A GENERALIZED BORSUK-ULAM THEOREM IN A REAL CLOSED FIELD

IKUMITSU NAGASAKI, TOMOHIRO KAWAKAMI, YASUHIRO HARA AND FUMIHIRO USHITAKI

Abstract. Let C_k be the cyclic group of order k and $\mathcal{N} = (R, +, \cdot, <, \dots)$ an o-minimal expansion of a real closed field R. Let X be a definably connected definable set with a free definable C_k-action. Assume that there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$. If Y is a definable set with a free definable C_k-action such that $H_{n+1}(Y/C_k, \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no definable C_k-map from X to Y. We also prove the topological version of this definable version.

1. Introduction

Let C_k be the cyclic group of order k. Let $\mathbb{S}^n$ be the n-dimensional unit sphere of the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with the antipodal C_2-action. From the viewpoint of transformation groups, the classical Borsuk-Ulam theorem states that if there exists a continuous C_2-map from $\mathbb{S}^n$ to $\mathbb{S}^m$, then $n \leq m$. There are several equivalent statements of it and many related generalizations (e.g. [2], [12], [13], [14], [16]).

The classical Borsuk-Ulam theorem is generalized to topological spaces by several authors. For example, J.W. Walker [20], Pedro L. Q Pergher, Denise de Mattos and Edivaldo L. dos Santos [17].

Several C_k-versions of the classical Borsuk-Ulam theorem are discussed in [10] and [7]. The following two theorems are C_k-versions for topological spaces which are generalizations of [20], [17], [10] and [7].

Theorem 1.1. *Let X be an arcwise connected free C_k-space and Y a Hausdorff free C_k-space. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$ and $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no continuous C_k-map from X to Y. Here this homology means the singular homology.*

Let k be a prime. For a topological space Y, let $D = \{(y_1, \dots, y_k) \in Y \times \cdots \times Y | y_1 = \cdots = y_k\}$ be the diagonal and write $Y^* = Y \times \cdots \times Y - D$ admitting the free C_k-action defined by $g(y_1, y_2, \dots, y_k) = (y_k, y_1, \dots, y_{k-1})$, where g generates C_k.

Theorem 1.2. *Let k be a prime and X an arcwise connected free C_k-space. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$ and Y is a Hausdorff free C_k-space with $H_{n+1}(Y^*/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then every continuous map $f : X \to Y$ has a*

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C_k-coincidence point, that is, a point x such that $f(x) = f(gx)$, where g is a generator of C_k.

The purpose of this paper is to consider the definable versions of Theorem 1.1 and Theorem 1.2

Let $\mathcal{N} = (R, +, \cdot, <, \dots)$ be an o-minimal expansion of a real closed field R. Any definable category is a generalization of the semialgebraic category. Many results in the semialgebraic geometry hold in the o-minimal setting and there exist uncountably many o-minimal expansions of the standard structure of the field $\mathbb{R}$ of real numbers ([18]). See also [4], [6], [11] for examples and constructions of o-minimal structures. General references on them are [3], [5], [19]. In this paper "definable" means "definable with parameters in $\mathcal{N}$", everything is considered in $\mathcal{N}$ and each definable map is continuous unless otherwise stated.

The singular definable homology is introduced in [21].

Theorem 1.3 (Definable Borsuk-Ulam Theorem)**.** *Let X be a definably connected definable set with a free definable C_k-action. If there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$ and Y is a definable set with a free definable C_k-action such that $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then there is no definable C_k-map from X to Y. Here this homology means the singular definable homology.*

If Y is a definable set with a definable C_k-action, then by 10.2.18 [3], Y/C_k is a definable set and the orbit map $\pi : Y \to Y/C_k$ is definable. If $\dim Y \leq n$, then by 4.1.6 [3] $\dim Y/C_r \leq n$. Thus if $\dim Y \leq n$, then $H_{n+1}(Y/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$.

Let S^n denote the n-dimensional unit sphere of R^{n+1}.

Corollary 1.4. *(1) Suppose that $k \geq 3$ and that C_k acts on S^{2m+1} and S^{2n+1} definably and freely. If there exists a definable C_k-map $f : S^{2m+1} \to S^{2n+1}$, then $m \leq n$.*

(2) If S^m and S^n have free definable C_2-actions and there exists a definable C_2-map $f : S^m \to S^n$, then $m \leq n$.

Corollary 1.4 is a generalization of 1.1 [15].

Theorem 1.5. *Let k be a prime and X a definably connected definable set with a free definable C_k-action. Assume that there exists a positive integer n such that $H_q(X; \mathbb{Z}/k\mathbb{Z}) = 0$ for $1 \leq q \leq n$. If Y is a definable set with $H_{n+1}(Y^*/C_k; \mathbb{Z}/k\mathbb{Z}) = 0$, then every definable map $f : X \to Y$ has a C_k-coincidence point, that is, a point x such that $f(x) = f(gx)$, where g is a generator of C_k.*

2. Proof of results

We first prove Theorem 1.3. Let $\mathbb{Z}/k\mathbb{Z}[C_k]$ denote the group ring of C_k over $\mathbb{Z}/k\mathbb{Z}$. For any $q \in \mathbb{N} \cup \{0\}$, the q-dimensional chain group $C_q(X; \mathbb{Z}/k\mathbb{Z})$ has the standard C_k-action. Then this action induces $\mathbb{Z}/k\mathbb{Z}[C_k]$-action on $C_q(X; \mathbb{Z}/k\mathbb{Z})$.

Let g be a generator of C_k, $\alpha = 1 + g + \cdots + g^{k-1}$, and $\beta = 1 - g$. Then by definition $\alpha\beta = \beta\alpha = 0$, for every q, $\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})$ and $\beta C_q(X; \mathbb{Z}/k\mathbb{Z})$ are $\mathbb{Z}/k\mathbb{Z}[C_k]$-submodules of $C_q(X; \mathbb{Z}/k\mathbb{Z})$ and $\alpha\partial = \partial\alpha, \beta\partial = \partial\beta$, where ∂ is the boundary operator of $\{C_q(X; \mathbb{Z}/k\mathbb{Z})\}$. Therefore $\{\alpha C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ and $\{\beta C_q(X; \mathbb{Z}/k\mathbb{Z})\}$ are subchain complexes of $\{C_q(X; \mathbb{Z}/k\mathbb{Z})\}$.

Proposition 2.1. *For every q, the following two sequences are exact.*

$$0 \to \alpha C_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{i} C_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta} \beta C_q(X;\mathbb{Z}/k\mathbb{Z}) \to 0,$$
$$0 \to \beta C_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{j} C_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha} \alpha C_q(X;\mathbb{Z}/k\mathbb{Z}) \to 0,$$

where i, j denote the inclusions and α (resp. β) stands for the multiplication of α (resp. β).

Proof. Since $\beta \circ i = 0, \alpha \circ j = 0$, $\mathrm{Im}\ i \subset \mathrm{Ker}\ \beta, \mathrm{Im}\ j \subset \mathrm{Ker}\ \alpha$.

Let $s = \sum_j \sum_{i=0}^{k-1} n_{ji} g^i \sigma_j \in \mathrm{Ker}\ \beta$, where g is a generator of C_k. If $l \neq l'$ and $0 \le i \le k-1$, then $g^i \sigma_l \neq \sigma_{l'}$. Since $\beta s = 0$, for any j, $\sum_{i=0}^{k-1} n_{ji} g^i (1-g)\sigma_j = 0$. Thus for every j, $\sum_{i=1}^{k-1}(n_{ji} - n_{j(i-1)})g^i\sigma_i + (n_{j0} - n_{j(k-1)})\sigma_j = 0$. Hence for each j, $n_{j0} = n_{j1} = \cdots = n_{jk-1}$. We set $n_j = n_{j0} (= n_{j1} = \cdots = n_{jk-1})$. Then We have $s = \sum_j n_j(1 + g + \cdots + g^{k-1})\sigma_j = \alpha \sum_j n_j \sigma_j \in \mathrm{Im}\ i$. Therefore $\mathrm{Ker}\ \beta = \mathrm{Im}\ i$.

Let $s = \sum_j \sum_{i=0}^{k-1} n_{ji} g^i \sigma_j \in \mathrm{Ker}\ \alpha$. Since $\alpha s = \sum_j (n_{j0} + \cdots + n_{j(k-1)})(1 + \cdots + g^{k-1})\sigma_j = 0$, $n_{j0} + \cdots + n_{j(k-1)} = 0$.

Thus $s = \sum_j (n_{j0}(1-g) + (n_{j0} + n_{j1})g(1-g) + (n_{j0} + n_{j1} + n_{j2})g^2(1-g) + \cdots + (n_{j0} + n_{j1} + \cdots + n_{j(k-2)})g^{k-2}(1-g))\sigma_j \in \mathrm{Im}\ j$. Therefore $\mathrm{Ker}\ \alpha = \mathrm{Im}\ j$. □

Let $H_q^\alpha(X, \mathbb{Z}/k\mathbb{Z})$ (resp. $H_q^\beta(X, \mathbb{Z}/k\mathbb{Z})$) denote the homology group induced from the chain complex $\{\alpha C_q(X;\mathbb{Z}/k\mathbb{Z})\}$ (resp. $\{\beta C_q(X;\mathbb{Z}/k\mathbb{Z})\}$).

By Proposition 2.1, we have the following theorem.

Theorem 2.2. *The following two sequences are exact.*

$$\cdots \to H_q^\alpha(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{i_*} H_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\beta_*} H_q^\beta(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial_*} H_{q-1}^\alpha(X;\mathbb{Z}/k\mathbb{Z}) \to \dots$$

$$\cdots \to H_q^\beta(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{j_*} H_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha_*} H_q^\alpha(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial'_*} H_{q-1}^\beta(X;\mathbb{Z}/k\mathbb{Z}) \to \dots\ .$$

In particular, if $p = 2$, then $\alpha = \beta$ and

$$\cdots \to H_q^\alpha(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{i_*} H_q(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\alpha_*} H_q^\alpha(X;\mathbb{Z}/k\mathbb{Z}) \xrightarrow{\partial_*} H_{q-1}^\alpha(X;\mathbb{Z}/k\mathbb{Z}) \to \dots$$

is exact.

Definable fiber bundles are introduced in [8].

Proposition 2.3. *Let X be a definable set with a free definable C_k-action. Then $(X, \pi, X/C_k, C_k)$ is a principal definable C_k-fiber bundle, where $\pi : X \to X/C_k$ denotes the orbit map. In particular $\pi : X \to X/C_k$ is a definable covering map.*

Let $p : E \to X$ be a definable map. We say that p has *the definable homotopy lifting property* if for any definable set Y, each definable homotopy $h : Y \times [0,1] \to X$ and a definable map $F : Y \to E$ such that $p \circ F(y) = h(y, 0)$ for all $y \in Y$, there exists a definable homotopy $H : Y \times [0,1] \to E$ such that $p \circ H = h$ and $H(y, 0) = F(y)$ for all $y \in Y$.

Theorem 2.4 (4.10 [1]). *Every definable covering map has the definable homotopy lifting property.*

Corollary 2.5. *Let X be a definable set with a free definable C_k-action. Then the orbit map $\pi : X \to X/C_k$ has the definable homotopy lifting property.*

Proposition 2.6. *Under the assumptions in Theorem 1.3, for every q, $H^\alpha_q(Y, \mathbb{Z}/k\mathbb{Z}) \cong H_q(Y/C_k, \mathbb{Z}/k\mathbb{Z})$.*

Proof. We first show that the map $\alpha : C(Y;\mathbb{Z}/k\mathbb{Z}) \to C(Y;\mathbb{Z}/k\mathbb{Z})$ and the map $\pi_* : C(Y;\mathbb{Z}/k\mathbb{Z}) \to C(Y/C_k;\mathbb{Z}/k\mathbb{Z})$ induced from the orbit map $\pi : Y \to Y/C_k$ have the same kernel. Let σ be a singular s-simplex of Y. We need only to consider elements of $C(C_k\sigma)$, since $C(Y) \cong \oplus_{[\sigma]\in\Delta(s)/C_k} C(C_k\sigma)$, where $\Delta(s)$ is the set of singular s-simplexes of Y and $\Delta(s)/C_k$ is its orbit set under the induced action.

Since $\alpha(\sum n_i g^i\sigma) = (\sum n_i)\alpha(\sigma)$, $\alpha(\sum n_i g^i \sigma) = 0$ if and only if $\sum n_i = 0$, and similarly $\pi_*(\sum n_i g^i\sigma) = (\sum n_i)\pi\circ\sigma = 0$ if and only if $\sum n_i = 0$; therefore, both kernels coincide.

We next show that π_* is surjective; namely, there is a definable lift $\tilde{\tau} : \Delta^s \to Y$ of $\tau : \Delta^s \to Y/C_k$, where Δ^s denotes the affine span of $(s+1)$-points which are affine independent. Since Δ^s is definably contractible, there is a definable homotopy $H' : \Delta^s \times [0,1] \to \Delta^s$ such that $H'(-,0) = c_{e_0}$ and $H'(-,1) = id_{\Delta^s}$, where c_{e_0} denotes the constant map whose value is $e_0 \in \Delta^s$. Then the composition $H = \tau \circ H'$ is a definable homotopy from the constant map $c_{\tau(e_0)}$ to τ. Let y_0 be a point of Y such that $\pi(y_0) = \tau(e_0)$, and $c_{y_0} : \Delta^s \to Y$ the constant map whose value is y_0. Since $H(-,0) = \pi \circ c_{y_0}$, it follows from Corollary 2.5 that there exists a definable lift $\tilde{H} : \Delta^s \times [0,1] \to Y$ of H such that $\tilde{H}(-,0) = c_{y_0}$. Then $\tilde{\tau} := \tilde{H}(-,1)$ is a definable lift of $\tau = H(-,1)$.

Since π_* is surjective, $\alpha C(Y;\mathbb{Z}/k\mathbb{Z})$ and $C(Y/C_p;\mathbb{Z}/k\mathbb{Z})$ are isomorphic as chain complexes. Accordingly their homology groups are also isomorphic. □

The topological version of Proposition 2.6 is studied in 5.33 [9].

Proof of Theorem 1.3. Assume that there exists a definable C_k-map $f : X \to Y$ under the conditions of Theorem 1.3. Since X is definably connected, $f(X)$ is definable connected. Hence $f(X)$ is contained in a definably connected component of Y. Therefore it is sufficient to prove the case where Y is definably connected.

We first prove the case where $k = 2$. Since f is a definable C_2-map, $\alpha f_\sharp = f_\sharp \alpha$.

For simplicity, we abbreviate the coefficient $\mathbb{Z}/2\mathbb{Z}$ in the definable homology. By Theorem 2.2, we have a commutative diagram

$$\begin{array}{cccccccccccc}
\to & H^\alpha_{n+1}(X) & \overset{\partial^X_*}{\to} & H^\alpha_n(X) & \overset{i^X_*}{\to} & H_n(X) & \overset{\alpha^X_*}{\to} & H^\alpha_n(X) & \overset{\partial^X_*}{\to} & H^\alpha_{n-1}(X) & \to & \dots \\
 & f^\alpha_* \downarrow & & f^\alpha_* \downarrow & & f_* \downarrow & & f^\alpha_* \downarrow & & f^\alpha_* \downarrow & & \\
\to & H^\alpha_{n+1}(Y) & \overset{\partial^Y_*}{\to} & H^\alpha_n(Y) & \overset{i^Y_*}{\to} & H_n(Y) & \overset{\alpha^Y_*}{\to} & H^\beta_n(Y) & \overset{\partial^Y_*}{\to} & H^\alpha_{n-1}(Y) & \to & \dots
\end{array}$$

$$\begin{array}{cccccccccccccc}
\to & H^\alpha_1(X) & \overset{i^X_*}{\to} & H_1(X) & \overset{\alpha^X_*}{\to} & H^\alpha_1(X) & \overset{\partial^X_*}{\to} & H^\alpha_0(X) & \overset{i^X_*}{\to} & H_0(X) & \overset{\alpha^X_*}{\to} & H^\alpha_0(X) & \to & 0 \\
 & f^\alpha_* \downarrow & & f_* \downarrow & & f^\alpha_* \downarrow & & f^\alpha_* \downarrow & & f_* \downarrow & & f^\alpha_* \downarrow & & f_* \downarrow \\
\to & H^\alpha_1(Y) & \overset{i^Y_*}{\to} & H_1(Y) & \overset{\alpha^Y_*}{\to} & H^\alpha_1(Y) & \overset{\partial^Y_*}{\to} & H^\alpha_0(Y) & \overset{i^Y_*}{\to} & H_0(Y) & \overset{\alpha^Y_*}{\to} & H^\alpha_0(Y) & \to & 0
\end{array}$$

with exact rows.

By definition, $(i^X_*)_0 = 0$ and $(i^Y_*)_0 = 0$. Thus $(\alpha^X_*)_0 : H_0(X) \to H^\alpha_0(X)$ and $(\alpha^Y_*)_0 : H_0(Y) \to H^\alpha_0(Y)$ are isomorphisms. By assumption, $H_0(X) \cong \mathbb{Z}/2\mathbb{Z}$. Hence $H_0(X) \cong$

$H_0^\alpha(X) \cong \mathbb{Z}/2\mathbb{Z}$. Similarly, $H_0(Y) \cong H_0^\alpha(Y) \cong \mathbb{Z}/2\mathbb{Z}$. Since $(f_*)_0 : H_0(X) \to H_0(Y)$ is an isomorphism and $(\alpha_*^Y)_0 \circ (f_*)_0 = (f_*^\alpha)_0 \circ (\alpha_*^X)_0$, $(f_*^\alpha)_0 : H_0^\alpha(X) \to H_0^\alpha(Y)$ is an isomorphism. Since $(i_*^X)_0 = 0$, we have $\mathrm{Im}\,(\partial_*^X)_1 = \mathrm{Ker}\,(i_*^X)_0 = H_0^\alpha(X)$. Thus we see that $(\partial_*^Y)_1 \circ (f_*^\alpha)_1 = (f_*^\alpha)_0 \circ (\partial_*^X)_1 : H_1^\alpha(X) \to H_0^\alpha(Y)$ is a non-zero homomorphism. Hence $(f_*^\alpha)_1 : H_1^\alpha(X) \to H_1^\alpha(Y)$ is a non-zero homomorphism. Using the assumptions on X, we see that $(\partial_*^X)_q : H_q^\alpha(X) \to H_{q-1}^\alpha(X)$ is an isomorphism for each $1 \le q \le n$. Using this fact and by induction, we have the claim that $(f_*^\alpha)_q : H_q^\alpha(X) \to H_q^\alpha(Y)$ is a non-zero homomorphism for each $0 \le q \le n$.

By Proposition 2.6, $H_{n+1}^\alpha(Y) \cong H_{n+1}(Y/C_p)$. Thus $H_{n+1}^\alpha(Y) = 0$. Hence $(i_*^Y)_n : H_n^\alpha(Y) \to H_n(Y)$ is injective and $(i_*^Y)_n \circ (f_*^\alpha)_n : H_n^\alpha(X) \to H_n(Y)$ is a non-zero homomorphism.

On the other hand, since $H_n(X) = 0$, $(i_*^Y)_n \circ (f_*^\alpha)_n = (f_*)_n \circ (i_*^X)_n = 0$. This contradiction proves the theorem in this case.

Next we prove the case where $k > 2$. For simplicity, we abbreviate the coefficient $\mathbb{Z}/k\mathbb{Z}$ in the definable homology. By Theorem 2.2, we have two commutative diagrams

$$
\begin{array}{ccccccccccc}
\to & H_n^\alpha(X) & \xrightarrow{i_*^X} & H_n(X) & \xrightarrow{\beta_*^X} & H_n^\beta(X) & \xrightarrow{\partial_*^X} & H_{n-1}^\alpha(X) & \to & \dots \\
 & f_*^\alpha \downarrow & & f_* \downarrow & & f_*^\beta \downarrow & & f_*^\alpha \downarrow & & \\
\to & H_n^\alpha(Y) & \xrightarrow{i_*^Y} & H_n(Y) & \xrightarrow{\beta_*^Y} & H_n^\beta(Y) & \xrightarrow{\partial_*^Y} & H_{n-1}^\alpha(Y) & \to & \dots
\end{array}
$$

$$
\begin{array}{cccccccccccccc}
\to & H_1^\alpha(X) & \xrightarrow{i_*^X} & H_1(X) & \xrightarrow{\beta_*^X} & H_1^\beta(X) & \xrightarrow{\partial_*^X} & H_0^\alpha(X) & \xrightarrow{i_*^X} & H_0(X) & \xrightarrow{\beta_*^X} & H_0^\beta(X) & \to & 0 \\
 & f_*^\alpha \downarrow & & f_* \downarrow & & f_*^\beta \downarrow & & f_*^\alpha \downarrow & & f_* \downarrow & & f_*^\beta \downarrow & & f_* \downarrow \\
\to & H_1^\alpha(Y) & \xrightarrow{i_*^Y} & H_1(Y) & \xrightarrow{\beta_*^Y} & H_1^\beta(Y) & \xrightarrow{\partial_*^Y} & H_0^\alpha(Y) & \xrightarrow{i_*^Y} & H_0(Y) & \xrightarrow{\beta_*^Y} & H_0^\beta(Y) & \to & 0
\end{array}
$$

and

$$
\begin{array}{ccccccccccccc}
\to & H_{n+1}^\alpha(X) & \xrightarrow{\partial_*'^X} & H_n^\beta(X) & \xrightarrow{j_*^X} & H_n(X) & \xrightarrow{\alpha_*^X} & H_n^\alpha(X) & \xrightarrow{\partial_*'^X} & H_{n-1}^\beta(X) & \to & \dots \\
 & f_*^\alpha \downarrow & & f_*^\beta \downarrow & & f_* \downarrow & & f_*^\alpha \downarrow & & f_*^\beta \downarrow & & \\
\to & H_{n+1}^\alpha(Y) & \xrightarrow{\partial_*'^Y} & H_n^\beta(Y) & \xrightarrow{j_*^Y} & H_n(Y) & \xrightarrow{\alpha_*^Y} & H_n^\alpha(Y) & \xrightarrow{\partial_*'^Y} & H_{n-1}^\beta(Y) & \to & \dots
\end{array}
$$

$$
\begin{array}{cccccccccccccc}
\to & H_1^\beta(X) & \xrightarrow{j_*^X} & H_1(X) & \xrightarrow{\alpha_*^X} & H_1^\alpha(X) & \xrightarrow{\partial_*'^X} & H_0^\beta(X) & \xrightarrow{j_*^X} & H_0(X) & \xrightarrow{\alpha_*^X} & H_0^\alpha(X) & \to & 0 \\
 & f_*^\beta \downarrow & & f_* \downarrow & & f_*^\alpha \downarrow & & f_*^\beta \downarrow & & f_* \downarrow & & f_*^\alpha \downarrow & & f_* \downarrow \\
\to & H_1^\beta(Y) & \xrightarrow{j_*^Y} & H_1(Y) & \xrightarrow{\alpha_*^Y} & H_1^\alpha(Y) & \xrightarrow{\partial_*'^Y} & H_0^\beta(Y) & \xrightarrow{j_*^Y} & H_0(Y) & \xrightarrow{\alpha_*^Y} & H_0^\alpha(Y) & \to & 0
\end{array}
$$

with exact rows.

We easily see that $(i_*^X)_0 = 0$ and $(i_*^Y)_0 = 0$. Thus $(\beta_*^X)_0 : H_0(X) \to H_0^\beta(X)$ and $(\beta_*^Y)_0 : H_0(Y) \to H_0^\beta(Y)$ are isomorphisms. Since $(f_*)_0 : H_0(X) \to H_0(Y)$ is an isomorphism, we have the claim that $(f_*^\beta)_0 : H_0^\beta(X) \to H_0^\beta(Y)$ is an isomorphism. Similarly we see that $(f_*^\alpha)_0 : H_0^\alpha(X) \to H_0^\alpha(Y)$ is an isomorphism from the second diagram. Since $H_1(X) = 0$ and $(i_*^X)_0 = 0$, $(\partial_*^X)_1 : H_1^\beta(X) \to H_0^\alpha(X)$ is an isomorphism. Similarly $(\partial_*'^X)_1 : H_1^\alpha(X) \to H_0^\beta(X)$ is an isomorphism. Since $(\partial_*^Y)_1 \circ (f_*^\beta)_1 = (f_*^\alpha)_0 \circ (\partial_*^X)_1$ and $(\partial_*'^Y)_1 \circ (f_*^\alpha)_1 = (f_*^\beta)_0 \circ (\partial_*'^X)_1$, $(f_*^\alpha)_1 : H_1^\alpha(X) \to H_1^\alpha(Y)$ and $(f_*^\beta)_1 : H_1^\alpha(X) \to H_1^\beta(Y)$ are non-zero homomorphisms. By induction, we have the claim that $(f_*^\alpha)_q : H_q^\alpha(X) \to H_q^\alpha(Y)$ and $(f_*^\beta)_q : H_1^\alpha(X) \to H_q^\beta(Y)$ are non-zero homomorphism for each $0 \le q \le n$. By Proposition

2.6, $H^{\alpha}_{n+1}(Y) \cong H_{n+1}(Y/C_p)$. Hence $H^{\alpha}_{n+1}(Y/C_p) = 0$ and $(j^Y_*)_n : H^{\beta}_n(Y) \to H_n(Y)$ is injective. Therefore $(j^Y_*)_n \circ (f^{\beta}_*)_n$ is a non-zero homomorphism.

On the other hand, $(j^Y_*)_n \circ (f^{\beta}_*)_n = (f_*)_n \circ (j^X_*)_n = 0$ because $H_n(X) = 0$. This is a contradiction. Therefore the proof is complete. □

Proof of Theorem 1.5. Suppose that $f(x) \neq f(gx)$ for any $x \in X$. Then the map $F : X \to Y^*$ defined by $F(x) = (f(x), f(gx), \ldots, f(g^{k-1}x))$ is a definable C_k-map. This contradicts Theorem 1.3. □

Proof of Theorem 1.1 *and Theorem* 1.2. Similar proofs of Theorem 1.3 and Theorem 1.5 prove Theorem 1.1 and Theorem 1.2, respectively. □

References

[1] E. Baro and M. Otero, *On o-minimal homotopy groups*, preprint.
[2] Carlos Biasi and Denise de Mattos, *A Borsuk-Ulam theorem for compact Lie group actions*, Bull. Braz. Math. Soc. **37** (2006), 127–137.
[3] L. van den Dries, *Tame topology and o-minimal structures*, Lecture notes series **248**, London Math. Soc. Cambridge Univ. Press (1998).
[4] L. van den Dries, A. Macintyre and D. Marker, *The elementary theory of restricted analytic field with exponentiation*, Ann. of Math. **140** (1994), 183–205.
[5] L. van den Dries and C. Miller, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497-540.
[6] L. van den Dries and P. Speissegger, *The real field with convergent generalized power series*, Trans. Amer. Math. Soc. **350**, (1998), 4377–4421.
[7] Y. Hemmi, T. Kobayashi and T. Yoshida, *The Borsuk-Ulam theorem for a $\mathbb{Z}_q$-map from S^{2m+1} to a $\mathbb{Z}_p$-complex*, Mem. Fac. Sci. Kochi Univ. Ser. A Math. **8** (1987), 27-30.
[8] T. Kawakami, *Homotopy property of definable fiber bundles*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **53** (2003), 1–6.
[9] K. Kawakubo, *The theory of transformation groups*, Oxford Univ. Press, (1991).
[10] T. Kobayashi, *The Borsuk-Ulam theorem for a $\mathbb{Z}_q$-map from a $\mathbb{Z}_q$-space to S^{2n+1}*, Proc. Amer. Math. Soc. 97 (1986), 714–716.
[11] C. Miller, *Expansion of the field with power functions*, Ann. Pure Appl. Logic **68**, (1994), 79–94.
[12] I. Nagasaki, *Isovariant Borsuk-Ulam results for pseudofree circle actions and their converse*, Trans. Amer. Math. Soc. **358** (2006), 743–757
[13] I. Nagasaki, *The converse of isovariant Borsuk-Ulam results for some abelian groups*, Osaka J. Math. **43** (2006), 689–710.
[14] I. Nagasaki, *The weak isovariant Borsuk-Ulam theorem for compact Lie groups*, Arch. Math. (Basel) **81** (2003), 348–359.
[15] I. Nagasaki, T. Kawakami, Y. Hara and F. Ushitaki, *The Borsuk-Ulam theorem in a real closed field*, Far East J. Math. Sci (FJMS) **33** (2009), 113-124.
[16] I. Nagasaki and F. Ushitaki, *Isovariant maps from free C_n-manifolds to representation spheres*, Topology Appl. **155** (2008), 1066-1076.
[17] Pedro L. Q Pergher, Denise de Mattos and Edivaldo L. dos Santos, *The Borsuk-Ulam theorem for general spaces*, Arch. Math. (Basel) **81** (2003), 96–102.
[18] J.P. Rolin, P. Speissegger and A.J. Wilkie, *Quasianalytic Denjoy-Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), 751-777.
[19] M. Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Math. **150** (1997), Birkhäuser.
[20] J.W. Walker, *A homology version of the Borsuk-Ulam theorem*, Amer. Math. Monthly **90** (1983), 466–468.
[21] A. Worheide, *O-minimal homology*, Ph.D. thesis (1996), University of Illinois at Urbana-Champaign.

Department of Mathematics, Kyoto Prefectural University of Medicine, 13 Nishi-Takatsukaso-Cho, Taishogun Kita-ku, Kyoto 603-8334, Japan

Department of Mathematics, Faculty of Education, Wakayama University, Sakaedani Wakayama 640-8510, Japan

Department of Mathematics, Graduate School of Science, Osaka University, Machikaneyama 1-1, Toyonaka, Osaka 560-0043, Japan

Department of Mathematics, Faculty of Science, Kyoto Sangyo University, Kamigamo Motoyama, Kita-ku, Kyoto 603-8555, Japan

E-mail address: nagasaki@koto.kpu-m.ac.jp

E-mail address: kawa@center.wakayama-u.ac.jp

E-mail address: hara@math.sci.osaka-u.ac.jp

E-mail address: ushitaki@ksuvx0.kyoto-su.ac.jp