constant $C' > O$,

$$\Big|(\zeta - \lambda_j(u)) D^\sigma \Big[\frac{1}{\lambda_{j+1} - \lambda_j}\Big](u) D^\rho(\beta_{j+1} - \beta_j)(u)\Big| \leqslant$$

$$\frac{C'}{\big(\sum_{\nu=l+1}^{n} c_\nu u_\nu\big)^{|\sigma|}} \Big(\sum_{\nu=l+1}^{n} u_\nu\Big)^{p-|\rho|} \sup_{\substack{|\mu| = p \\ \theta \in [0,1]}} |D^\mu(\beta_{j+1} - \beta_j)(\pi(u) + \theta(u - \pi(u)))|,$$

which tends to 0, when $u$ tends to $u_0$. Similarly in the second case.

We will also need some $\mathcal{C}^p$-extension result based on the following $\mathcal{C}^1$-extension theorem (cf. [Pa, Proposition 2]).

**Theorem 5.3** (**$\mathcal{C}^1$-Extension Theorem**). *Let $f : S \longrightarrow R$ be a $\mathcal{C}^1$-function defined on a cell*

$$S = \{(x', x_n) \in R^n : \ x' \in G, \ \ \varphi(x') < x_n < \psi(x')\}$$

*in $R^n$ such that $G$ is an open subset of $R^{n-1}$ and $\varphi < \psi : G \longrightarrow R$ are of class $\mathcal{C}^1$.*

*Assume that $\dfrac{\partial f}{\partial x_n}$ has a finite limit value*[2] *at (almost) each point of $\varphi$ ( for example, when $\dfrac{\partial f}{\partial x_n}$ is bounded ).*

*Then there is a closed nowhere dense subset $Z$ of $\varphi$ such that $f$ extends to a $\mathcal{C}^1$-function*

$$f : S \cup (\varphi \setminus Z) \longrightarrow R$$

*to $S \cup (\varphi \setminus Z)$ as a $\mathcal{C}^1$-submanifold of $R^n$ with boundary $\varphi \setminus Z$.*

*Proof.* With no loss of generality we can assume that $\varphi \equiv 0$; i.e. $\varphi = G \times \{0\}$. For each $a \in G$ the set

$$\operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_n}(x)$$

of all finite limit values of $\dfrac{\partial f}{\partial x_n}$ at point $(a, 0)$ is a closed non-empty interval, because $S$ satisfies the Łojasiewicz (s)-condition at points of $\varphi$. Since

$$\bigcup_{a \in G} \{a\} \times \operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_n}(x) = \overline{\frac{\partial f}{\partial x_n}} \setminus \frac{\partial f}{\partial x_n}$$

is of dimension $n - 1$, it follows that there exists a closed nowhere dense subset $E$ of $G$ such that there exists a finite limit

$$\lim_{x \to (a,0)} \frac{\partial f}{\partial x_n}(x), \quad \text{for each } a \in G \setminus E.$$

This implies in particular that for each $x' \in G \setminus E$ there exists a finite limit

(5.3.1) $$g(x') := \lim_{x_n \to 0} f(x', x_n) \in R.$$

There exists a closed nowhere dense subset $Z$ of $G$ containing $E$ such that $g$ is of class $\mathcal{C}^1$ on $G \setminus Z$. Hence, without any loss of generality we can assume that $g \equiv 0$ and $Z = \emptyset$. Repeating the previous argument with dimension we conclude that after removing a closed nowhere dense subset from $G$ $f$ extends by 0 to a continuous function on $S \cup \varphi$.

Now, we will show that for any $i \in \{1, \ldots, n-1\}$ the partial derivative $\partial f / \partial x_i$ extends by 0 to a continuous function defined on $S \setminus E$, where $E \subset \varphi$ and $\dim E < k$. With no loss of generality we assume that $i = n - 1$. First we will show that

(5.3.2) $$0 \in \operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_{n-1}}(x), \qquad \text{for each } a \in G.$$

---

[2]An element $\alpha \in \overline{R}$ is a limit value of a function $g : S \longrightarrow R$ at $a \in \overline{S}$ if and only if there is an arc $\gamma : (0, 1) \longrightarrow S$ such that $\lim_{t \to 0} \gamma(t) = a$ and $\lim_{t \to 0} g(\gamma(t)) = \alpha$.

To check this fix any arbitrarily small $\eta > 0$ such that $B(a,\eta) := \{u \in R^k : |u-a| \leqslant \eta\} \subset G$ and any $\varepsilon > 0$. There exists $\delta > 0$ such that $|f(x',x_n)| \leqslant \varepsilon\eta$, when $x' \in B(a,\eta)$ and $x_n \in (0,\delta)$. By the Mean Value Theorem there exists $\theta \in (0,1)$ such that

$$\Big|\frac{\partial f}{\partial x_{n-1}}(\tilde{a}, a_{n-1}+\theta\eta, x_n)\Big| = \Big|\frac{f(\tilde{a}, a_{n-1}+\eta, x_n) - f(a, x_n)}{\eta}\Big| \leqslant 2\varepsilon,$$

where $a = (\tilde{a}, a_{n-1})$. This ends the proof of (5.3.2). Repeating the previous argument we conclude that

$$\lim_{x\to(a,0)} \frac{\partial f}{\partial x_{n-1}}(x) = 0, \tag{5.3.3}$$

for $a \in G \setminus Z$, where $Z$ is a closed subset of $Z$ of dimension $< k$. This ends the proof of the theorem.

**Lemma 5.4 (basic $\mathcal{C}^p$-extension lemma).** *Let $\Omega \subset R^k$ be an open subset, where $k \in \{0,\dots,n-1\}$, and let $p$ be a positive integer.*
*Let*

$$\varphi_{k+1}, \psi_{k+1} : \Omega \longrightarrow R \quad \textit{be } \mathcal{C}^p\textit{-functions such that } \varphi_{k+1} < \psi_{k+1};$$

$$\varphi_{k+2}, \psi_{k+2} : [\varphi_{k+1}, \psi_{k+1}) \longrightarrow R \quad \textit{be } \mathcal{C}^p\textit{-functions such that } \varphi_{k+2} < \psi_{k+2}$$
$$\textit{on } (\varphi_{k+1}, \psi_{k+1}) \textit{ and } \varphi_{k+2} = \psi_{k+2} \textit{ on } \varphi_{k+1};$$

$$\varphi_{k+3}, \psi_{k+3} : [\varphi_{k+2}, \psi_{k+2}] \longrightarrow R \quad \textit{be } \mathcal{C}^p\textit{-functions such that } \varphi_{k+3} < \psi_{k+3}$$
$$\textit{on } (\varphi_{k+2}, \psi_{k+2}) \textit{ and } \varphi_{k+3} = \psi_{k+3} \textit{ on } \varphi_{k+2}|\varphi_{k+1};$$

$$\dots$$

$$\varphi_n, \psi_n : [\varphi_{n-1}, \psi_{n-1}] \longrightarrow R \quad \textit{be } \mathcal{C}^p\textit{-functions such that } \varphi_n < \psi_n$$
$$\textit{on } (\varphi_{n-1}, \psi_{n-1}) \textit{ and } \varphi_n = \psi_n \textit{ on } \varphi_{n-1}|(\dots(\varphi_{k+2}|\varphi_{k+1})\dots).$$

*Put*

$$\Sigma := \{(x_1,\dots,x_n) \in \Omega \times R^{n-k} : \varphi_j(x_1,\dots,x_{j-1}) = x_j \quad (j \in \{k+1,\dots,n\})\}.$$

*Let $f : [\varphi_n, \psi_n] \setminus \Sigma \longrightarrow R$ a $\mathcal{C}^p$-function such that all the partial derivatives*

$$\frac{\partial^p f}{\partial x_{k+1}^{\alpha_{k+1}} \dots \partial x_n^{\alpha_n}} \quad (|\alpha| = \alpha_{k+1} + \dots + \alpha_n = p) \quad \textit{have continuous} \tag{5.4.1}$$

*extensions to $\Sigma$.*

*Then there exists a closed subset $E$ of $\Sigma$ of dimension $< k$ such that $f$ extends to a $\mathcal{C}^p$-function defined on $[\varphi_n, \psi_n] \setminus E$.*

*Proof.* First assume that $p = 1$. With no loss of generality we can assume that

$$\varphi_{k+1} \equiv 0,\ \varphi_{k+2}|\varphi_{k+1} \equiv 0,\ \dots,\ \varphi_n|(\dots(\varphi_{k+2}|\varphi_{k+1})\dots) \equiv 0; \tag{5.4.2}$$

in other words $\Sigma = \Omega \times \{0\}^{n-k}$.

Put $y := (x_{k+1}, \dots, x_n)$. For any $a \in \Omega$ the function $f_a : [\varphi_n, \psi_n]_a \setminus \{0\} \longrightarrow R$ defined by $f_a(y) := f(a, y)$ on the set $[\varphi_n, \psi_n]_a \setminus \{0\} := \{y \neq 0 : (a, y) \in [\varphi_n, \psi_n]\}$ is a $\mathcal{C}^1$-function with bounded first order partial derivatives near 0. Since $[\varphi_n, \psi_n]_a \setminus \{0\}$ is quasi-convex[3] near 0, this implies that the limit

$$g(a) := \lim_{y \to 0} f_a(y)$$

exists in $R$ (cf. [Pa, Proposition 1]). Since there exists a closed subset $E$ of $\Omega$ of dimension $< k$ such that $g$ is of class $\mathcal{C}^1$ on $\Omega \setminus E$, with no loss of generality we can assume that $g$ is $\mathcal{C}^1$ and then that $g \equiv 0$.

For each $a \in \Omega$ the set $\operatorname*{Lim}_{x \to (a,0)} f(x)$ of all finite limit values of $f$ at point $(a, 0)$ is a closed interval containing 0, because $[\phi_n, \psi_n] \setminus \Sigma$ satisfies the Łojasiewicz (s)-condition at points of $\Sigma$. We want to check that $\operatorname*{Lim}_{x \to (a,0)} f(x) = \{0\}$, for almost all $a \in \Omega$. Suppose it is not so. Hence there exists a non-empty open subset $G$ of $\Omega$ and $\varepsilon > 0$ such that $[0, \varepsilon] \subset \operatorname*{Lim}_{x \to (a,0)} f(x)$ (or $[-\varepsilon, 0] \subset \operatorname*{Lim}_{x \to (a,0)} f(x)$) for each $a \in G$.

Then $G \times \{0\}^{n-k} \subset \overline{f^{-1}(\varepsilon/2, \infty)}$. It follows by the Cell Decomposition Theorem that there exists $a \in G$ such that $\{0\}^{n-k} \subset \overline{f^{-1}(\varepsilon/2, \infty)_a} = \overline{f_a^{-1}(\varepsilon/2, \infty)}$, a contradiction.

It follows that we can assume that $f$ extends by 0 to a continuous function defined on $[\varphi_n, \psi_n]$. Now, we will show that for any $i \in \{1, \dots, k\}$ the partial derivative $\partial f / \partial x_i$ extends by 0 to a continuous function defined on $[\varphi_n, \psi_n] \setminus E$, where $E \subset \Sigma$ and $\dim E < k$. With no loss of generality we assume that $i = k$. Suppose it is not so. Then there exists a non-empty open subset $G$ of $\Omega$ such that

$$\operatorname*{Lim}_{x \to (a,0)} \frac{\partial f}{\partial x_k}(x) \neq \{0\}, \qquad \text{for each} \quad a \in G. \tag{5.4.3}$$

It follows that there there exists a non-empty open subset $G$ of $\Omega$ and $\varepsilon > 0$ such that

$$G \times \{0\}^{n-k} \subset \overline{\Big(\frac{\partial f}{\partial x_k}\Big)^{-1}[\varepsilon, \infty)}$$

or

$$G \times \{0\}^{n-k} \subset \overline{\Big(\frac{\partial f}{\partial x_k}\Big)^{-1}(-\infty, -\varepsilon]}.$$

By an analogue of the Whitney Wing Lemma (cf. [Ł, Section 19]) or directly by the Cell Decomposition Theorem there exist a non-empty open subset $G'$ of $G$ and $\delta > 0$ such that $G' \times [0, \delta) \subset [\varphi_{k+1}, \psi_{k+1})$ and a continuous mapping

$$\alpha : G' \times [0, \delta) :\longrightarrow \Big(\frac{\partial f}{\partial x_k}\Big)^{-1}[\varepsilon, \infty), \tag{5.4.4}$$

[3] A subset $A$ of $R^m$ is called *quasi-convex* if there is a positive integer $M$ such that for any two points $a_1, a_2 \in A$ there exists a (definable) continuous arc $\lambda : [0, |a_1 - a_2|] \longrightarrow A$ such that $\lambda(0) = a_1$, $\lambda(|a_1 - a_2|) = a_2$ and $|\lambda'(t)| \leqslant M$, for any $t \in [0, |a_1 - a_2|]$ such that $\lambda'(t)$ exists. (Then $\lambda$ is necessarily piece-wise $\mathcal{C}^1$.)

such that

$$(5.4.5)\qquad \alpha(u,x_{k+1})=(u,x_{k+1},\alpha_{k+2}(u,x_{k+1}),\dots,\alpha_n(u,x_{k+1})),$$

where $\alpha_j(u,0)=0$, for each $j\in\{k+2,\dots,n\}$ and $u\in G'$, because of (5.4.2). Since

$$\varphi_{k+2}(u,x_{k+1})<\alpha_{k+2}(u,x_{k+1})<\psi_{k+2}(u,x_{k+1}),\quad \text{and}$$
$$\varphi_{j+1}(u,x_{k+1},\alpha_{k+2}(u,x_{k+1}),\dots,\alpha_j(u,x_{k+1}))<\alpha_{j+1}(u,x_{k+1})<$$
$$\psi_{j+1}(u,x_{k+1},\alpha_{k+2}(u,x_{k+1}),\dots,\alpha_j(u,x_{k+1})),\quad \text{for } j\in\{k+2,\dots,m\},$$

it follows that

$$(5.4.6)\qquad \lim_{x_{k+1}\to 0}\frac{\partial\alpha_j}{\partial x_{k+1}}(u,x_{k+1})\in R,\quad \text{for each } u\in G' \text{ and } j\in\{k+2,\dots,n\}.$$

By Theorem 5.3, at the expense of shrinking $G'$ and diminishing $\delta$ ,we can assume that $\alpha_j$ are $\mathcal{C}^1$ functions on $G'\times[0,\delta)$; in particular

$$(5.4.7)\qquad \lim_{x_{k+1}\to 0}\frac{\partial\alpha_j}{\partial x_k}(u,x_{k+1})=0,\quad \text{for } u\in G' \text{ and } j\in\{k+2,\dots,n\}.$$

It follows from (5.4.1) and (5.4.6) that for each $u\in G'$ the derivative

$$\frac{\partial(f\circ\alpha)}{\partial x_{k+1}}(u,x_{k+1})$$

is bounded when $x_{k+1}$ is near 0. Again by Theorem 5.3, after perhaps shrinking $G'$ and diminishing $\delta$ we can assume that $(f\circ\alpha)|G'\times[0,\delta)$ is of class $\mathcal{C}^1$; in particular

$$(5.4.8)\qquad \lim_{x_{k+1}\to 0}\frac{\partial(f\circ\alpha)}{\partial x_k}(u,x_{k+1})=0.$$

On the other hand,

$$\frac{\partial(f\circ\alpha)}{\partial x_k}(u,x_{k+1})=\frac{\partial f}{\partial x_k}(u,x_{k+1},\alpha_{k+2}(u,x_{k+1}),\dots,\alpha_n(u,x_{k+1}))+$$
$$\sum_{j=k+2}^{n}\frac{\partial f}{\partial x_j}(u,x_{k+1},\alpha_{k+2}(u,x_{k+1}),\dots,\alpha_n(u,x_{k+1}))\frac{\partial\alpha_j}{\partial x_k}(u,x_{k+1}),$$

which, in view of (5.4.8), (5.4.1) and (5.4.7), implies that

$$\lim_{x_{k+1}\to 0}\frac{\partial f}{\partial x_k}(u,x_{k+1},\alpha_{k+2}(u,x_{k+1}),\dots,\alpha_n(u,x_{k+1}))=0,$$

contradicting (5.4.4). This ends the proof in the case $p=1$.

Assume now that $p>1$ and the lemma is true for $p-1$. Since $[\varphi_n,\psi_n]\setminus\Sigma$ is locally quasi-convex near $\Sigma$[4] it suffices to check that all the partial derivatives

$$(5.4.9)\qquad \frac{\partial^{|\beta|}f}{\partial x_1^{\beta_1}\dots\partial x_n^{\beta_n}}\qquad (|\beta|:=\beta_1+\dots+\beta_n\leqslant p)$$

[4]It means that each point $u\in\Sigma$ admits arbitrarily small neighborhoods $U$ in $R^n$ such that $U\cap[\varphi_n,\psi_n]\setminus\Sigma$ is quasi-convex.

have continuous extensions to $\Sigma \setminus E$, where $E$ is a closed subset of $\Sigma$ of dimension $< k$ (cf. [T, p. 80]). By the induction hypothesis, there exists a closed subset of $\Sigma$ of dimension $< k$ such that for each $j \in \{k+1, \dots, n\}$ all the derivatives

$$\frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}} \Big( \frac{\partial f}{\partial x_j} \Big) = \frac{\partial}{\partial x_j} \Big( \frac{\partial^{|\gamma|}}{\partial x_1^{\gamma_1} \dots \partial x_n^{\gamma_n}} \Big) \quad (|\gamma| = p-1)$$

have continuous extensions to $\Sigma \setminus E$. It follows from the case $p = 1$, that there exists a closed subset $E'$ of $\Sigma$ containing $E$ of dimension $< k$ such that all the derivatives (5.4.9) have continuous extensions to $\Sigma \setminus E'$.

## 6. Existence of strict $\mathcal{C}^p$-triangulations orthogonally flat along simplexes.

Let $\Gamma$ be an open subset of $R^k = \{(x_1, \dots, x_n) \in R^n : x_{k+1} = \dots = x_n = 0\} \subset R^n$ and let $f : D \longrightarrow R^m$ be a $\mathcal{C}^p$-mapping defined on a non-necessarily open but locally closed subset $D$ of $R^n$ such that $D \subset \overline{\mathrm{int} D}$; i. e. there exists an open neighborhood $\Omega$ of $D$ in $R^n$ and a $\mathcal{C}^p$-mapping $\tilde{f} : \Omega \longrightarrow R^m$ such that $\tilde{f}|D = f$. Assume that $\Gamma \subset D$. We say that $f$ *is orthogonally $p$-flat along $\Gamma$* if

$$\frac{\partial^{|\alpha|} f}{\partial x_{k+1}^{\alpha_{k+1}} \dots \partial x_n^{\alpha_n}} (x_1, \dots, x_k, 0, \dots, 0) = \frac{\partial^{|\alpha|} f}{\partial x_{k+1}^{\alpha_{k+1}} \dots \partial x_n^{\alpha_n}} (u, 0) = 0$$

for each $u = (x_1, \dots, x_k) \in \Gamma$ and $\alpha = (\alpha_{k+1}, \dots, \alpha_n) \in \mathbb{N}^{n-k}$ such that $1 \leqslant |\alpha| \leqslant p$. This definition generalizes in a natural way to the case when $\Gamma$ is an open subset of any affine subspace $\mathrm{Aff}(\Gamma)$ of $R^n$ of dimension $k$.

*Remark 6.1.* If $f : D \longrightarrow R^m$ is a $\mathcal{C}^p$-mapping orthogonally $p$-flat along $\Gamma \subset D$ and $w_1 \in \mathbb{S}^{n-1}$ is a vector orthogonal to $\mathrm{Aff}(\Gamma)$, then for each $j \in \{0, \dots, p\}$ and arbitrary $w_2, \dots, w_j \in \mathbb{S}^{n-1}$

$$\frac{\partial^j f}{\partial w_1 \dots \partial w_j} \Big| \Gamma \equiv 0.$$

To prove the main theorem of this section we need the following lemma.

**Lemma 6.2.** *Let*

$$\Lambda = \{(x_1, \dots, x_k) \in R^k : \ \rho_i(x_1, \dots, x_k) > 0 \ (i \in \{0, \dots, k)\}$$

*be a simplex of dimension $k$ in $R^k$, where $\rho_i$ are nonzero affine forms. Put*

$$\omega(u) := \frac{(\rho_0 \cdot \dots \cdot \rho_k)(u)}{\sum_j (\rho_0 \dots \hat{\rho}_j \dots \rho_k)(u)}, \quad \textit{for each } u \in \Lambda.$$

*Then there exists constants* $\quad C_\alpha > 0 \ \ (\alpha \in \mathbb{N}^k)$ *such that*

$$C_0^{-1} d(u, \partial\Lambda) \leqslant \omega(u) \leqslant C_0 d(u, \partial\Lambda), \quad \textit{for each } u \in \Lambda$$

*and*

$$|D^\alpha \omega(u)| \leqslant \frac{C_\alpha}{\omega(u)^{|\alpha|-1}}, \quad \textit{for each } u \in \Lambda \textit{ and } \alpha \in \mathbb{N}^k \setminus \{0\}.$$

*Proof.* Put $H_i := \rho_i^{-1}(0) \quad (i \in \{0,\dots,k\})$. Then $d(u,\partial\Lambda) = \min_i d(u,H_i)$ and there exists $C>0$ such that $C^{-1}\rho_i(u) \leqslant d(u,H_i) \leqslant C\rho_i(u)$, for $u \in \Lambda$. Hence

$$C^{-1}\min_i \rho_i(u) \leqslant d(u,\partial\Lambda) \leqslant C\min_i \rho_i(u).$$

For a fixed $u \in \Lambda$ let $j$ be such that $\rho_j(u) = \min_i \rho_i(u)$. Then

$$\frac{1}{\rho_j(u)} \leqslant \frac{1}{\rho_0(u)} + \cdots + \frac{1}{\rho_k(u)} \leqslant \frac{k+1}{\rho_j(u)}; \quad \text{thus}$$

$$\frac{1}{k+1}\min_i \rho_i(u) \leqslant \omega(u) = \frac{1}{\frac{1}{\rho_0(u)} + \cdots + \frac{1}{\rho_k(u)}} \leqslant \min_i \rho; \tag{6.2.1}$$

finally, $\qquad \dfrac{1}{C(k+1)}\omega(u) \leqslant d(u,\partial\Lambda) \leqslant C(k+1)\omega(u).$

There are constants $a_j \quad (j \in \{0,\dots,k\}$ such that $\qquad \dfrac{\partial\omega}{\partial x_\nu} =$

$$\sum_i a_i \frac{(\rho_0 \dots \hat{\rho}_i \dots \rho_k)}{\sum_j (\rho_0 \dots \hat{\rho}_j \dots \rho_k)} - (\rho_0 \dots \rho_k)\Big(\sum_{i\neq j} a_i \frac{1}{\rho_i\rho_j}\rho_0 \dots \rho_k\Big)\frac{1}{\left[\sum_i \rho_0 \dots \hat{\rho}_i \dots \rho_k\right]^2} =$$

$$\sum_i \frac{a_i}{\rho_i}\cdot\omega - \sum_{i\neq j} a_i \frac{1}{\rho_i\rho_j}\omega^2.$$

By the Leibnitz formula $\quad D^\alpha\big(\dfrac{\partial\omega}{\partial x_\nu}\big) =$

$$\sum_i a_i \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta} D^\beta\big(\frac{1}{\rho_i}\big)D^{\alpha-\beta}\omega - \sum_{i\neq j} a_i \sum_{\alpha=\beta+\gamma+\delta+\epsilon}\frac{\alpha!}{\beta!\gamma!\delta!\epsilon!}D^\beta\big(\frac{1}{\rho_i}\big)D^\gamma\big(\frac{1}{\rho_j}\big)D^\delta\omega D^\epsilon\omega.$$

There exist constants $M_\beta > 0 \quad (\beta \in \mathbb{N}^n)$ such that

$$D^\beta\big(\frac{1}{\rho_i}\big) = \frac{M_\beta}{\rho_i^{|\beta|+1}}. \tag{6.2.2}$$

By (6.2.1) and (6.2.2) and the induction on the degree of the derivative

$$\big|D^\alpha\big(\frac{\partial\omega}{\partial x_\nu}\big)\big| \leqslant \sum_i |a_i| \sum_{\beta\leqslant\alpha}\binom{\alpha}{\beta}\frac{|M_\beta|}{\rho_i^{|\beta|+1}}\frac{|C_{\alpha-\beta}|}{\omega^{|\alpha|-|\beta|-1}} +$$

$$\sum_{i\neq j}|a_i| \sum_{\alpha=\beta+\gamma+\delta+\epsilon}\frac{\alpha!}{\beta!\gamma!\delta!\epsilon!}\frac{|M_\beta|}{\rho_i^{|\beta|+1}}\frac{|M_\gamma|}{\rho_j^{|\gamma|+1}}\frac{C_\delta}{\omega^{|\delta|-1}}\frac{C_\epsilon}{\omega^{|\epsilon|-1}}.$$

The lemma follows.

**Theorem 6.3.** *Let $\mathcal{K}$ be any finite simplicial complex in $R^n$ such that $|\mathcal{K}| = \overline{\mathrm{int}|\mathcal{K}|}$. Then there exists a homeomorphism $h : R^n \longrightarrow R^n$ of class $\mathcal{C}^p$ such that*

*(6.3.1) $h|\Gamma : \Gamma \longrightarrow \Gamma$ is a $\mathcal{C}^p$-diffeomorphism, for each $\Gamma \in \mathcal{K}$, and*

*(6.3.2) $h$ is orthogonally $p$-flat along each simplex $\Gamma \in \mathcal{K}$.*

*Proof.* Take a $\mathcal{C}^p$-function $\varphi : [0,\infty) \longrightarrow [0,1]$ such that $\varphi^{(i)}(0) = 0$ for each $i \in \{0,\dots,p\}$, $\varphi'(t) > 0$ for $t \in (0,1)$ and $\varphi(t) = 1$ for each $t \in [1,\infty)$.

We will prove by induction on $k \in \{0,\dots,n-1\}$ that there exists such a homeomorphism $h : R^n \longrightarrow R^n$ of class $\mathcal{C}^p$ that (6.3.1) is satisfied, while (6.3.2) is satisfied just for simplexes of dimension $\leqslant k$.

I. Let $k = 0$. Let $\{a\} \in \mathcal{K}$ and fix $r_a > 0$ such that $B(a, r_a) \cap |\mathcal{K}| \subset \bigcup \mathrm{St}\{a\}$. Define

$$h_a(x) := \varphi\Big(\frac{|x-a|^2}{r_a^2}\Big)(x-a) + a, \quad \text{for each } \ x \in R^n.$$

Then $h_a$ is of class $\mathcal{C}^p$ and $p$-flat at $a$. Besides, $h_a$ is a homeomorphism and $\mathcal{C}^p$-diffeomorphism on $R^n \setminus \{a\}$, because

$$x = a + \psi^{-1}(|h_a(x) - a|)\frac{h_a(x) - a}{|h_a(x) - a|}, \quad \text{for each } \ x \in R^n,$$

where $\psi(t) := \varphi\big(\frac{t^2}{r_a^2}\big) \cdot t$, $(t \in R)$ is an increasing homeomorphism of $R$ onto $R$.

It is clear that $h_a(\Gamma) = \Gamma$, for each $\Gamma \in \mathcal{K}$. Now, if $a_1, \dots, a_m$ are all vertices of $\mathcal{K}$, then we put

$$h := h_{a_m} \circ \dots \circ h_{a_1}.$$

II. Assume now that $0 < k \leqslant n-1$ and we have a $\mathcal{C}^p$-homeomorphism $h$ satisfying (1) and (2), for simplexes of dimension $< k$. Let $\Lambda \in \mathcal{K}$ and $\dim \Lambda = k$. With no loss of generality we can assume that $\Lambda$ is an open simplex in $R^k = \{(x_1,\dots,x_n) : \ x_{k+1} = \dots = x_n = 0\}$. Put $u = (u_1,\dots,u_k) = (x_1,\dots,x_k)$ and $v = (v_1,\dots,v_{n-k}) = (x_{k+1},\dots,x_n)$. Take $\omega : \Lambda \longrightarrow (0,\infty)$ as in Lemma 6.2. Since $\Omega := \bigcup \mathrm{St}(\Lambda)$ is an open neighborhood of $\Lambda$ in $|\mathcal{K}|$, there exists (by a kind of the Łojasiewicz inequality) a constant $r > 0$ such that

$$\{(u,v) \in \Lambda \times R^{n-k} : \ |v| \leqslant r\omega(u)\} \cap |\mathcal{K}| \subset \Omega.$$

Put $G := \{(u,v) \in \Gamma \times R^{n-k} : \ |v| < r\omega(u)\}$. The mapping

$$g(u,v) := \begin{cases} \big(u, \varphi\big(\frac{|v|^2}{r^2\omega^2(u)}\big) \cdot v\big) & ;\text{when } (u,v) \in G, \\ (u,v) & ;\text{when } (u,v) \in R^n \setminus G \end{cases}$$

is a homeomorphism of $R^n$ onto $R^n$ such that $g|\Gamma : \Gamma \longrightarrow \Gamma$ is a $\mathcal{C}^p$-diffeomorphism, for each $\Gamma \in \mathcal{K}$. Moreover, $g$ is of class $\mathcal{C}^p$ on $R^n \setminus \partial\Lambda$. Now define

$$H(u,v) := h(g(u,v)), \qquad \text{for each } \ (u,v) \in R^n.$$

For any $(u,v)\in G$ and $\nu\in\{1,\dots,n-k\}$

$$\frac{\partial}{\partial v_\nu}H(u,v)=\sum_{\mu=1}^{n-k}\frac{\partial h}{\partial v_\mu}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)v_\mu\frac{2v_\nu}{r^2\omega^2(u)}\varphi'\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)+$$

$$\frac{\partial h}{\partial v_\nu}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big).$$

It follows by induction on $|\alpha|\in\{1,\dots,p\}$, where $\alpha=(\alpha_1,\dots,\alpha_{n-k})$, that $\dfrac{\partial H}{\partial v^\alpha}$ expresses as a finite linear combination with real coefficients of the following functions

$$\frac{\partial^{|\beta|}h}{\partial v^\beta}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)\frac{v^\gamma}{r^{2s}\omega^{2s}(u)}\Big[\varphi^{(0)}\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)\Big]^{\nu_0}\dots\Big[\varphi^{(|\alpha|)}\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)\Big]^{\nu_{|\alpha|}},$$

where $|\beta|\in\{1,\dots,|\alpha|\}$, $|\beta|+2s-|\gamma|=|\alpha|$, $\nu_0+\dots+\nu_{|\alpha|}=|\beta|$ and $\nu_0+\nu_1+2\nu_2+\dots+|\alpha|\nu_{|\alpha|}\leqslant|\alpha|$.

Hence in particular

$$\frac{\partial^{|\alpha|}H}{\partial v^\alpha}(u,v)=0,\quad\text{when } u\in\Lambda,\ v=0,\ \alpha\in\mathbb{N}^{n-k},\ 1\leqslant|\alpha|\leqslant p. \tag{6.3.3}$$

Now in general, if $\alpha\in\mathbb{N}^{n-k}$ and $\varkappa\in\mathbb{N}^k$ and $|\alpha|+|\varkappa|\leqslant p$, then the derivative

$$\frac{\partial^{|\alpha|+|\varkappa|}H}{\partial v^\alpha\partial u^\varkappa}$$

is a finite linear combination with real coefficients of functions of the form

$$\frac{\partial^{|\beta|+|\lambda|}h}{\partial v^\beta\partial u^\lambda}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)\frac{v^\gamma}{\omega^d(u)}\times \tag{6.3.4}$$

$$\Big[\varphi^{(0)}\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)\Big]^{\nu_0}\dots\Big[\varphi^{(|\alpha|+|\varkappa|)}\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)\Big]^{\nu_{|\alpha|+|\varkappa|}}\times(D^{\varepsilon_1}\omega(u))\dots(D^{\varepsilon_q}\omega(u)),$$

where $0\leqslant q\leqslant|\alpha|+|\varkappa|$, $d\geqslant0$, $|\varepsilon_1|>0,\dots,|\varepsilon_q|>0$, $\lambda+\varepsilon_1+\dots+\varepsilon_q=\varkappa$, $|\beta|+d-|\gamma|=|\alpha|+q$, $\nu_0\geqslant,\dots,\nu_{|\alpha|+|\varkappa|}\geqslant0$, $d\geqslant|\gamma|$ and $|\beta|\geqslant|\varkappa|-|\lambda|$.

Assume now that $(u,v)\in G$ and $(u,v)$ tends to $(u_0,0)$ along some (definable) arc, where $u_0\in\partial\Lambda$. Let $\Gamma_0\in\mathcal{K}$ and $u_0\in\Gamma_0$. By an orthogonal change of coordinates $u_1,\dots,u_k$ one can assume that

$$d(u,\partial\Lambda)=d(u,\Gamma)=|u_1|,$$

where $\Gamma\in\mathcal{K}$, $\dim\Gamma=k-1$, $\Gamma\subset\{(u_1,\dots,u_k)\in R^k:\ u_1=0\}$ and $\Gamma_0\subset\{(u_1,\dots,u_k):u_1=\dots=u_l=0\}$ $(l\in\{1,\dots,k\})$.

When $\alpha\neq0$, in a product (6.3.4) we necessarily have $\beta\neq0$, therefore by the Taylor Formula,

$$\Big|\frac{\partial^{|\beta|+|\lambda|}h}{\partial v^\beta\partial u^\lambda}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)\Big|=$$

$$\Big|\frac{\partial^{|\beta|+|\lambda|}h}{\partial v^\beta \partial u^\lambda}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)-\frac{\partial^{|\beta|+|\lambda|}h}{\partial v^\beta \partial u^\lambda}\Big(0,u_2,\dots,u_k,0\Big)\Big| =$$

$$\Big|\sum_{\substack{\sigma+|\rho|=\\ p-|\beta|-|\lambda|}}\frac{1}{\sigma!\rho!}u_1^\sigma\Big[\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big]^\rho\frac{\partial^p h}{\partial v^{\beta+\rho}\partial u^\lambda \partial u_1^\sigma}\Big(\theta u_1,u_2,\dots,u_k,\theta\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)\Big|,$$

where $\theta\in(0,1)$. Hence

$$\Big|\frac{\partial^{|\beta|+|\lambda|}h}{\partial v^\beta \partial u^\lambda}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)\Big|\leqslant(\omega(u))^{p-|\beta|-|\lambda|}\mu(u,v),$$

where $\mu(u,v)\to 0$, when $(u,v)\to(u_0,0)$. Thus, there exists a constant $M>0$ such that

$$|(6.3.4)|\leqslant M\omega^{p-|\beta|-|\lambda|}\mu\frac{\omega^{|\gamma|}}{\omega^d}\omega^{-|\varepsilon_1|+1}\dots\omega^{-|\varepsilon_q|+1}=$$

$$M\mu\omega^{p-|\beta|-|\lambda|+|\gamma|-d+q-|\varepsilon_1|-\dots-|\varepsilon_q|}=M\mu\omega^{p-|\alpha|-|\varkappa|}\to 0,$$

when $(u,v)\to(u_0,0)$.

Suppose now that $\alpha=0$ and $\varkappa\neq 0$. Then, for each $(u,v)\in G$,

$$\frac{\partial^{|\varkappa|}H}{\partial u^\varkappa}(u,v)=\frac{\partial^{|\varkappa|}h}{\partial u^\varkappa}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)+\text{a linear combination with real}$$

$$\text{coefficients of functions of the form (6.3.4), where } \beta\neq 0.$$

It follows that

$$\lim_{(u,v)\to(u_0,0)}\frac{\partial^{|\varkappa|}H}{\partial u^\varkappa}(u,v)=\lim_{(u,v)\to(u_0,0)}\frac{\partial^{|\varkappa|}h}{\partial u^\varkappa}\Big(u,\varphi\Big(\frac{|v|^2}{r^2\omega^2(u)}\Big)v\Big)=\frac{\partial^{|\varkappa|}h}{\partial u^\varkappa}(u_0,0).$$

We have just checked that $H$ is of class $\mathcal{C}^p$ which is orthogonally $p$-flat along $\varGamma_0$ and (6.3.3) shows that it is orthogonally $p$-flat along $\varLambda$. We consecutively repeat the above construction for every simplex of dimension $k$.

**Corollary 6.4.** *If $\mathcal{K}$ is a finite simplicial complex in $R^n$ such that $|\mathcal{K}|=\overline{\mathrm{int}|\mathcal{K}|}$ and $f:|\mathcal{K}|\longrightarrow A$ is a homeomorphism such that for each $\varLambda\in\mathcal{K}$, $f|\varLambda$ is of class $\mathcal{C}^p$ and $f|\mathring{\varLambda}$ is a $\mathcal{C}^p$-embedding, then there exists a strict $\mathcal{C}^p$-triangulation $(f^*,\mathcal{K}^*)$ of $A$ orthogonally $p$-flat along simplexes such that $\mathcal{K}^*$ is a refinement of $\mathcal{K}$ and $f(\varLambda)=f^*(\varLambda)$, for each $\varLambda\in\mathcal{K}$. In particular, if $(\mathcal{K},f)$ is any strict $\mathcal{C}^p$-triangulation of $A$, there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{K}^*,f^*)$ of $A$ orthogonally $p$-flat along simplexes such that $\mathcal{K}^*$ is a refinement of $\mathcal{K}$ and $f(\varLambda)=f^*(\varLambda)$, for each $\varLambda\in\mathcal{K}$.*

## 7. Regular cells, $(k, f, q)$-proper regular cells and convex polyhedra $(k, f, q)$-well situated in $R^n$.

We define a notion of a *regular cell* in $R^n$, its *boundary cells* and its *boundary* inductively on $n$. If $n = 1$, a regular cell in $R$ is either a singleton or a closed bounded interval $[a, b]$, where $a < b$, and then its boundary cells are $\{a\}$ and $\{b\}$, while its boundary $\partial[a, b] := \{a, b\}$. Assume now that $n > 1$. A subset $C$ of $R^n$ is a regular cell if it is either a graph of a continuous function

$$C = \{(x', x_n) \in R^{n-1} \times R : \ x' \in C',\, x_n = \varphi(x')\}$$

defined on a regular cell $C'$ in $R^{n-1}$, and then a boundary cells of $C$ are exactly the graphs $\varphi|D'$, where $D'$ is a boundary cell of $C'$, while its boundary $\partial C$ is $\varphi|\partial C'$, or there are two continuous functions $\varphi_1 \leqslant \varphi_2 : C' \longrightarrow R$ defined on a regular cell $C'$ in $R^{n-1}$ such that

$$C = [\varphi_1, \varphi_2] := \{(x', x_n) \in R^{n-1} \times R : \ x' \in C',\, \varphi_1(x') \leqslant x_n \leqslant \varphi_2(x')\}$$

and the set $\{x' \in C' : \varphi_1(x') = \varphi_2(x')\}$ is a union of some boundary cells of $C'$, the boundary cells of $C$ are then exactly $\varphi_1$, $\varphi_2$, the boundary cells of $\varphi_1$ and those of $\varphi_2$ and finally all $[\varphi_1|D', \varphi_2|D']$, where $D'$ is a boundary cell of $C'$, while the boundary $\partial C$ of $C$ is the union of all its boundary cells.

Let now $C$ be a regular cell in $R^n$ of dimension $n$, let $k, q$ be non-negative integers and let $f : B \longrightarrow R^d$ be a continuous mapping defined on a subset $B$ of $R^n$ containing $C$. Then we say that $C$ is $(k, f, q)$-*proper (regular) cell* if either $f$ is of class $\mathcal{C}^q$ on the set[5] $C \setminus \bigcup\{D : D$ a boundary cell of $C$ of dimension $< k\}$ or there exists exactly one boundary cell $\Xi(C)$ of $C$ of dimension $k$ such that $f|C \setminus \Xi(C)$ is of class $\mathcal{C}^q$ and the projection $\pi^n_{k+1}|\Xi(C)$ is injective. In the first case we put $\Xi(C) = \emptyset$.

Let now $P$ be any convex polyhedron in $R^n$ of dimension $n$. Notice that it may not be a regular cell in $R^n$, but it becomes a regular cell after an arbitrarily small linear change of coordinates and then boundary cells are unions of some faces of $P$. Let $k, q$ be non-negative integers and let $f : B \longrightarrow R^d$ be a continuous mapping defined in a subset $B$ of $R^n$ containing $P$. We will say that *a convex polyhedron $P$ of dimension $n$ is $(k, f, q)$-well situated in $R^n$ (relative to the canonical basis)* if either $f$ is of class $\mathcal{C}^q$ on $P \setminus P^{(k-1)}$, where $P^{(k-1)}$ denotes the union of all faces of $P$ of dimension $\leqslant k-1$, or $f$ is not of class $\mathcal{C}^q$ on $P \setminus P^{(k-1)}$ but there exists exactly one face $\Sigma(P)$ of $P$ of dimension $k$ such that $f$ is of class $\mathcal{C}^q$ on $P \setminus \Sigma(P)$, and moreover

$$(\pi^n_{k+1})^{-1}(\pi^n_{k+1}(\Sigma(P))) \cap P = \Sigma(P) \tag{7.1}$$

and the restriction

$$\pi^n_{k+1}|\Sigma(P) : \Sigma(P) \longrightarrow R^{k+1} \quad \text{is injective.} \tag{7.2}$$

[5] A mapping $f : E \longrightarrow R^d$ defined on any subset $E$ of $R^n$ is called *of class $\mathcal{C}^q$*, if there exists an extension $\tilde{f} : \Omega \longrightarrow R^d$ of $f$ to an open neighborhood of $E$ in $R^n$ which is of class $\mathcal{C}^q$.

In the first case we put $\Sigma(P) = \emptyset$.

Notice that if $P$ is $(k, f, q)$-well situated in $R^n$ and if it is at the same time a regular cell in $R^n$, then it is as a cell $(k, f, q)$-proper and $\Sigma(P) \subset \Xi(P)$.

If $v = (v_1, \ldots, v_n) \in \boldsymbol{V}_n(R^n)$ is any orthonormal basis in $R^n$, we will say that *a convex polyhedron $P$ of dimension $n$ is $(k, f, q)$-well situated in $R^n$ relative to the basis $v$* if $\lambda(P)$ is $(k, f \circ \lambda^{-1}, q)$-well situated in $R^n$ relative to the canonical basis $e = (e_1, \ldots, e_n)$, where $\lambda$ stands for the linear automorphism of $R^n$ such that $\lambda(v_i) = e_i \quad (i \in \{1, \ldots, n\})$. Then we put $\Sigma(P) := \lambda^{-1}(\Sigma(\lambda(P)))$.

The following proposition is straightforward.

**Proposition 7.1.** *Let now $P$ be any convex polyhedron in $R^n$ of dimension $n$. Let $k, q$ be non-negative integers and let $f : B \longrightarrow R^d$ be a continuous mapping defined in a subset $B$ of $R^n$ containing $P$.*
*Then*

(7.1.1) *if there exists a face $\Sigma$ of $P$ of dimension $\leqslant k$, such that $f|P \setminus \Sigma$ is of class $C^q$, then there exists an orthonormal basis $v \in \boldsymbol{V}_n(R^n)$ such that $P$ is $(k, f, q)$-well situated in $R^n$ relative to $v$;*

(7.1.2) *the subset of all bases $v \in \boldsymbol{V}_n(R^n)$ such that $P$ is $(k, f, q)$-well situated in $R^n$ relative to $v$ is open;*

(7.1.3) *if $P$ is $(k, f, q)$-well situated in $R^n$ relative to a basis $v$ and* $\dim \Sigma(P) =$ *$k$, then changing this basis slightly we can assume additionally that for each $j \in \{n, \ldots, k+2\}$, the set $\pi^n_j(P)$ is a capsule in $R^j$ the rim of which contains $\pi^n_j(\Sigma(P))$ while $\pi^n_{k+1}(P)$ is a capsule in $R^{k+1}$ the boundary of which contains $\pi^n_{k+1}(\Sigma(P))$ and $\pi^n_{k+1}(\Sigma(P))$ is a graph of a linear function restricted to a polyhedron $\pi^n_k(\Sigma))$ of dimension $k$;*

(7.1.4) *if $P$ is $(k, f, q)$-well situated in $R^n$ relative to a basis $v$ and $Q$ is any polyhedron in $R^n$ of dimension $n$ and $Q \subset P$, then $Q$ is $(k, f, q)$-well situated in $R^n$ relative to a basis $v$ and $\Sigma(Q) \subset \Sigma(P)$.*

## 8. Main Theorem - proof in generic case.

**Proposition 8.1.** *Assume that our Main Theorem is true in dimensions $< n$. Let $\mathcal{P}$ be a finite polyhedral complex in $R^{n-1}$ and put $D := |\mathcal{P}|$. Let $q_1, q \in \mathbb{Z}$ and $q \geqslant q_1 \geqslant p + 1$.*

*Let $\alpha_0 \leqslant \cdots \leqslant \alpha_r : D \longrightarrow R$ be an increasing sequence of continuous* PL-*functions such that the family*

$$\mathcal{K} := \left\{\overline{(\alpha_i, \alpha_{i+1})} : \quad i \in \{0, \ldots, r-1\}\right\}$$

*is a family of capsules in $R^n$. Let $\mathcal{K}_1 \subset \mathcal{K}$, $A := |\mathcal{K}|$ and $A_1 := |\mathcal{K}_1|$. Let $f = (f_1, \ldots, f_d) : A_1 \longrightarrow R^d$ be a continuous mapping such that $f|\mathring{K}$ is of class $\mathcal{C}^{q_1}$ for each $K \in \mathcal{K}_1$. Let $\mathcal{E}$ be any finite family of subsets of $D$.*

*Then there exist*

(8.1.1) *a strict $\mathcal{C}^q$-triangulation $(\mathcal{M}, h)$ of $D$ compatible with $\mathcal{E}$ such that $|\mathcal{M}| = D$ and $h(\Gamma) = \Gamma$, for every face of each polyhedron $P \in \mathcal{P}$,*

(8.1.2) *an increasing sequence of continuous* PL-*functions*

$$\eta_0 \leqslant \cdots \leqslant \eta_k : D \longrightarrow R,$$

*which is a refinement of $\alpha_0, \ldots, \alpha_r$ such that the family $\mathcal{C} := \left\{\overline{(\eta_j, \eta_{j+1})} : \ j \in \{0, \ldots, k-1\}\right\}$ is a family of capsules refining the family $\mathcal{K}$,*

(8.1.3) *a homeomorphism $\Psi : [\alpha_0, \alpha_r] \longrightarrow [\alpha_0, \alpha_r]$ of the form $\Psi(u, \zeta_n) = (h(u), \psi(u, \zeta_n))$, for each $(u, \zeta_n) \in [\alpha_0, \alpha_r]$,*

*such that*

(8.1.4) $\Psi(u, \alpha_i(u)) = (h(u), \alpha_i(h(u)))$ *for each* $u \in D$ *and* $i \in \{0, \ldots, r\}$;

(8.1.5) *if $a \in C \in \mathcal{C}$, where $C \subset K \in \mathcal{K}_1$ and $f|K$ is of class $\mathcal{C}^{q_1}$ in a neighborhood of $\Psi(a)$ in $K$, then $\Psi|C$ and $f \circ \Psi|C$ are of class $\mathcal{C}^{q_1}$ in a neighborhood of $a$ in $C$;*

(8.1.6) $\Psi|\mathring{C}$ *and* $f \circ \Psi|\mathring{C}$ *are of class* $\mathcal{C}^{q_1}$*, for each* $C \in \mathcal{C}$ *such that* $C \subset K \in \mathcal{K}_1$;

(8.1.7) $\Psi|C$ *is of class* $\mathcal{C}^q$ *for each* $C \in \mathcal{C}$ *such that* $C \subset K \in \mathcal{K} \setminus \mathcal{K}_1$ *and*

$$\frac{\partial^\sigma(\psi|C)}{\partial \zeta_n^\sigma} = 0 \quad \text{on } \partial C \ \text{ for } \ \sigma \in \{1, \ldots, p\};$$

(8.1.8) *if $C \in \mathcal{C}$ and $C \subset K \in \mathcal{K}_1$, then the derivatives*

$$\frac{\partial^\sigma(\Psi|\mathring{C})}{\partial \zeta_n^\sigma} \quad \text{and} \quad \frac{\partial^\sigma(f \circ \Psi|\mathring{C})}{\partial \zeta_n^\sigma} \qquad (\sigma \in \{1, \ldots, p\})$$

*have continuous extensions by zero to the whole $C$;*

(8.1.9) $$\frac{\partial(\psi|\mathring{C})}{\partial \zeta_n} > 0, \ \text{ for each } \ C \in \mathcal{C}.$$

*Proof.* By a refinement of $\mathcal{P}$ one can assume that

(8.1.10) every function $\alpha_i$ is affine on each $P \in \mathcal{P}$, and

(8.1.11) $\mathcal{P}$ is *compatible* with each of the sets $\{x' \in D : \ \alpha_i(x') = \alpha_{i+1}(x')\}$ $(i \in \{0, \ldots, r-1\})$; i.e. each of these sets is a union of some $P \in \mathcal{P}$.

By Lemma 5.1, we get a sequence of continuous functions

$$\delta_0 \leqslant \cdots \leqslant \delta_k : D \longrightarrow R$$

and a homeomorphism $\Phi : [\delta_0, \delta_k] \longrightarrow [\alpha_0, \alpha_r]$ with the properties (5.1.1)-(5.1.6).

Now we apply the induction hypothesis. We get a strict $\mathcal{C}^q$-triangulation $(\mathcal{M}, h)$ of the set $D$ such that

(8.1.12) $\mathcal{M}$ is a finite simplicial complex in $R^{n-1}$ such that $|\mathcal{M}| = D$;

(8.1.13) $(\mathcal{M}, h)$ is compatible with each $E \in \mathcal{E}$ and with each $P \in \mathcal{P}$ (the latter follows from (8.1.14) below);

(8.1.14) $h(P) = P$, for each $P \in \mathcal{P}$; hence, each of the sets $\{x' \in D : \ \alpha_i(x') = \alpha_{i+1}(x')\}$ $(i \in \{0, \dots, r-1\})$ is $h$-invariant (see (8.1.11));

(8.1.15) $\delta_j \circ h, \ \theta_j \circ h : D \longrightarrow R$ are of class $\mathcal{C}^q$ $(j \in \{0, \dots, k\})$;

(8.1.16) for all the functions $a_1, \dots, a_m$ from condition (5.1.6) the compositions $a_1 \circ h, \dots, a_m \circ h : D \longrightarrow R$ are of class $\mathcal{C}^q$, and

(8.1.17) $(\mathcal{M}, h)$ is compatible with each of the sets $\{x' \in D : \ \delta_j(x') = \delta_{j+1}(x')\}$ $(j \in \{0, \dots, k-1\})$.

By passing to the barycentric subdivision we can have in addition

(8.1.18) for each $j \in \{0, \dots, k-1\}$ and each simplex $\Delta \in \mathcal{M}$, if $\delta_j \circ h \not\equiv \delta_{j+1} \circ h$ on $\Delta$, then $\delta_j(h(w)) < \delta_{j+1}(h(w))$, for some vertex $w$ of $\Delta$

and by (6.4)

(8.1.19) $(\mathcal{M}, h)$ is a strict $\mathcal{C}^q$-triangulation orthogonally $C^q$-flat along simplexes.

Define the following homeomorphism

$$\Phi^* : [\delta_0 \circ h, \delta_k \circ h] \longrightarrow [\alpha_0, \alpha_r]$$

by the formula

$$\Phi^*(u, \xi_n) := (h(u), \varphi(h(u), \xi_n)) = (h(u), \varphi^*(u, \xi_n)). \tag{8.1.20}$$

Then

(8.1.21) the sequence $\theta_j \circ h$ $(j \in \{0, \dots, k\})$ is a refinement of $\alpha_0 \circ h, \dots, \alpha_r \circ h$;

(8.1.22) $\mathcal{L}^* := \{\overline{(\delta_j \circ h, \delta_{j+1} \circ h)} : \ j \in \{0, \dots, k-1\}\}$ is a family of capsules in $R^n$ such that $\{\Phi^*(L^*) : \ L^* \in \mathcal{L}^*\} = \{\Phi(L) : \ L \in \mathcal{L}\}$ is a refinement of $\mathcal{K}$.

Put $\mathcal{L}_1^* := \{L^* \in \mathcal{L}^* : \Phi^*(L^*) \subset K, \text{ for some } K \in \mathcal{K}_1\}$. Then

(8.1.23) for any $L^* \in \mathcal{L}_1^*$, $\Phi^*|\mathring{L}^*$ and $f \circ \Phi^*|\mathring{L}^*$ are of class $\mathcal{C}^{q_1}$ (by (5.1.6) and (8.1.16)),

$$\frac{\partial \varphi^*}{\partial \xi_n} > 0 \quad \text{on } \mathring{L}^* \text{ and all the derivatives} \qquad \frac{\partial^\sigma(\Phi^*|\mathring{L}^*)}{\partial \xi_n^\sigma}, \quad \frac{\partial^\sigma(f \circ \Phi^*|\mathring{L}^*)}{\partial \xi_n^\sigma},$$

where $\sigma \in \{1, \ldots, p\})$ have continuous extensions by zero to $L^*$;

(8.1.24) for any $L^* \in \mathcal{L}^* \setminus \mathcal{L}_1^*$, $\Phi^*|L^*$ is of class $\mathcal{C}^q$ (by (5.1.6) and (8.1.16)),

$$\frac{\partial \varphi^*}{\partial \xi_n} > 0 \quad \text{on } \overset{\circ}{L^*} \quad \text{and the derivatives } \frac{\partial^\sigma (\Phi^*|L^*)}{\partial \xi_n^\sigma} \quad (\sigma \in \{1, \ldots, p\})$$

are equal zero on $\partial L^*$;

(8.1.25) if $L^* \in \mathcal{L}_1^*$, $b \in \partial L^*$ and $\Phi^*(L^*) \subset K \in \mathcal{K}_1$ and $f|K$ is of class $\mathcal{C}^p$ in a neighborhood of $\Phi^*(b)$ in $K$, then $\Phi^*|L^*$ and $f \circ \Phi^*|L^*$ are of class $\mathcal{C}^p$ in a neighborhood of $b$ in $L^*$.

Now we want to replace the $\mathcal{C}^q$-functions $\delta_j \circ h$ by continuous PL-functions defined on $D$ by using Lemma 5.2. Therefore we want to find continuous PL-functions, affine in restriction to any simplex $S \in \mathcal{M}$

$$\eta_0 \leqslant \cdots \leqslant \eta_k : D \longrightarrow R$$

such that for each $j \in \{0, \ldots, k-1\}$

$$\{u \in D : \ (\delta_j \circ h)(u) = (\delta_{j+1} \circ h)(u)\} = \{u \in D : \ \eta_j(u) = \eta_{j+1}(u)\}. \tag{8.1.26}$$

For any continuous function $\beta : D \longrightarrow R$ define the continuous PL-function $\beta^\sharp : D \longrightarrow R$ by the formula

$$\beta^\sharp(\lambda_0 v_0 + \cdots + \lambda_s v_s) := \lambda_0 \beta(v_0) + \cdots + \lambda_s \beta(v_s),$$

where $(v_0, \ldots, v_s) \in \mathcal{M}$ is a simplex with vertices $v_0, \ldots, v_s$ $\lambda_0, \ldots, \lambda_s \geqslant 0$ and $\lambda_0 + \cdots + \lambda_s = 1$.

In view of (8.1.17) and (8.1.18)

$$\delta_j \circ h(u) < \delta_{j+1} \circ h(u) \Longleftrightarrow \theta_j \circ h(u) < \theta_{j+1} \circ h(u) \Longleftrightarrow (\theta_j \circ h)^\sharp(u) < (\theta_{j+1} \circ h)^\sharp(u), \tag{8.1.27}$$

for any $u \in D$ and $j \in \{0, \ldots, k-1\}$.

By (8.1.27) $(\theta_j \circ h)^\sharp$ are continuous PL-functions, affine on simplexes and satisfying (8.1.26). However they might not be a refinement of $\alpha_0, \ldots, \alpha_r$, so some improvement is necessary.

Of course, $(\theta_j \circ h)^\sharp$ $(j \in \{0, \ldots, k\})$ are a refinement of $(\alpha_i \circ h)^\sharp$ $(i \in \{0, \ldots, r\})$. By (8.1.14) and (8.1.27), for each $i \in \{0, \ldots, r-1\}$

$$\{u \in D : (\alpha_i \circ h)^\sharp(u) = (\alpha_{i+1} \circ h)^\sharp(u)\} = \{u \in D : (\alpha_i \circ h)(u) = (\alpha_{i+1} \circ h)(u)\} = \{u \in D : \alpha_i(u) = \alpha_{i+1}(u)\}.$$

This shows that we can define the following homeomorphisms

$$H_i : [(\alpha_i \circ h)^\sharp, (\alpha_{i+1} \circ h)^\sharp] \longrightarrow [\alpha_i, \alpha_{i+1}],$$

$$H_i\big(u, \tau((\alpha_{i+1}\circ h)^\sharp(u) - (\alpha_i \circ h)^\sharp(u)) + (\alpha_i \circ h)^\sharp(u)\big) = \big(u, \tau(\alpha_{i+1}(u) - \alpha_i(u)) + \alpha_i(u)\big),$$

where $\tau \in [0,1]$, $i \in \{0, \dots, r-1\}$. Gluing them together gives us a homeomorphism

$$H := \bigcup_{i=0}^{r-1} H_i : [(\alpha_0 \circ h)^\sharp, (\alpha_r \circ h)^\sharp] \longrightarrow [\alpha_0, \alpha_r]$$

strictly increasing with respect to the last variable. Finally we put $\eta_j := \big(H((\theta_j \circ h)^\sharp)\big)^\sharp$, $(j \in \{0, \dots, k\})$, which are a refinement of $\alpha_0, \dots, \alpha_r$, according to (8.1.10).

**Corollary 8.2.** *Assume the Main Theorem is proved in dimensions $< n$. Let $\alpha_0 \leqslant \dots \leqslant \alpha_r : D \longrightarrow R$ be an increasing sequence of continuous* PL*-functions such that*

$$\mathcal{K} := \big\{\overline{(\alpha_i, \alpha_{i+1})} : \quad i \in \{0, \dots, r-1\}\big\}$$

*is a family of capsules in $R^n$ such that $D = \{\pi^n_{n-1}(K) : K \in \mathcal{K}\}$ and $[\alpha_0, \alpha_r]$ is a convex polyhedron. Let $\mathcal{P}$ be a polyhedral complex in $R^{n-1}$ such that $|\mathcal{P}| = D$. Let $\mathcal{V}$ be a finite family of open subsets of $R^n$ covering $\bigcup\{\mathring{K} : K \in \mathcal{K}\}$.*

*Then there exists a sequence of continuous* PL*-functions $\beta_0 \leqslant \dots \leqslant \beta_s : D \longrightarrow R$ which is a refinement of the previous one and a homeomorphism $G : [\alpha_0, \alpha_r] \longrightarrow [\alpha_0, \alpha_r]$ of the form $G(u, x_n) = (g(u), \tilde{g}(u, x_n))$ such that, for each $j \in \{0, \dots, s-1\}$, $G|\overline{(\beta_j, \beta_{j+1})}$ is of class $\mathcal{C}^q$ and such that $G(\beta_i, \beta_{i+1}) \subset V$, for some $V \in \mathcal{V}$ and $G(\alpha_i|P) = \alpha_i|P$ for each $i \in \{0, \dots, r\}$ and $P \in \mathcal{P}$. Moreover, $\partial\tilde{g}/\partial x_n > 0$ on each $(\beta_j, \beta_{j+1})$ and*

$$\frac{\partial^\sigma G}{\partial x_n^\sigma}(u, \beta_j(u)) = 0, \quad \textit{for each } u \in D,\ j \in \{0, \dots, s\} \textit{ and } \sigma \in \{1, \dots, q\}.$$

*Proof.* By Proposition 2.5, there is a refinement

$$\beta_0 \leqslant \dots \leqslant \beta_s : D \longrightarrow R$$

of the sequence $\alpha_0, \dots, \alpha_r$ such that each $(\beta_j, \beta_{j+1})$ is contained in some $V \in \mathcal{V}$. Now it suffices to use Proposition 8.1, where we put

$$\mathcal{K} = \big\{\overline{(\beta_j, \beta_{j+1})} : j \in \{0, \dots, s-1\}\big\}$$

and $\mathcal{K}_1 = \emptyset$.

**Proposition 8.3.** *Assume that the Main Theorem is proved in dimensions* $< n$. *Let* $0 \leqslant k < n$. *Fix an integer* $q \geqslant (n-1-k)\binom{p}{2} + p$. *Assume that*

$$\alpha_0^n \leqslant \cdots \leqslant \alpha_{r_n}^n : D_{n-1} \longrightarrow R$$

*is a sequence of continuous* PL-*functions such that*

$$\mathcal{K}^n := \left\{\overline{(\alpha_i^n, \alpha_{i+1}^n)} : \ i \in \{0, \dots, r_n - 1\}\right\}$$

*is a family of convex* PL-*capsules in* $R^n$, *where* $D_{n-1} = \bigcup\{\pi_{n-1}^n(K_n) : K_n \in \mathcal{K}^n\}$. *Let* $\mathcal{K}_1^n \subset \mathcal{K}^n$. *Assume that* $D_{n-1}$ *is a closed convex polyhedron in* $R^{n-1}$ *of dimension* $n-1$. *Put* $D_n := |\mathcal{K}^n|$.

*Let* $f : |\mathcal{K}_1^n| \longrightarrow R^d$ *be continuous and such that* $f|\mathring{K}_n$ *is of class* $\mathcal{C}^q$ *for each* $K_n \in \mathcal{K}_1^n$. *Assume that each* $K_n \in \mathcal{K}_1^n$ *is* $(k, f, q)$*-well situated in* $R^n$ *and that all the derivatives*

(8.3.1) $$\frac{\partial^i (f|K_n \setminus \Sigma(K_n))}{\partial x_n^i} \quad (i \in \{1, \dots, q\}) \quad \textit{have continuous extensions}$$

*by zero to all* $K_n$.

*Assume that* $k \leqslant l \leqslant n-1$ *and* $m \in \{1, \dots, p\}$. *Put*

$$\lambda(l,m) := \begin{cases} q, & \textit{when } \ l = n-1 \\ q - (n-2-l)\binom{p}{2} - (p-1) - \cdots - (p-m), & \textit{when } k \leqslant l \leqslant n-2. \end{cases}$$

*Then, after some arbitrarily small linear change of coordinates in* $R^{n-1}$:

(8.3.2) *for each* $j$ *such that* $l \leqslant j \leqslant n-1$ *there exists a sequence of continuous* PL-*functions*

$$\alpha_0^j \leqslant \cdots \leqslant \alpha_{r_j}^j : D_{j-1} \longrightarrow R,$$

*such that* $\mathcal{K}^j := \left\{\overline{(\alpha_i^j, \alpha_{i+1}^j)} : \ i \in \{0, \dots, r_j - 1\}\right\}$ *is a family of convex capsules in* $R^j$ *which is a refinement of* $\{\pi_j^{j+1}(K_{j+1}) : \ K_{j+1} \in \mathcal{K}^{j+1}\}$, $D_{j-1} = \bigcup\{\pi_{j-1}^j(K_j) : K_j \in \mathcal{K}^j\}$, *every* $\alpha_i^{j+1}$ *is affine over each* $K_j \in \mathcal{K}^j$ *and there exists a homeomorphism* $\Phi_j : D_j \longrightarrow D_j$ *of the form*

$$\Phi_j(x_1, \dots, x_j) = (\tilde{\Phi}_j(x_1, \dots, x_{j-1}), \varphi_j(x_1, \dots, x_j)), \quad \textit{such that}$$

(8.3.3) $\tilde{\Phi}_j : D_{j-1} \longrightarrow D_{j-1}$ *is of class* $\mathcal{C}^q$;

(8.3.4) $\Phi_j(\pi_j^{j+1}(L)) = \pi_j^{j+1}(L)$, *for every face* $L$ *of any polyhedron* $K_{j+1} \in \mathcal{K}^{j+1}$;

(8.3.5) *each* $K_j \in \mathcal{K}^j$ *is* $(k, \varphi_j, \lambda(l,m))$*-well situated in* $R^j$ *and* $\Sigma(K_j) \subset \pi_j^n(\Sigma(K_n))$;

(8.3.6) *consider homeomorphisms* $\Psi_l, \ldots, \Psi_n$ *defined inductively as follows:*
$\Psi_l := id_{D_l}$, $\Psi_j := \Phi_j(\Psi_{j-1}, \rho_{x_j}) : (\Psi_{j-1}, \rho_{x_j})^{-1}(D_j) \longrightarrow D_j$, *for* $j \in \{l+1, \ldots, n\}$, *where* $\rho_{x_j}$ *denotes the projection of* $R^j$ *onto* $x_j$*-axis and* $\Phi_n := id_{D_n}$;

(8.3.7) *for* $j \in \{l+1, \ldots, n\}$, $(\Psi_{j-1}, \rho_{x_j})^{-1}(D_j) = \Psi_j^{-1}(D_j)$ *is the union of the capsules*

$$\overline{(\alpha_i^j \circ \Psi_{j-1}, \alpha_{i+1}^j \circ \Psi_{j-1})}, \qquad (i \in \{0, \ldots, r_j - 1\}),$$

*where* $\alpha_0^j \circ \Psi_{j-1} \leqslant \cdots \leqslant \alpha_{r_j}^j \circ \Psi_{j-1} : \Psi_{j-1}^{-1}(D_{j-1}) \longrightarrow R$ *and at the same time it is a union of some cells of the form*

$$Q^j(i_{l+1}, \ldots, i_j) =$$

$$\{(x_1, \ldots, x_j) \in R^j : (x_1, \ldots, x_\nu) \in \overline{(\alpha_{i_\nu}^\nu \circ \Psi_{\nu-1}, \alpha_{i_\nu+1}^\nu \circ \Psi_{\nu-1})}, \text{ when } l+1 \leqslant \nu \leqslant j\}$$

*for some* $i_\nu \in \{0, \ldots, r_\nu - 1\}$, *where* $l+1 \leqslant \nu \leqslant j$;

(8.3.8) *each of the cells* $Q^n = Q^n(i_{l+1}, \ldots, i_n)$ *such that* $\Psi_n(Q^n) \subset K_n \in \mathcal{K}_1^n$ *and* $\dim\big(\Psi_n(Q^n) \cap \Sigma(K_n)\big) = k$ *is* $(k, f \circ \Psi_n, \lambda(l,m))$*-proper and all the derivatives*

$$\frac{\partial^{|\varkappa|}(f \circ \Psi_n | \overset{\circ}{Q^n})}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_n^{\varkappa_n}}, \quad \text{where } 1 \leqslant |\varkappa| = \varkappa_{l+1} + \cdots + \varkappa_n \leqslant p \text{ and } \varkappa_{l+1} \leqslant m$$

*have continuous extensions by zero to* $\Xi(Q^n)$ *and at the same time each of the cells* $Q^j = Q^j(i_{l+1}, \ldots, i_j) = \pi_j^n(Q^n)$ $(j \in \{l+1, \ldots, n-1\})$ *is* $(k, \{\Psi_{l+1}, \ldots, \Psi_j\}, \lambda(l,m))$*-proper and all the derivatives*

$$\frac{\partial^{|\varkappa|}(\Psi_j | \overset{\circ}{Q^j})}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_j^{\varkappa_j}}, \quad \text{where } 1 \leqslant |\varkappa| = \varkappa_{l+1} + \cdots + \varkappa_j \leqslant p \text{ and } \varkappa_{l+1} \leqslant m$$

*have continuous extensions by zero to* $\Xi(Q^j)$.

*Proof.* We will use the descending induction on $l$ and the ascending induction on $m$.

Assume first that $l = n-1$. There exists a polyhedral complex $\mathcal{P}$ in $R^{n-1}$ such that $|\mathcal{P}| = D_{n-1}$, $\mathcal{P}$ is a refinement of

$$\{\pi_{n-1}^n(L) : \ L \text{ a face of some } K_n \in \mathcal{K}^n\};$$

hence, all the functions $\alpha_i^n$ $(i \in \{0, \ldots, r_n\})$ are affine over each $P \in \mathcal{P}$. Moreover, we assume that each $P \in \mathcal{P}$ has a face, say $M$, of dimension $\leqslant k$ such that if $K_n \in \mathcal{K}_1^n$ and $P \subset \pi_{n-1}^n(K_n)$, then $\pi_{n-1}^n(\Sigma(K_n)) \cap P$ is empty or a face of $M$. By an arbitrarily small linear change of coordinates we can assume that both $D_{n-1}$ and all $P \in \mathcal{P}$ are capsules in $R^{n-1}$ and if $K_n \in \mathcal{K}_1^n$ and $P \subset \pi_{n-1}^n(K_n)$, then

$\pi^n_{n-1}(\Sigma(K_n)) \cap P$ is contained in the rim of $P$ if $k < l = n-1$. Hence, by Remark 2.3 there exists a sequence of continuous PL-functions

$$(8.3.9) \qquad \alpha_0^{n-1} \leqslant \cdots \leqslant \alpha_{r_{n-1}}^{n-1} : D_{n-2} \longrightarrow R$$

such that $\mathcal{K}^{n-1} := \{\overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})} : i \in \{0, \ldots, r_{n-1} - 1\}\}$ is a family of convex capsules in $R^{n-1}$ which is a refinement of $\mathcal{P}$ and $D_{n-2} = \bigcup\{\pi^{n-1}_{n-2}(K_{n-1}) : K_{n-1} \in \mathcal{K}^{n-1}\}$. We put $\Phi_{n-1} = id_{D_{n-1}}$. Then the first part of (8.3.8) is satisfied due to (8.3.1) and the second part is emptily satisfied.

Assume now that $l = n-2$ and $m = 1$[6]. Fix any $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})$ and any $K_n \in \mathcal{K}^n_1$ such that $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1}) \subset \pi^n_{n-1}(\overset{\circ}{K}_n)$. Fix any $\varkappa \in \{1, \ldots, p\}$. The function

$$(\alpha_i^{n-1}, \alpha_{i+1}^{n-1}) \ni (x', x_{n-1}) \longmapsto \sup\Big\{\Big|\frac{\partial^{\varkappa} f}{\partial x_{n-1} \partial x_n^{\varkappa-1}}(x', x_{n-1}, x_n)\Big| : (x', x_{n-1}, x_n) \in K_n\Big\} \in [0, \infty),$$

where $x' = (x_1, \ldots, x_{n-2})$, is continuous. It follows that $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})$ can be covered by a finite family $\mathcal{V}$ of open subsets, which do not depend on $\varkappa$, such that for each $V \in \mathcal{V}$ the norm over $\overline{V} \setminus \pi^n_{n-1}(\Sigma(K_n))$ of the derivative

$$(8.3.10) \qquad \frac{\partial^{\varkappa}(f|K_n \setminus \Sigma(K_n))}{\partial x_{n-1} \partial x_n^{\varkappa-1}}$$

is either bounded from above (the first case) or bounded from below (the second case) by a positive constant. In the second case we can take detectors $\{\omega_\mu\}_\mu$ of class $\mathcal{C}^q$ on $R^{n-1} \setminus \pi^n_{n-1}(\Sigma(K_n))$ for the derivative (8.3.10) over $\overline{V} \setminus \pi^n_{n-1}(\Sigma(K_n))$. It follows from Corollary 8.2 (for $n-1$ in the place of $n$) that there exists a refinement $\overline{(\beta_j, \beta_{j+1})}$ of $\overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})}$ and a homeomorphism $G : D_{n-1} \longrightarrow D_{n-1}$ of class $\mathcal{C}^q$ of the form $G(\xi', \xi_{n-1}) = (g(\xi'), \tilde{g}(\xi', \xi_{n-1}))$, where $\xi' := (\xi_1, \ldots, \xi_{n-2})$, such that every $G((\beta_j, \beta_{j+1}))$ is contained in some $V \in \mathcal{V}$ and $G$ preserves the faces of polyhedrons $K_i^{n-1}$. Then we replace our function $f$ by

$$F(\xi', \xi_{n-1}, x_n) := f(g(\xi'), \tilde{g}(\xi', \xi_{n-1}), x_n).$$

If now $G((\beta_j, \beta_{j+1})) \subset V$ and we have the first case, then the derivative

$$(8.3.11) \qquad \frac{\partial^{\varkappa} F}{\partial \xi_{n-1} x_n^{\varkappa-1}}(\xi', \xi_{n-1}, x_n) = \frac{\partial^{\varkappa}(f|K_n \setminus \Sigma(K_n))}{\partial x_{n-1} x_n^{\varkappa-1}}(g(\xi'), \tilde{g}(\xi', \xi_{n-1}), x_n)\frac{\partial \tilde{g}}{\partial \xi_{n-1}}(\xi', \xi_{n-1})$$

is bounded and if $G((\beta_j, \beta_{j+1})) \subset V$ and we have the second case, then $\tilde{\omega}_\mu(\xi', \xi_{n-1}) := \omega_\mu(g(\xi'), \tilde{g}(\xi', \xi_{n-1})$ are detectors of class $\mathcal{C}^q$ for the derivative (8.3.11) over $\overline{(\beta_j, \beta_{j+1})} \setminus \pi^n_{n-1}(\Sigma(K_n))$.

[6]From the formal point of view it is not necessary to analyze this case separately, but in this simple case it is easy to present the general idea of the proof.

The above argument shows that coming back to the initial derivative (8.3.10), we can assume with no loss in generality that for any $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})$ and any $K_n \in \mathcal{K}_1^n$ such that $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1}) \subset \pi_{n-1}^n(\overset{\circ}{K}_n)$ either (8.3.10) is bounded over $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})$ or there are detectors $\{\omega_\mu\}_\mu$ of class $\mathcal{C}^q$ for (8.3.10) over $\overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})} \setminus \pi_{n-1}^n(\Sigma(K_n))$ which have continuous extensions to $\overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})} \cap \pi_{n-1}^n(\Sigma(K_n))$. Now we apply Proposition 8.1 in dimension $n-1$ in the place of $n$. Hence, there exists a refinement $\overline{(\gamma_j^{n-1}, \gamma_{j+1}^{n-1})}$ of the system of capsules $\overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})}$ and a homeomorphism $\Phi_{n-1} : D_{n-1} \longrightarrow D_{n-1}$, preserving faces of $K_i^{n-1}$, satisfying (8.3.3)-(8.3.5), for $j = n-1$, and such that:

if $L^{n-1} = \overline{(\gamma_j^{n-1}, \gamma_{j+1}^{n-1})} \subset \overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})} \subset \pi_{n-1}^n(K_n)$, where $K_n \in \mathcal{K}_1^n$ and (8.3.10) is bounded on $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})$, then $\Phi_{n-1}|\overset{\circ}{L}{}^{n-1}$ is of class $\mathcal{C}^q$ with

$$\text{(8.3.12)} \qquad \frac{\partial(\varphi_{n-1}|\overset{\circ}{L}{}^{n-1})}{\partial \zeta_{n-1}} \qquad \text{extending continuously by zero to } \partial L^{n-1};$$

consequently, when $\zeta \in \overset{\circ}{L}{}^{n-1}$ and $(\Phi_{n-1}(\zeta), x_n) \in \overset{\circ}{K}_n$

$$\frac{\partial^\varkappa}{\partial \zeta_{n-1} \partial x_n^{\varkappa-1}} f(\Phi_{n-1}(\zeta), x_n) = \frac{\partial^\varkappa f}{\partial x_{n-1} \partial x_n^{\varkappa-1}} (\Phi_{n-1}(\zeta), x_n) \frac{\partial \varphi_{n-1}}{\partial \zeta_{n-1}}$$

extends continuously by zero to $\{(\zeta, x_n) \in C \times R : \ (\Phi_{n-1}(\zeta), x_n) \in K_n\}$;

if $L^{n-1} = \overline{(\gamma_j^{n-1}, \gamma_{j+1}^{n-1})} \subset \overline{(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})} \subset \pi_{n-1}^n(K_n)$, where $K_n \in \mathcal{K}_1^n$ and (7.3.10) is unbounded on $(\alpha_i^{n-1}, \alpha_{i+1}^{n-1})$, then $\Phi_{n-1}|\overset{\circ}{L}{}^{n-1}$, $\omega_\mu \circ \Phi_{n-1}|\overset{\circ}{L}{}^{n-1}$ and $\frac{\partial^{\varkappa-1} f}{\partial x_n^{\varkappa-1}}(\Phi_{n-1}, \omega_\mu \circ \Phi_{n-1})|\overset{\circ}{C}$ are of class $\mathcal{C}^{q-(p-1)}$ and all the derivatives,

$$\frac{\partial(\Phi_{n-1}|\overset{\circ}{L}{}^{n-1})}{\partial \zeta_{n-1}}, \ \frac{\partial(\omega_\mu \circ \Phi_{n-1}|\overset{\circ}{C})}{\partial \zeta_{n-1}} \quad \text{and} \quad \frac{\partial}{\partial \zeta_{n-1}} \Big[ \frac{\partial^{\varkappa-1} f}{\partial x_n^{\varkappa-1}} (\Phi_{n-1}, \omega_\mu \circ \Phi_{n-1})|\overset{\circ}{C} \Big],$$

extend continuously by zero to $L^{n-1}$; it follows that if $(\Phi_{n-1}(\zeta), x_n) \in \overset{\circ}{K}_n$

$$\Big| \frac{\partial^\varkappa}{\partial \zeta_{n-1} \partial x_n^{\varkappa-1}} f(\Phi_{n-1}(\zeta), x_n) \Big| = \Big| \frac{\partial^\varkappa f}{\partial x_{n-1} \partial x_n^{\varkappa-1}} (\Phi_{n-1}(\zeta), x_n) \frac{\partial \varphi_{n-1}}{\partial \zeta_{n-1}} \Big| \leqslant$$

$$2 \sup_\mu \Big| \frac{\partial^\varkappa f}{\partial x_{n-1} \partial x_n^{\varkappa-1}} (\Phi_{n-1}(\zeta), \omega_\mu(\Phi_{n-1}(\zeta)) \frac{\partial \varphi_{n-1}}{\partial \zeta_{n-1}} \Big| \leqslant$$

$$2 \sup_\mu \Big| \frac{\partial}{\partial \zeta_{n-1}} \Big[ \frac{\partial^{\varkappa-1} f}{\partial x_n^{\varkappa-1}} (\Phi_{n-1}, \omega_\mu \circ \Phi_{n-1}) \Big] \Big| + 2 \sup_\mu \Big| \frac{\partial^\varkappa f}{\partial x_n^\varkappa} (\Phi_{n-1}, \omega_\mu \circ \Phi_{n-1})) \frac{\partial(\omega_\mu \circ \Phi)}{\partial \zeta_{n-1}} \Big|,$$

which extends continuously by zero to $\{(\zeta, x_n) \in C \times R : \ (\Phi_{n-1}(\zeta), x_n) \in K_n\}$;

finally, if $L^{n-1} = \overline{(\gamma_j^{n-1}, \gamma_{j+1}^{n-1})} \not\subset \pi_{n-1}^n(K_n)$, for any $K_n \in \mathcal{K}_1^n$, then $\Phi_{n-1}|C$ is of class $\mathcal{C}^q$. Now $\mathcal{L}^{n-1} = \{\overline{(\gamma_j^{n-1}, \gamma_{j+1}^{n-1})}\}_j$ is a new family of capsules in dimension $n-1$. In a similar way, as in the case $l = n-1$, after some arbirarily small linear change of coordinates in $R^{n-2}$, we get a system of convex PL-capsules $\mathcal{K}^{n-2}$ which is a refinement of $\{\pi_{n-2}^{n-1}(C) : C \text{ a face of some } L^{n-1} \in \mathcal{L}^{n-1}\}$.

Assume now that we have our proposition proved for some $l$ such that $k \leqslant l < n-1$ and for some $m \in \{1, \ldots, p-1\}$. Fix any

$$Q^n = Q^n(i_{l+1}, \ldots, i_n) = \{(x_1, \ldots, x_n) \in R^n :$$

$$(x_1, \ldots, x_\nu) \in \overline{(\alpha_{i_\nu}^\nu \circ \Psi_{\nu-1}, \alpha_{i_\nu+1}^\nu \circ \Psi_{\nu-1})}, \quad \text{when} \quad l+1 \leqslant \nu \leqslant n\},$$

such that $Q^n \subset K_n \in \mathcal{K}_1^n$.
Put

$$Q^j = Q^j(i_{l+1}, \ldots, i_j) = \{(x_1, \ldots, x_j) \in R^j :$$

$$(x_1, \ldots, x_\nu) \in \overline{(\alpha_{i_\nu}^\nu \circ \Psi_{\nu-1}, \alpha_{i_\nu+1}^\nu \circ \Psi_{\nu-1})}, \quad \text{when} \quad l+1 \leqslant \nu \leqslant j\}.$$

For any $\varkappa = (\varkappa_{l+1}, \ldots, \varkappa_n) \in \mathbb{N}^{n-l}$ such that $|\varkappa| \leqslant p$ and $\varkappa_{l+1} = m+1$ the functions

$$(\alpha_{i_{l+1}}^{l+1}, \alpha_{i_{l+1}+1}^{l+1}) \ni (x', x_{l+1}) \longmapsto \sup \Big\{ \Big| \frac{\partial^{|\varkappa|}(\Psi_j | Q^j)}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_j^{\varkappa_j}}(x', x_{l+1}, \ldots, x_j) \Big| :$$

$$(x', x_{l+1}, \ldots, x_j) \in Q^j \Big\} \in [0, \infty)$$

and

$$(\alpha_{i_{l+1}}^{l+1}, \alpha_{i_{l+1}+1}^{l+1}) \ni (x', x_{l+1}) \longmapsto \sup \Big\{ \Big| \frac{\partial^{|\varkappa|}(f \circ \Psi_n | Q^j)}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_n^{\varkappa_n}}(x', x_{l+1}, \ldots, x_n) \Big| :$$

$$(x', x_{l+1}, \ldots, x_n) \in Q^n \Big\} \in [0, \infty),$$

where $x' = (x_1, \ldots, x_l)$, are continuous. It follows that $(\alpha_{i_{l+1}}^{l+1}, \alpha_{i_{l+1}+1}^{l+1})$ can be covered by a finite family $\mathcal{V}$ of open subsets such that for each $V \in \mathcal{V}$ the norm over $\overline{V} \setminus \pi_{l+1}^j(\Xi(Q^j))$ of each of the derivatives

$$\text{(8.3.13)} \qquad \frac{\partial^{|\varkappa|}(\Psi_j | Q^j \setminus \pi_{l+1}^j(\Xi(Q^j)))}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_j^{\varkappa_j}}(x', x_{l+1}, \ldots, x_j) \quad (|\varkappa| \leqslant p,\ \varkappa_{l+1} = m+1)$$

and, similarly, the norm over $\overline{V} \setminus \pi_{l+1}^n(\Xi(Q^n))$ of each of the derivatives

$$\text{(8.3.14)} \qquad \frac{\partial^{|\varkappa|}(f \circ \Psi_n | Q^n \setminus \pi_{l+1}^n(\Xi(Q^n)))}{\partial x_{l+1}^{\varkappa_{l+1}} \ldots \partial x_n^{\varkappa_n}}(x', x_{l+1}, \ldots, x_n)$$

$$(|\varkappa| \leqslant p,\ \varkappa_{l+1} = m+1)$$

is either bounded from above (the first case) or bounded from below (the second case) by a positive constant. In the second case we can take detectors $\{\omega^j_\mu\}_\mu$ (respectively $\{\omega_\mu\}_\mu$) of class $\mathcal{C}^q$ on $R^{l+1} \setminus \pi^j_{l+1}(\Xi(Q^j))$ (respectively of class $\mathcal{C}^q$ on $R^{l+1} \setminus \pi^n_{l+1}(\Xi(Q^n)))$ for the derivative (8.3.13) (respectively (8.3.14)) over $\overline{V} \setminus \pi^j_{l+1}(\Xi(Q^j))$ (respectively over $\overline{V}\setminus\pi^n_{l+1}(\Xi(Q^n)))$. It follows from Corollary 8.2 (for $l+1$ in the place of $n$) that there exists a refinement $\overline{(\beta_{j_{l+1}}, \beta_{j_{l+1}+1})}$ of $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})}$ and a homeomorphism $G : D_{l+1} \longrightarrow D_{l+1}$ of class $\mathcal{C}^q$ of the form $G(\xi', \xi_{l+1}) = (g(\xi'), \tilde{g}(\xi', \xi_{l+1}))$ such that every $G((\beta_{j_{l+1}}, \beta_{j_{l+1}+1}))$ is contained in some $V \in \mathcal{V}$ and $G$ preserves all faces of any $K_{l+1} \in \mathcal{K}^{l+1}$. Then we replace the functions $\Psi_j$ by

$$\tilde{\Psi}_j(\xi', \xi_{l+1}, x_{l+2}, \dots, x_j) := \Psi_j(g(\xi'), \tilde{g}(\xi', \xi_{l+1}), x_{l+2}, \dots, x_j).$$

and $f\circ\Psi_n$ by $F := f\circ\tilde{\Psi}_n$. If now $G((\gamma_{j_{l+1}}, \gamma_{j_{l+1}+1})) \subset V$ and we have the first case, then for each $(\xi', \xi_{l+1}) \in (\gamma_{j_{l+1}}, \gamma_{j_{l+1}+1})$ such that $(G(\xi', \xi_{l+1}), x_{l+2}, \dots, x_n) \in Q^n$

$$\frac{\partial^{|\varkappa|}\tilde{\Psi}_j}{\partial\xi^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_j}_j}(\xi', \xi_{l+1}, x_{l+2}, \dots, x_j) = \tag{8.3.15}$$

$$\frac{\partial^{|\varkappa|}\Psi_j}{\partial x^{\varkappa_{l+1}}_{l+1} x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_j}_j}(G(\xi', \xi_{l+1}), x_{l+2}\dots, x_j)\Big(\frac{\partial\tilde{g}}{\partial\xi_{l+1}}\Big)^{m+1} + \text{a bounded function}$$

and/or, similarly

$$\frac{\partial^{|\varkappa|}F}{\partial\xi^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_n}_n}(\xi', \xi_{l+1}, x_{l+2}, \dots, x_n) = \tag{8.3.16}$$

$$\frac{\partial^{|\varkappa|}(f\circ\Psi_n)}{\partial x^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2},\dots\partial x^{\varkappa_n}_n}(G(\xi', \xi_{l+1}), x_{l+2}, \dots, x_n)\Big(\frac{\partial\tilde{g}}{\partial\xi_{l+1}}\Big)^{m+1} + \text{a bounded function}$$

is bounded and if $G((\beta_{j_{l+1}}, \beta_{j_{l+1}+1})) \subset V$ and we have the second case, then putting $\tilde{\omega}^j_\mu(\xi', \xi_{l+1}) := \omega^j_\mu(g(\xi'), \tilde{g}(\xi', \xi_{l+1}))$ and $\tilde{\omega}_\mu(\xi', \xi_{l+1}) := \omega_\mu(g(\xi'), \tilde{g}(\xi', \xi_{l+1}))$, by (8.3.15), we have for each $(\xi', \xi_{l+1}) \in (\beta_{j_{l+1}}, \beta_{j_{l+1}+1})$ such that $(G(\xi', \xi_{l+1}), x_{l+2}, \dots, x_n) \in Q^n$

$$\Big|\frac{\partial^{|\varkappa|}\tilde{\Psi}_j}{\partial\xi^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_j}_j}(\xi', \xi_{l+1}, x_{l+2}, \dots, x_j)\Big| \leqslant \tag{8.3.17}$$

$$2\sup_\mu\Big|\frac{\partial^{|\varkappa|}\tilde{\Psi}_j}{\partial\xi^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_n}_n}(\xi', \xi_{l+1}, \tilde{\omega}^j_\mu(\xi', \xi_{l+1}))\Big| + \text{a bounded function}$$

and/or, similarly

$$\Big|\frac{\partial^{|\varkappa|}F}{\partial\xi^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_n}_n}(\xi', \xi_{l+1}, x_{l+2}, \dots, x_n)\Big| \leqslant \tag{8.3.18}$$

$$2\sup_\mu\Big|\frac{\partial^{|\varkappa|}F}{\partial\xi^{\varkappa_{l+1}}_{l+1}\partial x^{\varkappa_{l+2}}_{l+2}\dots\partial x^{\varkappa_n}_n}(\xi', \xi_{l+1}, \tilde{\omega}_\mu(\xi', \xi_{l+1}))\Big| + \text{a bounded function.}$$

(Notice, that now $\tilde{\omega}^j_\mu$ are not necessarily detectors in the previous sense, but still will play the role of detectors as we will see in a moment.) It follows that coming back to the initial family of capsules we can assume without any loss in generality that for each $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})} \subset \pi^n_{l+1}(Q^n)$ and $j \in \{l+1, \ldots, n\}$ each of the derivatives (8.3.13) is either bounded (the first case) or there exists a finite family $\{\omega^j_\mu\}$ of continuous maps on $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})}$ which are of class $\mathcal{C}^q$ on $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})} \setminus \pi^j_{l+1}(\Xi(Q^j))$ and such that

$$\left|\frac{\partial^{|\varkappa|}\Psi_j}{\partial x^{\varkappa_{l+1}}_{l+1} \ldots \partial x^{\varkappa_j}_j}(x', x_{l+1}, \ldots, x_j)\right| \leqslant \tag{8.3.19}$$

$$2\sup_\mu \left|\frac{\partial^{|\varkappa|}\Psi_j}{\partial x^{\varkappa_{l+1}}_{l+1} \ldots \partial x^{\varkappa_j}_j}(x', x_{l+1}, \omega^j_\mu(x', x_{l+1}))\right| + \text{a bounded function}$$

(the second case).

Similarly, each of the derivatives (8.3.14) is either bounded on $Q^n$ (the first case) or (in the second case) there exists a finite family $\{\omega_\mu\}$ of continuous maps on $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})}$ which are of class $\mathcal{C}^q$ on $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})} \setminus \pi^n_{l+1}(\Xi(Q^n))$ and such that

$$\left|\frac{\partial^{|\varkappa|}(f \circ \Psi_n)}{\partial x^{\varkappa_{l+1}}_{l+1} \ldots \partial x^{\varkappa_n}_n}(x', x_{l+1}, \ldots, x_n)\right| \leqslant \tag{8.3.20}$$

$$2\sup_\mu \left|\frac{\partial^{|\varkappa|}(f \circ \Psi_n)}{\partial x^{\varkappa_{l+1}}_{l+1} \ldots \partial x^{\varkappa_n}_n}(x', x_{l+1}, \omega_\mu(x', x_{l+1}))\right| + \text{a bounded function.}$$

In this way, to every capsule $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})}$, there corresponds a finite number of continuous maps $\omega^j_\mu$, $\omega_\mu$ (depending also on the choice of cell $Q^n$, which is not reflected in the notation in order not to overcharge it), which are of class $\mathcal{C}^q$ on $(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})$. Now, we apply Proposition 8.1 to all the functions

$$\frac{\partial^{|\varkappa|-m-1}\Psi_j}{\partial x^{\varkappa_{l+2}}_{l+2} \ldots \partial x^{\varkappa_j}_j}(x', x_{l+1}, \omega^j_\mu(x', x_{l+1})) \tag{8.3.21}$$

and

$$\frac{\partial^{|\varkappa|-m-1}(f \circ \Psi_n)}{\partial x^{\varkappa_{l+2}}_{l+2} \ldots \partial x^{\varkappa_n}_n}(x', x_{l+1}, \omega_\mu(x', x_{l+1})), \tag{8.3.22}$$

which are continuous on $\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})}$ and of class $\mathcal{C}^{\lambda(l,m)-(p-m-1)} = \mathcal{C}^{\lambda(l,m+1)}$ on $(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})$. Hence, there exists a refinement $\mathcal{L}^{l+1} = \{\overline{(\gamma^{l+1}_{j_{l+1}}, \gamma^{l+1}_{j_{l+1}+1})}\}_{j_{l+1}}$ of $\mathcal{K}^{l+1} = \{\overline{(\alpha^{l+1}_{i_{l+1}}, \alpha^{l+1}_{i_{l+1}+1})}\}_{i_{l+1}}$ and a homeomorphism $\Phi_{l+1} : D_{l+1} \longrightarrow D_{l+1}$ of the form

$$\Phi_{l+1}(\zeta_1, \ldots, \zeta_l, \zeta_{l+1}) = (\tilde{\Phi}_{l+1}(\zeta_1, \ldots, \zeta_l), \varphi_{l+1}(\zeta_1, \ldots, \zeta_l, \zeta_{l+1})),$$

where $\tilde{\Phi}_{l+1} : D_l \longrightarrow D_l$ is a homeomorphism of class $\mathcal{C}^q$, $\Phi_{l+1}(S) = S$, for every face of any $K_{l+1} \in \mathcal{K}^{l+1}$ and if $L_{l+1} \subset K_{l+1} \subset \pi^n_{l+1}(Q^n)$, then $L_{l+1}$ is $(k, \varphi_{l+1}, \lambda(l, m+1))$-well situated in $R^{l+1}$ with $\Sigma(L_{l+1}) \subset \Sigma(K_{l+1})$,

$$\frac{\partial(\varphi_{l+1}|\overset{\circ}{L}_{l+1})}{\partial \zeta_{l+1}}, \dots, \frac{\partial^{m+1}(\varphi_{l+1}|\overset{\circ}{L}_{l+1})}{\partial \zeta_{l+1}^{m+1}} \tag{8.3.23}$$

extend continuously by zero to $\partial L_{l+1}$;

consequently, when (8.3.13) (respectively, (8.3.14)) is bounded on $\overset{\circ}{Q^j}$ (respectively on $\overset{\circ}{Q^n}$), which is the first case, then obviously

$$\frac{\partial^{|\varkappa|}}{\partial \zeta_{l+1}^{\varkappa_{l+1}} x_{l+2}^{\varkappa_{l+2}} \dots \partial x_j^{\varkappa_j}} \Psi_j(\Phi_{l+1}(\zeta', \zeta_{l+1}), x_{l+2}, \dots, x_j) \tag{8.3.24}$$

$$(|\varkappa| \leqslant p,\ \varkappa_{l+1} = m+1)$$

extend continuously by zero to the cell

$$\{(\zeta, x_2, \dots, x_j) \in L_{l+1} \times R^{j-l-1} : (\Phi_{l+1}(\zeta), x_2, \dots, x_j) \in Q^j\} \tag{8.3.25}$$

(respectively,

$$\frac{\partial^{|\varkappa|}}{\partial \zeta_{l+1}^{\varkappa_{l+1}} x_{l+2}^{\varkappa_{l+2}} \dots \partial x_n^{\varkappa_n}} (f \circ \Psi_n)(\Phi_{l+1}(\zeta', \zeta_{l+1}), x_{l+2}, \dots, x_n) \tag{8.3.26}$$

$$(|\varkappa| \leqslant p,\ \varkappa_{l+1} = m+1)$$

extend continuously by zero to the cell

$$\{(\zeta, x_2, \dots, x_n) \in L_{l+1} \times R^{n-l-1} : (\Phi_{l+1}(\zeta), x_2, \dots, x_n) \in Q^n\}.) \tag{8.3.27}$$

In the second case, we can have not only (8.3.23) extending continuously by zero to $L_{l+1}$, but also the derivatives

$$\frac{\partial}{\partial \zeta_{l+1}} (\omega^j_\mu \circ \Phi_{l+1})|\overset{\circ}{L}_{l+1}, \dots, \frac{\partial^{m+1}}{\partial \zeta_{l+1}^{m+1}} (\omega^j_\mu \circ \Phi_{l+1})|\overset{\circ}{L}_{l+1} \tag{8.3.28}$$

and

$$\frac{\partial^{m+1}}{\partial \zeta_{l+1}^{m+1}} \Big[ \frac{\partial^{|\varkappa|-m-1} \Psi_j}{\partial x_{l+2}^{\varkappa_{l+2}} \dots \partial x_j^{\varkappa_j}} \big(\Phi_{l+1}, \omega^j_\mu \circ \Phi_{l+1}\big) \Big] \Big| \overset{\circ}{L}_{l+1} \tag{8.3.29}$$

$\big($respectively, the derivatives

$$\frac{\partial}{\partial \zeta_{l+1}} (\omega_\mu \circ \Phi_{l+1})|\overset{\circ}{L}_{l+1}, \dots, \frac{\partial^{m+1}}{\partial \zeta_{l+1}^{m+1}} (\omega_\mu \circ \Phi_{l+1})|\overset{\circ}{L}_{l+1} \tag{8.3.30}$$

and

$$\frac{\partial^{m+1}}{\partial\zeta_{l+1}^{m+1}}\Big[\frac{\partial^{|\varkappa|-m-1}(f\circ\Psi_n)}{\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_n^{\varkappa_n}}(\Phi_{l+1},\omega_\mu\circ\Phi_{l+1})\Big]\Big|\overset{\circ}{L}_{l+1}\ \Big). \tag{7.3.31}$$

By the induction hypothesis, (8.3.8), (8.3.23), (8.3.19) and (8.3.29) we have on the cell (8.3.25)

$$\Big|\frac{\partial^{|\varkappa|}}{\partial\zeta_{l+1}^{\varkappa_{l+1}}x_{l+2}^{\varkappa_{l+2}}\dots\partial x_j^{\varkappa_j}}\Psi_j(\Phi_{l+1}(\zeta',\zeta_{l+1}),x_{l+2},\dots,x_j)\Big|\leqslant$$

$$\Big|\frac{\partial^{|\varkappa|}\Psi_j}{\partial x_{l+1}^{\varkappa_{l+1}}\dots\partial x_j^{\varkappa_j}}(\Phi_{l+1}(\zeta',\zeta_{l+1}),x_{l+2}\dots,x_j)\Big|\Big|\frac{\partial\varphi_{l+1}}{\partial\zeta_{l+1}}\Big|^{m+1}+$$

$$+\text{ a continuous function equal 0 at the boundary of (8.3.25) }\leqslant$$

$$2\sup_\mu\Big|\frac{\partial^{|\varkappa|}\Psi_j}{\partial x_{l+1}^{\varkappa_{l+1}}\dots\partial x_j^{\varkappa_j}}(\Phi_{l+1}(\zeta',\zeta_{l+1}),\omega_\mu^j(\Phi_{l+1}(\zeta',\zeta_{l+1}))\Big|\Big|\frac{\partial\varphi_{l+1}}{\partial\zeta_{l+1}}\Big|^{m+1}+$$

$$+\text{ a continuous function equal 0 at the boundary of (8.3.25) }\leqslant$$

$$2\sup_\mu\Big|\frac{\partial^{m+1}}{\partial\zeta_{l+1}^{m+1}}\Big[\frac{\partial^{|\varkappa|-m-1}\Psi_j}{\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_j^{\varkappa_j}}(\Phi_{l+1},\omega_\mu^j\circ\Phi_{l+1})\Big]+\text{a cont. funct. equal 0 at }\partial L_{l+1}\Big|$$

$$+\text{ a continuous function equal 0 at the boundary of (8.3.25),}$$

which finally is a function extending by zero to the boundary of (8.3.25).

Similarly, by the induction hypothesis (8.3.8), (8.3.23) and (8.3.20), we have on the cell (8.3.27)

$$\Big|\frac{\partial^{|\varkappa|}}{\partial\zeta_{l+1}^{\varkappa_{l+1}}x_{l+2}^{\varkappa_{l+2}}\dots\partial x_n^{\varkappa_n}}(f\circ\ \Psi_n)(\Phi_{l+1}(\zeta',\zeta_{l+1}),x_{l+2},\dots,x_n)\Big|\leqslant$$

$$\Big|\frac{\partial^{|\varkappa|}(f\circ\Psi_n)}{\partial x_{l+1}^{\varkappa_{l+1}}\dots\partial x_j^{\varkappa_j}}(\Phi_{l+1}(\zeta',\zeta_{l+1}),x_{l+2}\dots,x_n)\Big|\Big|\frac{\partial\varphi_{l+1}}{\partial\zeta_{l+1}}\Big|^{m+1}+$$

$$+\text{ a continuous function equal 0 at the boundary of (8.3.27) }\leqslant$$

$$2\sup_\mu\Big|\frac{\partial^{|\varkappa|}(f\circ\Psi_n)}{\partial x_{l+1}^{\varkappa_{l+1}}\dots\partial x_n^{\varkappa_n}}(\Phi_{l+1}(\zeta',\zeta_{l+1}),\omega_\mu(\Phi_{l+1}(\zeta',\zeta_{l+1}))\Big|\Big|\frac{\partial\varphi_{l+1}}{\partial\zeta_{l+1}}\Big|^{m+1}+$$

$$+\text{ a continuous function equal 0 at the boundary of (8.3.27) }\leqslant$$

$$2\sup_\mu\Big|\frac{\partial^{m+1}}{\partial\zeta_{l+1}^{m+1}}\Big[\frac{\partial^{|\varkappa|-m-1}(f\circ\Psi_n)}{\partial x_{l+2}^{\varkappa_{l+2}}\dots\partial x_n^{\varkappa_n}}(\Phi_{l+1},\omega_\mu\circ\Phi_{l+1})\Big]+\text{a cont.funct.equal 0 at }\partial L_{l+1}\Big|$$

$$+\text{ a continuous function equal 0 at the boundary of (8.3.27),}$$

which finally is a function extending by zero to the boundary of the cell (8.3.27).

To finish the proof it suffices now to assume that we have proved our proposition for some $l\in\{k+1,\dots,n-2\}$ and $m=p$ and to derive it for $l'=l-1$ and $m'=1$. We start as in the case $l=n-2$ and $m=1$ and then continue as for the case when $l\in\{k,\dots,n-2\}$ and $m\in\{1,\dots,p-1\}$ is augmented by 1, by a simple modification.

**Proposition 8.4.** *Assume that the Main Theorem is proved in dimensions* $< n$. *Let* $0 \leqslant k < n$. *Fix an integer* $q \geqslant (n-1-k)\binom{p}{2} + p$. *Assume that*

$$\alpha_0^n \leqslant \cdots \leqslant \alpha_{r_n}^n : D_{n-1} \longrightarrow R$$

*is a sequence of continuous* PL*-functions such that*

$$\mathcal{K}^n := \left\{\overline{(\alpha_i^n, \alpha_{i+1}^n)} : \; i \in \{0, \ldots, r_n - 1\}\right\}$$

*is a family of convex* PL*-capsules in* $R^n$, *where* $D_{n-1} = \bigcup\{\pi_{n-1}^n(K_n) : K_n \in \mathcal{K}^n\}$. *Let* $\mathcal{K}_1^n \subset \mathcal{K}^n$. *Assume that* $D_{n-1}$ *is a closed convex polyhedron in* $R^{n-1}$ *of dimension* $n-1$ *and* $\mathcal{A}$ *is a finite family of subsets of* $D_{n-1}$. *Put* $D_n := |\mathcal{K}^n|$.

*Let* $f : |\mathcal{K}_1^n| \longrightarrow R^d$ *be continuous and such that* $f|\overset{\circ}{K}_n$ *is of class* $\mathcal{C}^q$ *for each* $K_n \in \mathcal{K}_1^n$. *Assume that each* $K_n \in \mathcal{K}_1^n$ *is* $(k, f, q)$*-well situated in* $R^n$ *and that all the derivatives*

$$\text{(8.4.1)} \qquad \frac{\partial^i (f|K_n \setminus \Sigma(K_n))}{\partial x_n^i} \quad (i \in \{1, \ldots, q\}) \quad \textit{have continuous extensions}$$

*by zero to all* $K_n$.

*Let* $\tilde{q}$ *be any integer* $\geqslant q$.

*Then there exists a strict* $\mathcal{C}^{\tilde{q}}$*-triangulation* $(\mathcal{T}, h)$ *of* $D_n$ *compatible with all sets* $D_n \cap (A \times R)$ $(A \in \mathcal{A})$, *such that* $\mathcal{T}$ *is a simplicial complex in* $R^n$ *which is a refinement of* $\mathcal{K}^n$, $h(\Gamma) = \Gamma$, *for any face* $\Gamma$ *of any* $K_n \in \mathcal{K}^n$, *and each* $\Delta \in \mathcal{T}$ *such that* $\Delta \subset K_n \in \mathcal{K}_1^n$ *is* $(k-1, f \circ h, p)$*-well situated in* $R^n$.

*Proof.* Apply first Proposition 8.3 for $l = k$. Hence, by Lemma 5.4, for every $Q^n$ such that $\Psi_n(Q^n) \subset K_n \in \mathcal{K}_1^n$ there exists a closed subset $E(Q^n)$ of $\Xi(Q^n)$ of dimension $< k$ such that consecutively all mappings

$$\Psi_{k+1}|Q^n \setminus \Xi(Q^{k+1}), \ldots, \Psi_n|Q^n \setminus \Xi(Q^n) \text{ and } f \circ \Psi_n|Q^n \setminus \Xi(Q^n)$$

extend respectively to $\mathcal{C}^p$-mappings

$$\Psi_{k+1}|Q^n \setminus \pi_{k+1}^n(E(Q^n)), \ldots, \Psi_n|Q^n \setminus E(Q^n) \text{ and } f \circ \Psi_n|Q^n \setminus E(Q^n).$$

By induction hypothesis there exists a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_{k+1}, h_{k+1})$ of $D_{k+1}$ compatible with all $\pi_{k+1}^n(E(Q^n))$ such that $\mathcal{T}_{k+1}$ is a refinement of $\mathcal{K}^{k+1}$, $h_{k+1}$ preserving all faces of any $K_{k+1} \in \mathcal{K}^{k+1}$ and such that all $\alpha_{i_{k+2}}^{k+2} \circ \Psi_{k+1} \circ h_{k+1}$ are of class $\mathcal{C}^{\tilde{q}}$. By using Corollary 6.4 and Lemma 5.2 this allows us to define a polyhedral complex $\mathcal{P}_{k+2}$ in $R^{k+2}$, which is a refinement of $\mathcal{K}^{k+2}$ and a homeomorphism $H_{k+2} : D_{k+2} \longrightarrow \Psi_{k+2}^{-1}(D_{k+2})$, such that $\Psi_{k+2} \circ H_{k+2}$ is preserving all faces of any $K_{k+2} \in \mathcal{K}^{k+2}$ and for each $P_{k+2} \in \mathcal{P}_{k+2}$, $H_{k+2}|P_{k+2}$ is of class $\mathcal{C}^{\tilde{q}}$. Now we take a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_{k+2}, h_{k+2})$ of $D_{k+2}$ such that $\mathcal{T}_{k+2}$ is a refinement of $\mathcal{P}_{k+2}$ and such that all $\alpha_{i_{k+3}}^{k+3} \circ \Psi_{k+2} \circ H_{k+2} \circ h_{k+2}$ are of class $\mathcal{C}^{\tilde{q}}$. Again, by Corollary 6.4 and Lemma 5.2 this allows us to define a polyhedral complex $\mathcal{P}_{k+3}$ in $R^{k+3}$, which is a refinement of $\mathcal{K}^{k+3}$ and a homeomorphism $H_{k+3} : D_{k+3} \longrightarrow \Psi_{k+3}^{-1}(D_{k+3})$, such that $\Psi_{k+3} \circ H_{k+3}$ is preserving all faces of any $K_{k+3} \in \mathcal{K}^{k+3}$ and for each $P_{k+3} \in \mathcal{P}_{k+3}$, $H_{k+3}|P_{k+3}$ is of class $\mathcal{C}^{\tilde{q}}$. We continue this process, finally obtaining a strict $\mathcal{C}^{\tilde{q}}$-triangulation $(\mathcal{T}_{n-1}, h_{n-1})$ of $D_{n-1}$, which is compatible with all $\pi^n_{n-1}(E(Q^n))$, such $h_{n-1}$ is preserving all faces of any $K_{n-1} \in \mathcal{K}^{n-1}$, and such that $\Psi_{n-1} \circ h_{n-1}$ and all $\alpha^n_{i_n} \circ \Psi_{n-1} \circ h_{n-1}$ are of class $\mathcal{C}^{\tilde{q}}$. Again, due to Corollary 6.4 and Lemma 5.2 with allows us to define a polyhedral complex $\mathcal{P}_n$ in $R^n$, which is a refinement of $\mathcal{K}^{n-1}$ and a homeomorphism $H_n : D_n \longrightarrow \Psi_n^{-1}(D_n)$ of the form $H_n(x', x_n) = (h_{n-1}(x'), \tilde{H}_n(x', x_n))$ such that $\mathcal{P}_n$ is a refinement of $\mathcal{K}^n$, $\Psi_n \circ H_n$ is preserving all faces of each $K_n \in \mathcal{K}^n$ and such that for each $P_n \in \mathcal{P}_n$, both $H_{n-1}|P$ is of class $\mathcal{C}^{\tilde{q}}$. Then $h := \Psi_n \circ H_n = (\Psi_{n-1} \circ h_{n-1}, \tilde{H}_{n-1})$ is of class $\mathcal{C}^{\tilde{q}}$ at the restriction to any $P_n \in \mathcal{P}_n$. Passing to a simplicial subdivision of $\mathcal{P}_n$ and using once more Corollary 6.4 finishes the proof.

**Proposition 8.5.** *Assume that the Main Theorem is proved in dimensions $< n$. Let $0 \leqslant k < n$. Fix an integer $q \geqslant (n-1-k)\binom{p}{2} + p + 1$. Let $\mathcal{P}$ be a polyhedral complex in $R^n$, such that $|\mathcal{P}|$ is a convex polyhedron and each $P \in \mathcal{P}$ of dimension $n$ is a capsule in $R^n$ and let $\mathcal{P}_1 \subset \mathcal{P}$. Assume that $f : |\mathcal{P}_1| \longrightarrow R^d$ is a continuous mapping such that each $P \in \mathcal{P}_1$ is $(k, f, q)$-well situated in $R^n$. Let $\tilde{q}$ be any integer $\geqslant q$.*

*Then there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{P}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, each $\Delta \in \mathcal{T}$ of dimension $n$ such that $\Delta \subset P \in \mathcal{P}_1$ is well $(k-1, (f \circ h, h), p)$-situated in $R^n$, for each $\Delta \subset P \in \mathcal{P} \setminus \mathcal{P}_1$ the restriction $h|\Delta$ is of class $\mathcal{C}^{\tilde{q}}$ and $h(\Gamma) = \Gamma$, for any face $\Gamma$ of any polyhedron $P \in \mathcal{P}$.*

*Proof.* Since all $P \in \mathcal{P}$ are PL-capsules, by Remark 2.3 and Lemma 2.6, there exists a sequence of continuous PL-functions

$$\alpha_0 \leqslant \cdots \leqslant \alpha_r : D \longrightarrow R,$$

where $D = \pi^n_{n-1}(|\mathcal{P}|)$ such that

$$\mathcal{K} := \left\{\overline{(\alpha_i, \alpha_{i+1})} : \; i \in \{0, \ldots, r-1\}\right\}$$

is a family of convex PL-capsules in $R^n$, which is a refinement of $\mathcal{P}$. Put $\mathcal{K}_1 := \{K \in \mathcal{K} : \; K \subset P \in \mathcal{P}_1\}$. It is clear that all $K \in \mathcal{K}_1$ are $(k, f, q+1)$-well situated in $R^n$. By Proposition 8.1, there exists an increasing sequence of continuous PL-functions

$$\eta_0 \leqslant \cdots \leqslant \eta_s : D \longrightarrow R,$$

which is a refinement of $\alpha_0, \ldots, \alpha_r$ such that the family $\mathcal{C} := \left\{\overline{(\eta_j, \eta_{j+1})} : \; j \in \{0, \ldots, s-1\}\right\}$ is a family of capsules which is a refinement of the family $\mathcal{K}$ and moreover there exists a homeomorphism $\Psi : |\mathcal{P}| \longrightarrow |\mathcal{P}|$ preserving all faces of each $K \in \mathcal{K}$ and such that each $C \in \mathcal{C}_1 := \{C \in \mathcal{C} : \; C \subset K \in \mathcal{K}_1\}$ is $(k, (f \circ \Psi, \Psi), q)$-well situated in $R^n$, the derivatives

$$\frac{\partial^\sigma(\Psi|\mathring{C})}{\partial x_n^\sigma} \quad \text{and} \quad \frac{\partial^\sigma(f \circ \Psi|\mathring{C})}{\partial x_n^\sigma} \qquad (\sigma \in \{1, \ldots, q\})$$

have continuous extensions by zero to the whole $C$ and, finally, for each $C \in \mathcal{C} \setminus \mathcal{C}_1$, $\Psi|C$ is of class $\mathcal{C}^{\tilde{q}}$. Now the conclusion follows from Proposition 8.4.

## 9. Main Theorem - proof in general case.

**Proposition 9.1.** *Assume that the Main Theorem is proved in dimensions* $< n$. *Let* $0 \leqslant k < n$. *Fix an integer* $q \geqslant (n-1-k)\binom{p}{2} + p + 1$. *Let* $\mathcal{P}$ *be a polyhedral complex in* $R^n$ *such that* $|\mathcal{P}|$ *is a convex polyhedron of dimension* $n$. *Let* $f : |\mathcal{P}| \longrightarrow R^d$ *be a continuous mapping such that for each* $P \in \mathcal{P}$, *the restriction* $f|P \setminus P^{(k)}$ *is of class* $\mathcal{C}^q$.

*Then there exists a* $\mathcal{C}^p$*-triangulation* $(\mathcal{T}, h)$ *of* $|\mathcal{P}|$ *such that* $\mathcal{T}$ *is a refinement of* $\mathcal{P}$, $h(\Gamma) = \Gamma$, *for each face* $\Gamma$ *of any polyhedron* $P \in \mathcal{P}$ *and for each* $\Delta \in \mathcal{T}$ *of dimension* $n$, $h|\Delta \setminus \Delta^{(k-1)}$ *and* $f \circ h|\Delta \setminus \Delta^{(k-1)}$ *are of class* $\mathcal{C}^p$.

*Proof.* By a barycentric subdivision we reduce the situation to the case where each $P \in \mathcal{P}$ has only one face $\Sigma$ of dimension $\leqslant k$, such that $f|P \setminus \Sigma$ is of class $\mathcal{C}^q$. By Proposition 7.1, there exists a finite number of orthonormal bases $v_1, \ldots, v_s \in \boldsymbol{V}_n(R^n)$ such that each $P \in \mathcal{P}$ is $(k, f, q)$-well situated in $R^n$ relative to some $v_i$, where $i \in \{1, \ldots, s\}$. Hence we can represent (the set of polyhedra of dimension $n$ belonging to) $\mathcal{P}$ as a pair-wise disjoint union

$$\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_s,$$

where each $P \in \mathcal{P}_i$ is $(k, f, q)$-well situated in $R^n$ relative to $v_i$ ($i \in \{1, \ldots, s\}$). By Proposition 8.5 there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}_1, h_1)$ of $|\mathcal{P}|$ such that $\mathcal{T}_1$ is a refinement of $\mathcal{P}$, for each $\Delta_1 \in \mathcal{T}_1$ of dimension $n$, if $\Delta_1 \subset P \in \mathcal{P}_1$, then the restrictions $h_1|\Delta_1 \setminus \Delta_1^{(k-1)}$ and $f \circ h_1|\Delta_1 \setminus \Delta_1^{(k-1)}$ are of class $\mathcal{C}^p$, and if $\Delta_1 \subset P \in \mathcal{P} \setminus \mathcal{P}_1$ the restriction $h_1|\Delta_1$ is of class $\mathcal{C}^q$ and $h_1(\Gamma) = \Gamma$, for any face $\Gamma$ of any polyhedron $P \in \mathcal{P}$. Put

$$\mathcal{T}_{1i} := \{\Delta_1 \in \mathcal{T}_1 : \dim \Delta_1 = n,\ \Delta_1 \subset P \in \mathcal{P}_i\} \qquad (i \in \{1, \ldots, s\}).$$

Observe now that if $\Delta_1 \in \mathcal{T}_{1i}$ ($i \geqslant 2$), then $\Delta_1$ is $(k, (f \circ h_1, h_1), q)$-well situated in $R^n$ relative to $v_i$ and then $\Sigma(\Delta_1) \subset h_1^{-1}(\Sigma(P)) = \Sigma(P)$, where $\Delta_1 \subset P \in \mathcal{P}_i$. By Proposition 8.5, there exists a $\mathcal{C}^p$-triangulation $(\mathcal{T}_2, h_2)$ of $|\mathcal{P}|$ such that $\mathcal{T}_2$ is a refinement of $\mathcal{T}_1$, for each $\Delta_2 \in \mathcal{T}_{12}$ of dimension $n$, if $\Delta_2 \subset \Delta_1 \in \mathcal{T}_{12}$, then the restrictions $h_1 \circ h_2|\Delta_2 \setminus \Delta_2^{(k-1)}$ and $f \circ h_1 \circ h_2|\Delta_2 \setminus \Delta_2^{(k-1)}$ are of class $\mathcal{C}^p$, and if $\Delta_2 \subset \Delta_1 \in \mathcal{T}_1 \setminus \mathcal{T}_{12}$ the restriction $h_2|\Delta_2$ is of class $\mathcal{C}^q$ and $h_2(\Gamma_1) = \Gamma_1$, for any face $\Gamma_1$ of any simplex $\Delta_1 \in \mathcal{T}_1$. Clearly, $h_1 \circ h_2|\Delta_2 \setminus \Delta_2^{(k-1)}$ and $f \circ h_1 \circ h_2|\Delta_2 \setminus \Delta_2^{(k-1)}$ are of class $\mathcal{C}^p$, when $\Delta_2 \subset \Delta_1 \in \mathcal{T}_{11}$. Put

$$\mathcal{T}_{2i} := \{\Delta_2 \in \mathcal{T}_2 : \dim \Delta_2 = n,\ \Delta_2 \subset \Delta_1 \in \mathcal{T}_{1i}\} \qquad (i \in \{1, \ldots, s\}).$$

Observe now that if $\Delta_2 \in \mathcal{T}_{2i}$ ($i \geqslant 3$), then $\Delta_2$ is $(k, (f \circ h_1 \circ h_2, h_1 \circ h_2), q)$-well situated in $R^n$ relative to $v_i$ and then $\Sigma(\Delta_2) \subset h_2^{-1}(\Sigma(\Delta_1)) = \Sigma(\Delta_1)$, where $\Delta_2 \subset \Delta_1 \in \mathcal{T}_{1i}$.

It is clear how to continue this process which at the final $s$-th step gives the required triangulation $(\mathcal{T}, h) = (\mathcal{T}_s, h_1 \circ \cdots \circ h_s)$.

**Proposition 9.2.** *Let $p$ be a positive integer and let integers $q_1, \ldots, q_n$ be such that*

$$q_1 \geqslant (n-1)\binom{p}{2} + p + 1, q_2 \geqslant (n-2)\binom{q_1}{2} + q_1 + 1, \ldots, q_n \geqslant 0\binom{q_{n-1}}{2} + q_{n-1} + 1 = q_{n-1} + 1.$$

*Let $\mathcal{P}$ be a polyhedral complex in $R^n$ such that $|\mathcal{P}|$ is a convex polyhedron of dimension $n$. Let $f : |\mathcal{P}| \longrightarrow R^d$ be a continuous mapping such that for each $P \in \mathcal{P}$, the restriction $f|P \setminus P^{(n-1)}$ is of class $\mathcal{C}^{q_n}$.*

*Then there exists a strict $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{P}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, $h(\Gamma) = \Gamma$, for each face $\Gamma$ of any polyhedron $P \in \mathcal{P}$ and $f \circ h$ is of class $\mathcal{C}^p$.*

*Proof.* By Proposition 9.1 applied $n$ times, we obtain a $\mathcal{C}^p$-triangulation $(\mathcal{T}, h)$ of $|\mathcal{P}|$ such that $\mathcal{T}$ is a refinement of $\mathcal{P}$, $h(\Gamma) = \Gamma$, for each face $\Gamma$ of any $P \in \mathcal{P}$ and such that for each simplex $\Delta \in \mathcal{T}$ of dimension $n$ the restrictions $h|\Delta$ and $f \circ h|\Delta$ are of class $\mathcal{C}^p$. We now improve $h$, using Corollary 6.4.

## 10. An application to approximation theory.

Fernando and Ghiloni proved in [FG] the following approximation theorem.

**Theorem 10.1 ([FG, Corollary 1.5]).** *Let $A$ be a definable, closed and bounded subset of $R^n$ and let $\mathcal{T}$ be a finite simplicial complex in $R^m$. Let $f : A \longrightarrow |\mathcal{T}|$ be a definable continuous mapping.*

*Then for any positive integer $p$ and any $\varepsilon \in R$ such that $\varepsilon > 0$ there exists a $\mathcal{C}^p$-mapping $g : A \longrightarrow |\mathcal{T}|$ such that*

$$|f(x) - g(x)| \leqslant \varepsilon, \qquad \textit{for each } x \in A,$$

*where* $|(y_1, \ldots, y_m)| := \big(\sum_{i=1}^{m} y_i^2\big)^{\frac{1}{2}}$.

In fact [FG] contains a proof of Theorem 10.1 only in the semialgebraic case and $R = \mathbb{R}$ (the field of real numbers), but it is easy to check that the same proof, with obvious modifications, holds true in our general context.

The existence of strict $\mathcal{C}^p$-triangulations allows us to improve the last theorem.

**Theorem 10.2.** *Let $A$ and $B$ be any definable, closed and bounded subsets of $R^n$ and of $R^m$, respectively. Let $f : A \longrightarrow B$ be a definable continuous mapping.*

*Then for any positive integer $p$ and any $\varepsilon \in R$ such that $\varepsilon > 0$ there exists a $\mathcal{C}^p$-mapping $g : A \longrightarrow B$ such that*

$$|f(x) - g(x)| \leqslant \varepsilon, \qquad \textit{for each } x \in A.$$

*Proof.* Let $(\mathcal{T}, h)$ be a strict $\mathcal{C}^p$-triangulation of $B$; hence $h : |\mathcal{T}| \longrightarrow B$ is a homeomorphism of class $\mathcal{C}^p$. Since $h$ is uniformly continuous, there exists $\delta > 0$

such that for each pair $u, w \in |\mathcal{T}|$, if $|u - w| \leqslant \delta$, then $|h(u) - h(w)| \leqslant \varepsilon$. By Theorem 10.1 there exists a $\mathcal{C}^p$-mapping $g : A \longrightarrow |\mathcal{T}|$ such that

$$|h^{-1} \circ f(x) - g(x)| \leqslant \delta, \qquad \text{for each } x \in A.$$

Hence,

$$|f(x) - h \circ g(x)| \leqslant \varepsilon, \qquad \text{for each } x \in A,$$

and $h \circ g : A \longrightarrow B$ is of class $\mathcal{C}^p$ as a composition of two mappings of class $\mathcal{C}^p$.

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Wiesław Pawłucki; Instytut Matematyki Uniwersytetu Jagiellońskiego, ul. Prof. St. Łojasiewicza 6, 30-348 Kraków, Poland

e-mail: Wieslaw.Pawlucki@im.uj.edu.pl