THE LATTICE COORDINATIZED BY A RING

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Abstract. A coordinatization functor $\mathbb{L}(-,1)\colon \mathbf{Ring} \to \mathbf{Latt}$ is defined from the category of rings to the category of modular lattices. The main features of this coordinatization functors are 1) that it extends the functor $R \mapsto \mathbb{L}(R)$ of von Neumann that associates to a regular ring its lattice of principal right ideals; 2) it respects the respective $(-)^{\mathrm{op}}$ endofunctors on **Ring** and **Latt**; and 3) it admits localization at a left R-module. The complemented elements of $\mathbb{L}(R,1)$ form a partially ordered set $\mathbb{S}(R)$ isomorphic to the space of direct summands of R_R.

The right nonsingular rings for which the embedding of $\mathbb{S}(R)$ into the localization $\mathbb{L}(R,1)_Q$ at the right maximal ring of quotients Q_R is an isomorphism are characterized by a property that every finite matrix subgroup $\varphi({}_RR)$ of the left R-module ${}_RR$ is essential in an element of $\mathbb{S}(R)$. In that case, the space $\mathbb{S}(R)$ obtains the structure of a complemented modular lattice coordinatized by the dominion, or equivalently, the ring of definable scalars, of the maximal ring of quotients. The class of rings with this property is elementary, in contrast to the class of rings whose space of right summands is coordinatized by the maximal ring of quotients.

Contents

1. Introduction

The theory of rings of operators [3, 14] has its origins in Von Neumann's analysis [17], [6, §5.1], of the functor $R \mapsto \mathbb{L}(R)$ that associates to a (von Neumann) regular ring the complemented modular lattice of its principal right ideals. A regular ring R *coordinatizes* the lattice $\mathbb{L}(R)$ in the sense of the Fundamental Theorem of Projective Geometry [13, §8.4], [1, Chapter 4], where $R = M_n(\Delta)$ is the regular ring of $n \times n$ matrices over a division ring Δ and $\mathbb{L}(R) \cong \mathbb{L}(\Delta, n)$, the complemented modular lattice of subspaces of an n-dimensional Δ-vector space.

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In this paper, we propose an extension $\mathbb{L}(-,1)\colon \mathbf{Ring} \to \mathbf{Latt}$ of the coordinatization functor $R \mapsto \mathbb{L}(R)$ to the entire category **Ring** of rings, taking values in the category **Latt** of (not necessarily complemented) modular lattices. The value of the functor at R is given by $\mathbb{L}(R,1)$, the lattice of positive primitive formulae in one free variable, modulo equivalence relative to the theory of left R-modules. *Positive primitive formulae* [18, Chapter 1] come from model theory, a branch of mathematical logic, but in the context of R-modules, they are equivalent to the p-functors of the form $M \mapsto [A,i]M$, A a finite matrix, used by Huisgen-Zimmermann and Zimmermann [21] in their study of purity. The lattice $\mathbb{L}(R,1)$ also arises in Auslander's theory [2] of coherent functors as the lattice of subobjects of the forgetful functor; Freyd [7] proved that the category of coherent functors is *the free abelian category* over the forgetful functor. The importance of the lattice $\mathbb{L}(R,1)$ is therefore established by a consensus from diverse points of view.

Our lattice theoretic treatment of the lattice $\mathbb{L}(R,1)$, which originates in the Model Theory of Modules [18], was suggested to us by Christian Herrmann. Its main features are:

Duality (§2.3): The Prest dual $\varphi \mapsto \varphi^{\otimes}$ defines an anti-isomorphism $(-)^{\otimes}\colon \mathbb{L}(R,1)^{\mathrm{op}} \to \mathbb{L}(R^{\mathrm{op}},1)$ that respects the respective op endofunctors of **Ring** and **Latt**;

Localization (§2.4): Associated to every module ${}_RM$, there is a congruence $\Theta_1(M)$ on the lattice $\mathbb{L}(R,1)$, whose quotient lattice $\mathbb{L}(R,1)_M = \mathbb{L}(R,1)/\Theta_1(M)$ is the *localization* of $\mathbb{L}(R,1)$ at ${}_RM$;

The Regular Case (§2.5): The lattice $\mathbb{L}(R,1)$ is complemented if and only if R is a regular ring, in which case it is isomorphic to the lattice $\mathbb{L}(R)$ of principal right ideals of R.

Guided by the work of Handelman [11] we focus on the case when R is right nonsingular, and consider the localization $\mathbb{L}(R,1)_Q$ at the (right) maximal R-ring $q\colon R \to Q$ of quotients. The third section of the paper is devoted to the model theory of the left R-module ${}_RQ$. Theorem 3.2 shows that every finite matrix subgroup $\varphi({}_RQ) = E(\varphi({}_RR))$ of ${}_RQ$ as the (unique) injective envelope in Q_R of the corresponding finite matrix subgroup $\varphi({}_RR)$ of ${}_RR$. This is then used to characterize when ${}_RQ$ is flat or FP-injective in terms of the definable structure of ${}_RR$. Another consequence is that the congruence $\Theta_1({}_RQ)$ may be described as $\varphi \equiv \psi \ \ (\Theta_1({}_RQ))$ if and only if the finite matrix subgroups $\varphi({}_RR)$ and $\psi({}_RR)$ have a common essential extension in Q_R.

Having extended the functor $R \mapsto \mathbb{L}(R,1)$ to the entire category of rings, we denote by $\mathbb{S}(R)$ the partially ordered space of right summands of R (i.e., summands of R_R). This may be regarded as a subfunctor of $\mathbb{S}(-) \subseteq \mathbb{L}(-,1)$, whose R-component is given by the map $eR \mapsto e|u$. In the spirit of von Neumann's original program, we look for conditions (cf. [3, §34]) sufficient for $\mathbb{S}(R)$ to be a modular lattice, and a regular ring that coordinatizes it. Consider the sequence of maps

$$\begin{array}{ccccccc} \mathbb{S}(R) & \longrightarrow & \mathbb{L}(R,1) & \longrightarrow & \mathbb{L}(R,1)_Q & \longrightarrow & \mathbb{S}(Q) \\ eR & \longmapsto & e|u;\ \varphi & \longmapsto & (\varphi)_Q & \longmapsto & \varphi(Q), \end{array}$$

whose composition is given by the embedding $eR \mapsto eQ$. The first map $\mathbb{S}(R) \to \mathbb{L}(R,1)$ is an isomorphism if and only if R is a von Neumann regular ring, in which case $\mathbb{S}(R) = \mathbb{L}(R,1)$ is, by definition, coordinatized by R. Because Q is regular, the space $\mathbb{S}(Q) = \mathbb{L}(Q,1)$ is a modular lattice, so if the composition $\mathbb{S}(R) \to \mathbb{S}(Q)$ is an isomorphism, then Q coordinatizes $\mathbb{S}(R)$, the case considered in [11]. The remaining case, when the embedding $\mathbb{S}(R) \to \mathbb{L}(R,1)_Q$ is an isomorphism, is one of the principal subjects of the paper.

The class of rings R for which the first map above $\mathbb{S}(R) \to \mathbb{L}(R,1)$ is an isomorphism is clearly elementary, for these rings are the regular rings axiomatized by the sentence $\forall x \exists y\,(xyx \doteq x)$ in the language $\mathcal{L}(\mathbf{Ring}) = (+,\cdot,-,0,1)$ of rings. The class of right nonsingular rings of infinite right Goldie dimension is also elementary, but the class of right nonsingular rings of infinite Goldie dimension whose space $\mathbb{S}(R)$ is coordinatized by the right maximal R-ring of quotients is not (Theorem 4.15). One of the main results of the paper is that the class of right nonsingular ring for which the composition $\mathbb{S}(R) \to \mathbb{L}(R,1) \to \mathbb{L}(R,1)_Q$ is an isomorphism, is an elementary class. This is equivalent to the condition that the dominion of $q\colon R \to Q$ is regular and coordinatizes the space $\mathbb{S}(R)$. The axioms for this class of rings are obtained by a formal translation into $\mathcal{L}(\mathbf{Ring})$ of the following characterization.

Corollary 3.4 *Let R be a right nonsingular ring with (right) maximal R-ring of quotients $q\colon R \to Q$. The embedding $\mathbb{S}(R) \to \mathbb{L}(R,1)_Q$ is an isomorphism if and only if for every $\varphi \in \mathbb{L}(R,1)$, there is an essential inclusion $\varphi({}_RR) \subseteq eR$ of right ideals with $eR \in \mathbb{S}(R)$.*

The condition of Corollary 3.4 is a restricted form of the *extending* property, introduced by Chatters and Hajarnavis [4], on ${}_RR$, considered as a module over its endomorphism ring; this property is also known as that *CS* condition [15].

The last section of the paper is devoted to a treatment of the lattice $\mathbb{L}(R,2)$ of definable relations. It carries the structure of a relational algebra which contains the ring R as the subalgebra of definable scalars. The localization $\mathbb{L}(R,2) \to \mathbb{L}(R,2)_N$ at a module N contains the R-ring $\delta_N\colon R \to R_N$ of N-definable scalars [18, Chapter 6] as a subalgebra. We regard the process of extracting R_N from N as a kind of decoordinatization, in view of the canonical map $\mathbb{L}(R_N,1) \to \mathbb{L}(R,1)_N$ that arises. If $\mathbb{L}(R,1)_N$ is complemented, as in the case when R is right nonsingular and the map $\mathbb{S}(R) \to \mathbb{L}(R,1)_Q$ is an isomorphism, then the ring R_N of N-definable scalars is regular and the canonical map is an isomorphism: R_N coordinatizes the localization $\mathbb{L}(R,1)_N$.

Throughout the paper, R will denote an associative ring with identity $1 \in R$. An R-ring is a morphism of rings $f\colon R \to S$, and if ${}_RM$ is a left R-module, then $\mathrm{Latt}({}_\mathrm{R}\mathrm{M})$ will denote the lattice of R-submodules of ${}_RM$.

2. Coordinatization

Let R be an associative ring R with identity $1 \in R$. The language for left R-modules is $\mathcal{L}(R\text{-Mod}) = (+,-,0) \cup R$, where $(+,-,0) = \mathcal{L}(\mathrm{Ab})$ is the language for abelian groups and R is the ring of unary function symbols; if $r \in R$ the corresponding unary function symbol is denoted by the same. The nonlogical symbols of the language are precisely those needed to express a linear equation

$$r_1u_1 + r_2u_2 + \cdots + r_nu_n \doteq 0.$$

The classical axioms for a left R-module are expressible in the language $\mathcal{L}(R)$; the collection of consequences of these axioms is the theory $\mathrm{Th}(R\text{-Mod})$ of left R-modules.

2.1. **Positive primitive formulae.** A conjunction, or *system,* of linear equations is denoted as in **linear algebra:** if $A = (a_{ij})$ is an $m \times n$ matrix with entries in R, then

$$A\mathbf{u}^t \doteq 0 := \bigwedge_{i=1}^{m} \sum_{j=1}^{n} a_{ij}u_i \doteq 0,$$

where $\mathbf{u} = (u_1, \ldots, u_n)$ is the n-tuple of free variables of the formula. By a *positive primitive* (pp-) formula in $\mathcal{L}(R)$, we mean any formula that is equivalent, relative to the theory Th(R-Mod), to an existentially quantified system of linear equations

$$\varphi(\mathbf{u}) \;:=\; \exists \mathbf{v}\ (A\mathbf{u}^t + B\mathbf{v}^t \doteq 0) \;=\; \exists \mathbf{v}\ (A, B) \begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{pmatrix} \doteq 0. \tag{1}$$

Precisely, a pp-formula $\varphi(u_1, \ldots, u_n)$ in the free variables $\mathbf{u} = (u_1, \ldots, u_n)$ is given, for some m and k, by an $m \times n$ matrix A and an $m \times k$ matrix B; the k-tuple of existentially bound variables is seen to be $\mathbf{v} = (v_1, \ldots, v_k)$.

Two positive primitive formulae $\varphi(\mathbf{u})$ and $\psi(\mathbf{u})$ are identified if they are equivalent, relative to the theory of left R-modules,

$$R\text{-Mod} \models \forall \mathbf{u}\ (\varphi(\mathbf{u}) \leftrightarrow \psi(\mathbf{u})).$$

By the Completeness Theorem, this may be phrased in terms of pp-definable, or finite matrix, subgroups. If $M \in R$-Mod is a left R-module, then the positive primitive formula $\varphi(\mathbf{u})$ shown in (1) defines in M a subgroup

$$\varphi(M) = \{\mathbf{a} \in M^n \mid \text{ there exists a } \mathbf{b} \in M^k \text{ such that } A\mathbf{a}^t + B\mathbf{b}^t = 0 \text{ in } {}_RM\} \tag{2}$$

of M^n, called the *subgroup of* M^n*, pp-definable in* ${}_RM$ by $\varphi(\mathbf{u})$; alternatively, it is also called the *finite matrix subgroup* of M^n defined by $\varphi(\mathbf{u})$ in ${}_RM$. Then two positive primitive formulae $\varphi(\mathbf{u})$ and $\psi(\mathbf{u})$ are identified if $\varphi(M) = \psi(M)$, for every left R-module ${}_RM$. Indeed, the assignment $M \mapsto \varphi(M)$ defines a functor $\varphi(-) : R\text{-Mod} \to \text{Ab}$, the category of abelian groups, which is a subfunctor of the n-th power of the forgetful functor. The foregoing considerations indicate that two pp-formulae are identified if and only if they define the same functor.

If A is an $m \times n$ matrix with entries in R, and ${}_RM$ is a left R-module, then multiplication on the left by A yields a short exact sequence of abelian groups,

$$0 \longrightarrow \sigma(M) \longrightarrow M^n \xrightarrow{L_A} M^m \longrightarrow M^m/\tau(M) \longrightarrow 0,$$

where $\sigma(\mathbf{u}) = A\mathbf{u}^t \doteq 0$ is the system of linear equations with matrix of coefficients A, and $\tau(\mathbf{v})$ is the pp-formula in the m free variables $\mathbf{v} = (v_1, v_2, \ldots, v_m)$ given by $\exists \mathbf{u}\ (A\mathbf{u}^t \doteq \mathbf{v}^t)$. This latter formula may be abbreviated to $A \,|\, \mathbf{v}^t$, so that the general positive primitive formula (1) is abbreviated as $B \,|\, A\mathbf{u}^t$.

Example 2.1. *Let $r \in R$. The pp-formulae $ru \doteq 0$ and $r|u = \exists v\ (rv \doteq u)$ have 1 free variable; they define in a left R-module ${}_RM$ the pp-definable subgroups* $\text{ann}_M(r)$ *and rM, respectively. More generally, if $I = \sum_i r_iR$ is a finitely generated right ideal, then we denote by $I|u$ the formula $\sum_i r_i|u$, which defines in ${}_RM$ the subgroup $IM = \sum_i r_iM$, and if $J = \sum_j Rs_j$ is a finitely generated* left *ideal, then $Ju \doteq 0$ denotes the formula $\bigwedge_j s_ju \doteq 0$, which defines in ${}_RN$ the annihilator* $\text{ann}_N(J)$.

Example 2.2. [18, §2.3.4] *If R is a regular ring, then every positive primitive formula* (1) *is equivalent to a system of linear equations. By* [8, Lemma 1.6], *every matrix over R is regular: there is a $k \times m$ matrix C such that $BCB = B$, and we have that*

$$R\text{-Mod} \models B \,|\, A\mathbf{u}^t \to BCB \,|\, A\mathbf{u}^t \to BC \,|\, A\mathbf{u}^t \to B \,|\, A\mathbf{u}^t.$$

The given positive primitive formula $B \,|\, A\mathbf{u}^t$ is thus equivalent to $BC \,|\, A\mathbf{u}^t$, where BC is an $m \times m$ idempotent matrix. It follows that $R\text{-Mod} \models BC \,|\, A\mathbf{u}^t \leftrightarrow ((I_m - BC)A\mathbf{u}^t \doteq 0)$.

2.2. **Definition of the functor.** The positive primitive formulae $\varphi(u_1, \ldots, u_n)$ in the free variables $\mathbf{u} = (u_1, \ldots, u_n)$ form a bounded modular lattice $(\mathbb{L}(R, \mathbf{u}), \wedge, +, 0, 1)$. The infimum operation is given by conjunction $(\varphi \wedge \psi)(\mathbf{u}) := \varphi(\mathbf{u}) \wedge \psi(\mathbf{u})$; the supremum by $(\varphi + \psi)(\mathbf{u}) := \exists \mathbf{x}, \mathbf{y}\ (\varphi(\mathbf{x}) \wedge \psi(\mathbf{y}) \wedge (\mathbf{u} \doteq \mathbf{x} + \mathbf{y}))$. The supremum can be expressed more tersely, $R\text{-Mod} \models (\varphi + \psi)(\mathbf{u}) \leftrightarrow \exists \mathbf{x}\ (\varphi(\mathbf{x}) \wedge \psi(\mathbf{u} - \mathbf{x}))$, but if the matrices are made explicit, $\psi(\mathbf{u}) = \exists \mathbf{w}\ (C, D) \begin{pmatrix} \mathbf{u}^t \\ \mathbf{w}^t \end{pmatrix} \doteq 0$, then

$$(\varphi \wedge \psi)(\mathbf{u}) = \exists \mathbf{v}, \mathbf{w} \begin{pmatrix} A & B & 0 \\ C & 0 & D \end{pmatrix} \begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \\ \mathbf{w}^t \end{pmatrix} \doteq 0 \text{ and}$$

$$(\varphi + \psi)(\mathbf{u}) = \exists \mathbf{x}, \mathbf{y}, \mathbf{v}, \mathbf{w} \begin{pmatrix} I_n & -I_n & -I_n & 0 & 0 \\ 0 & A & 0 & B & 0 \\ 0 & 0 & C & 0 & D \end{pmatrix} \begin{pmatrix} \mathbf{u}^t \\ \mathbf{x}^t \\ \mathbf{y}^t \\ \mathbf{v}^t \\ \mathbf{w}^t \end{pmatrix} \doteq 0.$$

The bottom element 0 of the lattice is the formula $I_n\mathbf{u}^t \doteq 0$; the top element 1 is given by $I_n \,|\, \mathbf{u}^t$. Only the arity n of the tuple $\mathbf{u} = (u_1, \ldots, u_n)$ is material, so we will set $\mathbb{L}(R, n) := \mathbb{L}(R, \mathbf{u})$.

A morphism $f : R \to S$ of rings induces a map $\mathcal{L}(f) : \mathcal{L}(R) \to \mathcal{L}(S)$ of languages that restricts to a morphism

$$\mathbb{L}(f, n) : \mathbb{L}(R, n) \to \mathbb{L}(S, n),\ \ \exists \mathbf{v}\ (A, B) \begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{pmatrix} \doteq 0 \mapsto \exists \mathbf{v}\ (f(A), f(B)) \begin{pmatrix} \mathbf{u}^t \\ \mathbf{v}^t \end{pmatrix} \doteq 0, \tag{3}$$

of bounded lattices; if $A = (a_{ij})$ is a matrix over R, then $f(A) = (f(a_{ij}))$ is a matrix over S. In practice, we abbreviate this map by $\mathbb{L}(f, n) \colon \varphi \mapsto \varphi^f$.

Example 2.3. *If $R = \Delta$ is a division ring, then $\mathbb{L}(\Delta, n) \cong V_n(\Delta)$, the lattice of subspaces of an n-dimensional right vector space over Δ, via the map $\varphi(\mathbf{u}) \mapsto \varphi({}_\Delta\Delta)$. The map is injective, because every left vector space over Δ is a direct sum of the 1-dimensional one ${}_\Delta\Delta$, so that that two positive primitive formulae $\varphi(\mathbf{u})$ and $\psi(\mathbf{u})$ in $\mathbb{L}(R, n)$ are equivalent if and only if $\varphi(\Delta) = \psi(\Delta)$. On the other hand, Example 2.2 implies that every positive primitive formula is given by a system of linear equations with coefficients acting on the left. Every subspace of $(\Delta_\Delta)^n$ is therefore pp-definable, and the map is onto. The Fundamental Theorem of Projective Geometry* [13, §8.4] *states that if Δ and Δ' are division rings and $\mathbb{L}(\Delta, n) \cong \mathbb{L}(\Delta', n')$, with $n \geq 3$, then $\Delta \cong \Delta'$ and $n = n'$.*

Proposition 2.4. *There is a natural isomorphism* $\mathbb{L}(R,n) \cong \mathbb{L}(M_n(R),1)$, *of functors from* **Ring** *to* **Latt**, *where* $M_n(R)$ *is the ring of* $n \times n$ *matrices over* R.

Proof. An isomorphism of lattices $\mathbb{L}(R,n) \to \mathbb{L}(M_n(R),1)$ is defined in the following natural way. Given a positive primitive formula (1) in the language $\mathcal{L}(R\text{-Mod})$ of left R-modules, we may add some zero rows to the matrix (A,B) and some zero columns to B without changing its equivalence class in $\mathbb{L}(R,n)$. By doing so, we can assume, without loss of generality, that both m and k are multiples of n, say $m = an$ and $k = bn$, and then rewrite $A = \begin{pmatrix} A_1 \\ \vdots \\ A_a \end{pmatrix}$ as a column of $n \times n$ matrices, and $B = \begin{pmatrix} B_{11} & \cdots & B_{1b} \\ \vdots & \vdots & \vdots \\ B_{a1} & \cdots & B_{ab} \end{pmatrix}$ as an $a \times b$ matrix of $n \times n$ matrices. Associate to $\varphi(\mathbf{u})$ the positive primitive formula

$$\varphi_n(u) := \exists \mathbf{w} \bigwedge_{i=1}^{a} A_i u + B_{i1} w_1 + \cdots B_{ib} w_b \doteq 0$$

in one free variable from the language of modules over $M_n(R)$, and verify that the rule $\varphi(\mathbf{u}) \mapsto \varphi_n(u)$ is indeed a natural isomorphism of lattices. □

Definition 2.5. *The functor* $\mathbb{L}(-,1)\colon$ **Ring** $\to$ **Latt** *is called the* coordinatization functor; *the ring* R *is said to* coordinatize *the bounded modular lattice* $\mathbb{L}(R,1)$.

More generally, we say that R *coordinatizes* any bounded modular lattice isomorphic to $\mathbb{L}(R,1)$; Proposition 2.4 implies that the lattice $\mathbb{L}(R,n)$ is coordinatized by the ring $M_n(R)$ of $n \times n$ matrices over R.

Definition 2.6. *For any ring* R, *let* $\mathbb{L}(R)$ *be the upper semi-lattice of finitely generated right ideals of* R, *and let* $\mathbb{S}(R) \subseteq \mathbb{L}(R)$ *denote the partially ordered subset of right summands of* R, *that is, summands of* R_R. *Both of these definitions are natural, and so define functors* $R \mapsto \mathbb{S}(R)$ *and* $R \mapsto \mathbb{R}(R)$ *from the category* **Ring** *of rings to that of partial orders and upper semi-lattices, respectively,* $\mathbb{S}(-) \subseteq \mathbb{L}(-)$.

If $I \in \mathbb{S}(R)$, then $I = eR$ for some idempotent $e \in R$, which gives an anti-isomorphism $\mathbb{S}(R)^{\mathrm{op}} \to \mathbb{S}(R^{\mathrm{op}})$, $I \mapsto \mathrm{l.ann}_R(I) = R(1-e)$. This shows that the functor $\mathbb{S}$ respects the $(-)^{\mathrm{op}}$ endofunctors of the category **Ring** and that of partially ordered sets. Recall [10, §I.6] that a *complement* of φ in $\mathbb{L}(R,1)$ is an element φ' satisfying $\varphi \wedge \varphi' = 0$ and $\varphi + \varphi' = 1$, abbreviated by $\varphi \oplus \varphi' = 1$.

Proposition 2.7. *The map* $\mathbb{L}(R) \to \mathbb{L}(R,1)$, $I \mapsto I|u$, *is an embedding of bounded partial orders, as is the map* $\mathbb{L}(R^{\mathrm{op}})^{\mathrm{op}} \to \mathbb{L}(R,1)$ *given by* $J \mapsto Ju \doteq 0$. *The pullback of these embeddings is* $\mathbb{S}(R)$,

$$\begin{array}{ccc} \mathbb{S}(R) & \longrightarrow & \mathbb{L}(R) \\ {\scriptstyle (\mathrm{l.ann}(-))^{\mathrm{op}}}\downarrow & \lrcorner & \downarrow{\scriptstyle I|u} \\ \mathbb{L}(R^{\mathrm{op}})^{\mathrm{op}} & \xrightarrow{Ju \doteq 0} & \mathbb{L}(R,1), \end{array} \tag{4}$$

whose image in $\mathbb{L}(R,1)$ *is the space of complemented elements.*

Proof. The map $I \mapsto I|u$ is well-defined, because if $I_1 \subseteq I_2$ in $\mathbb{L}(R)$, then for every left R-module $I_1 M \subseteq I_2 M$, whence $I_1|u \leq I_2|u$ in $\mathbb{L}(R,1)$. This map is injective, because if $\varphi(u) = I|u$, then $\varphi({}_R R) = IR = I$. Similarly, if $J_1 \subseteq J_2 = \sum_j Rs_j$ then $\mathrm{ann}_M(J_2) \subseteq \mathrm{ann}_M(J_1)$. To see that the map is injective, take $M = R/J_2$; the inclusion $\mathrm{ann}_M(J_2) \subseteq \mathrm{ann}_M(J_1)$ implies that $J_1 \subseteq J_2$.

Next, we verify that the image of $\mathbb{S}(R)$ consists of the complemented elements of $\mathbb{L}(R,1)$. If a right ideal I_R is a direct summand of R_R, then $I = eR$, where $e \in R$ is idempotent, and $e|u = eR|u = (1-e)u \doteq 0 = R(1-e)u \doteq 0$. Indeed, we have that for every module ${}_RM$, $eM \oplus (1-e)M$, whence $e|u \oplus (1-e)|u = 1$ in $\mathbb{L}(R,1)$. Conversely, if $\varphi \oplus \varphi' = 1$ in $\mathbb{L}(R,1)$, then, in particular, $\varphi({}_RR) \oplus \varphi'({}_RR) = {}_RR$ may be viewed as a decomposition of the right R-module R_R, whence $\varphi(R) = eR$ and $\varphi'(R) = (1-e)R$ for some idempotent $e \in R$. This implies that $e|u \leq \varphi$ in $\mathbb{L}(R,1)$, as well as $(1-e)|u \leq \varphi'$, which must therefore both be equalities.

To verify the pullback property of $\mathbb{S}(R)$, suppose that $I|u = Ju \doteq 0$, with $I = \sum_{i=0}^n r_iR$. Using the notion of free realization [18, p. 23], we get that the map $R/J \to R^n$, $1 + J \mapsto (r_1, \dots, r_n)$, has a left inverse. This implies that R/J is a projective left R-module, and that $J \in \mathbb{S}(R^{\mathrm{op}})$, and $I = \mathrm{r.ann}_R(J)$. $\square$

The pushout of the diagram of Proposition 2.7 induces a map $\mathbb{L}(R^{\mathrm{op}})^{\mathrm{op}} \otimes_{\mathbb{S}(R)} \mathbb{L}(R) \to \mathbb{L}(R,1)$ with the property that $I|u \leq Ju \doteq 0$ in $\mathbb{L}(R,1)$ whenever $J \subseteq \mathrm{l.ann}_R\, I$.

2.3. **Prest duality.** The *Prest dual* of a positive primitive formula $\varphi \in \mathbb{L}(R,n)$, as displayed in (1) is the positive primitive formula

$$\varphi^{\otimes}(\mathbf{u}) = \exists \mathbf{w}\ (\mathbf{u}, \mathbf{w}) \begin{pmatrix} I_n & 0 \\ A & B \end{pmatrix} \doteq 0 \tag{5}$$

in the language $\mathcal{L}(R^{\mathrm{op}})$ of right R-modules. Prest [18, §1.3] proved that for every $n > 0$, the Prest dual $\varphi \mapsto \varphi^{\otimes}$ is a well-defined natural anti-isomorphism $(-)^{\otimes}_n : \mathbb{L}(R,n)^{\mathrm{op}} \to \mathbb{L}(R^{\mathrm{op}}, n)$. The inverse of the Prest dual of a positive primitive formula in the language of right R-modules is defined similarly. The definition is natural, in the sense that if $f : R \to S$ is a morphism of rings, then the diagram

$$\begin{array}{ccc} \mathbb{L}(R,n)^{\mathrm{op}} & \xrightarrow{(-)^{\otimes}_n} & \mathbb{L}(R^{\mathrm{op}},n) \\ {\scriptstyle \mathbb{L}(f,n)^{\mathrm{op}}}\downarrow & & \downarrow{\scriptstyle \mathbb{L}(f^{\mathrm{op}},n)} \\ \mathbb{L}(S,n)^{\mathrm{op}} & \xrightarrow{(-)^{\otimes}_n} & \mathbb{L}(S^{\mathrm{op}},n) \end{array} \tag{6}$$

of modular lattices commutes. The coordinatization functor $\mathbb{L}(-,1)\colon (\mathbf{Ring}, \mathrm{op}) \to (\mathbf{Latt}, \mathrm{op})$ therefore respects the *opposite* endofunctors of the categories **Ring** and **Latt**.

Example 2.8. *If $r \in R$, then $(r|u)^{\otimes} = ur \doteq 0$. More generally, if $I = \sum_i r_iR$ is a finitely generated right ideal, then $(I|u)^{\otimes} = uI \doteq 0$.*

The following criterion explains the notation for $\varphi^{\otimes}$.

Proposition 2.9. ([18, Theorem 1.3.7]) *Let M_R and ${}_RN$ be R-modules and suppose that $\mathbf{a} = (a_1, a_2, \dots, a_n) \in M^n$ and $\mathbf{b} = (b_1, b_2, \dots, b_n) \in N^n$ are n-tuples. Then $\mathbf{a} \otimes \mathbf{b} := \sum_i a_i \otimes b_i = 0$ in $M \otimes_R N$ if and only if there exists a $\varphi \in \mathbb{L}(R,n)$ such that $\mathbf{a} \in \varphi^{\otimes}(M)$ and $\mathbf{b} \in \varphi(N)$.*

For example, if M_R is a right R-module, and $m \in \varphi^{\otimes}(M_R)$ and $r \in \varphi({}_RR)$, then $m \otimes r \in M \otimes_R R \cong M_R$ is 0, so that $\varphi^{\otimes}(M_R) \subseteq \mathrm{ann}_M(\varphi({}_RR))$. Recall that a right module M_R is called *FP-injective,* or, *absolutely pure,* [18, §2.3.1] if for every $\varphi \in \mathbb{L}(R,1)$, $\varphi^{\otimes}(M_R) = \mathrm{ann}_M(\varphi({}_RR))$.

Corollary 2.10. *If $f\colon R \to S$ is an R-ring, then for every $\varphi \in \mathbb{L}(R,1)$, $\varphi^{\otimes}(S_R) \subseteq \mathrm{l.ann}_S\, \varphi({}_R S)$.*

Proof. If $t \in \varphi({}_R S)$ and $s \in \varphi^{\otimes}(S_R)$, then $s \otimes t = 0$ in $S \otimes_R S$; apply the map $S \otimes_R S \to S$, $a \otimes b \mapsto ab$. $\square$

2.4. **Localization.** Consider left R-module ${}_R M$ as an R-T-bimodule, for some ring T, for example, $T = \mathbb{Z}$ or $T = \mathrm{End_R\, {}_R M}$, the endomorphism ring. The evaluation map $\mathrm{Ev}_R({}_R M)\colon \mathbb{L}(R,n) \to \mathrm{Latt}((\mathrm{M^n})_\mathrm{T})$, $\varphi \mapsto \varphi({}_R M)$, is a morphism of lattices that determines the congruence $\Theta_n({}_R M)$ on $\mathbb{L}(R,n)$ by $\varphi \equiv \psi\ (\Theta_n(M))$ provided that $\varphi(M) = \psi(M)$. The quotient lattice modulo this congruence is the *localization* $\mathbb{L}(R,n) \to \mathbb{L}(R,n)_M := \mathbb{L}(R,n)/\Theta_n(M)$, $\varphi \mapsto \varphi_M$, whose image is isomorphic via the map $\varphi_M \mapsto \varphi(M)$ to the image of the evaluation map $\mathrm{Ev}_n({}_R M)$. We may identify the localization $\mathbb{L}(R,n)_M \subseteq \mathrm{Latt}((\mathrm{M^n})_\mathrm{T})$ and its image along this map.

If $f : R \to S$ is a morphism of rings, then the functor $\mathbb{L}(f,n) : \mathbb{L}(R,n) \to \mathbb{L}(S,n)$ defined by (3) respects localization: if ${}_S M$ is a left S-module, then $\varphi^f({}_S M) = \varphi({}_R M)$, where ${}_R M$ is obtained by restriction of scalars along f. This implies that $\mathbb{L}(f,n)(\Theta_n^R(M)) \subseteq \Theta_n^S(M)$ preserves the respective congruences and induces, for every $n > 0$, an embedding $\mathbb{L}(f,n)_M\colon \mathbb{L}(R,n)_M \to \mathbb{L}(S,n)_M$ of localizations, which is part of the commutative diagram

$$\begin{array}{ccc} \mathbb{L}(R,n) & \xrightarrow{\ \mathbb{L}(f,n)\ } & \mathbb{L}(S,n) \\ \Big\downarrow{\scriptstyle \mathrm{Ev}({}_R M)} & & \Big\downarrow{\scriptstyle \mathrm{Ev}({}_S M)} \\ \mathbb{L}(R,n)_M & \xrightarrow{\ \mathbb{L}(f,n)_M\ } & \mathbb{L}(S,n)_M. \end{array} \tag{7}$$

Definition 2.11. *A module ${}_R M$ is* congruence faithful *if $\Theta_1(M) = 0$, the minimum congruence on $\mathbb{L}(R,1)$. Equivalently, the localization $\mathbb{L}(R,1) \to \mathbb{L}(R,1)_M$ is an isomorphism of lattices.*

Congruence faithful modules do exist. For if $\psi < \varphi \in \mathbb{L}(R,1)$ is a proper inequality, it is because there is a module $M_{\varphi/\psi}$ for which the inclusion $\psi(M_{\varphi/\psi}) \subseteq \varphi(M_{\varphi/\psi})$ is proper. The coproduct $M = \amalg_{\psi<\varphi} M_{\varphi/\psi}$ is then congruence faithful.

If Θ is a congruence on $\mathbb{L}(R,1)$ the *dual* congruence $\Theta^{\otimes}$ on $\mathbb{L}(R^{\mathrm{op}},1)$ is defined by $\varphi^{\otimes} \equiv \psi^{\otimes}\ (\Theta^{\otimes})$ if and only if $\psi \equiv \varphi\ (\Theta)$. If $\Theta = \Theta_1(M)$ for some R-module ${}_R M$, then $\Theta^{\otimes} = \Theta_1(M^+)$, where M^+ is the *character module* $\mathrm{Hom}_{\mathbb{Z}}({}_R M, \mathbb{Q}/\mathbb{Z})$ whose right R-module is given by the action $(\zeta \cdot r)(m) := \zeta(rm)$. This follows from the fact that $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator in Ab and the following [18, Lemma 1.3.12].

Proposition 2.12. *If $\varphi \in \mathbb{L}(R,1)$, then $\varphi^{\otimes}(M^+) = \{\zeta\colon M \to \mathbb{Q}/\mathbb{Z} \mid \zeta(\varphi(M)) = 0\}$.*

2.5. **Regular rings.** The next proposition implies that a regular ring R "coordinatizes" the lattice $\mathbb{L}(R)$ in the sense of Definition 2.5, which therefore conforms to von Neumann's usage of the term.

Proposition 2.13. *The following are equivalent for a ring R :*

(1) *it is regular;*
(2) *the embedding $\mathbb{L}(R) \to \mathbb{L}(R,1)$, $I \mapsto I|u$, is an isomorphism;*
(3) *$\mathbb{L}(R,1)$ is a complemented lattice.*

Proof. (1) $\Rightarrow$ (2). If R is regular, Example 2.2 implies that every positive primitive formula is, up to equivalence, of the form $Ju \doteq 0$. Since $J = Re$, $Ju \doteq 0 = eu \doteq 0 = (1-e)|u$.
(2) $\Rightarrow$ (1). By hypothesis, the image of $\mathbb{L}(R^{\mathrm{op}})^{\mathrm{op}} \to \mathbb{L}(R,1)$ is contained in the image of $\mathbb{L}(R) \to \mathbb{L}(R,1)$. By the property of the pullback, $\mathbb{L}(R^{\mathrm{op}}) = \mathbb{S}(R^{\mathrm{op}})$ is complemented, so that R is regular.
(1) $\Leftrightarrow$ (3). Obviously, (1) and (2) imply (3). If $\mathbb{L}(R,1)$ is complemented, then the image of $\mathbb{L}(R)$ is that of $\mathbb{S}(R)$, so that $\mathbb{L}(R) = \mathbb{S}(R)$. □

If R is regular, then every $\varphi \in \mathbb{L}(R,1)$ is of the form $e|u$ with $e = e^2 \in R$ idempotent, so that the Prest dual is given by $ue \doteq 0 = (1-e)|u$ and we have that

$$\varphi^{\otimes}(R_R) = R(1-e) = \mathrm{l.ann}_R(eR) = \mathrm{l.ann}_R\, \varphi({}_RR). \tag{8}$$

Let us observe that the congruence $\Theta_1^R(M)$ is determined by the annihilator $\mathrm{ann}_R(M)$ of the module ${}_RM$. We only need to consider two formulae of the form $e|u \leq g|u$, where e, $g \in R$ are idempotent. But $eM \leq gM$ in $\mathbb{L}(R,1)_M$ if and only if $(1-g)eM = 0$ if and only if $(1-g)e \in \mathrm{ann}_R(M)$. It follows that ${}_RM$ is congruence faithful if and only if it is faithful.

2.6. **Regular R-rings.** Because the coordinatization functor $\mathbb{L}(-,1)\colon (\mathbf{Ring},\mathrm{op}) \to (\mathbf{Latt},\mathrm{op})$ respects the op endofunctors, every R-ring $f\colon R \to S$ gives rise an obviously commutative diagram

$$\begin{array}{ccc} \mathbb{L}(R,1)^{\mathrm{op}} & \xrightarrow{(-)^{\otimes}} & \mathbb{L}(R^{\mathrm{op}},1) \\ \Big\downarrow{\scriptstyle \mathbb{L}(f,1)^{\mathrm{op}}} & & \Big\downarrow{\scriptstyle \mathbb{L}(f^{\mathrm{op}},1)} \\ \mathbb{L}(S,1)^{\mathrm{op}} & \xrightarrow{(-)^{\otimes}} & \mathbb{L}(S^{\mathrm{op}},1). \end{array} \tag{9}$$

If S is a regular R-ring, then S is a faithful, and therefore congruence faithful, S-module on either side. Thus $\mathbb{L}(S,1) = \mathbb{L}(S,1)_S$ and $\mathbb{L}(S^{\mathrm{op}},1) = \mathbb{L}(S^{\mathrm{op}},1)_S$ and we obtain the following.

Proposition 2.14. *If $f\colon R \to S$ is a regular R-ring and $\varphi \in \mathbb{L}(R,1)$, then $\varphi^{\otimes}(S_R) = \mathrm{l.ann}_S\, \varphi({}_RS)$. Therefore $\Theta_1({}_RS)^{\otimes} = \Theta_1(S_R)$.*

Proof. Use the commutativity of Diagram (9) and Equation (8) to get

$$\varphi^{\otimes}(S_R) = (\varphi^{\otimes})^f(S_S) = (\varphi^f)^{\otimes}(S_S) = \mathrm{l.ann}_S\, \varphi^f({}_SS) = \mathrm{l.ann}_S\, \varphi({}_RS).$$

This implies that if $\varphi \equiv \psi$ $(\Theta_1({}_RS))$, then $\psi^{\otimes} \equiv \varphi^{\otimes}$ $(\Theta_1(S_R))$, and therefore $\Theta_1(S_R)^{\otimes} \subseteq \Theta_1({}_RS)$; the opposite inclusion is verified similarly. □

3. The maximal ring of quotients

Recall that a ring R is *right nonsingular* if there exists no nonzero element $r \in R$ for which $\mathrm{r.ann}_R(r) \subseteq R_R$ is essential. Right nonsingular rings have the property (see [20, Corollary VI.6.8]) that the hereditary torsion pair $(\mathcal{T}, \mathrm{Cogen}\, E(R_R))$ in Mod-R cogenerated by the injective envelope $E(R_R)$ of the right regular representation R_R is generated by the *singular* modules, the modules of the form $E(M_R)/M_R$.

3.1. **The maximal R-ring of quotients.** The endomorphism ring $Q = \mathrm{End_R\, E(R_R)}$, which acts on the left, is a regular ring with the property that an endomorphism $f\colon E(R_R) \to E(R_R)$ is determined by its restriction to R_R, and therefore the value $m = f(1)$; this is carefully explained in the last section of [9, Chapter 4]. If we denote this morphism by f_m, then the map $Q \to E(R_R)$, $f \mapsto f(1)$, is seen to be an isomorphism of right R-modules, with inverse $m \mapsto f_m$. It is customary to identify Q and $E(R_R)$ along this isomorphism, so that Q is seen to be an R-ring $q\colon R \to Q$, called the *(right) maximal R-ring of quotients,* with the property that the morphism $q_R\colon R_R \to Q_R$ in Mod-R is an injective envelope of R_R. The following lemma an important property of the maximal ring of quotients useful in our study of the elementary properties of the modules ${}_RQ$ and Q_R.

Lemma 3.1. *If R is a right nonsingular and $M_R \subseteq Q_R$, there exists an idempotent $g \in Q$ such that:*

(1) *$M_R \subseteq gQ_R$ is a maximum (containing all others) essential extension of M_R in Q_R, and*
(2) $\mathrm{l.ann}_Q\, M_R = \mathrm{l.ann}_Q\, gQ = Q(1-g)$.

Proof. Because Q_R is injective, there exists an injective envelope $M_R \subseteq E(M_R)$, which is a summand of $Q_R = E(M_R) \oplus N_R$. Let $g\colon Q_R \to Q_R$ be the projection onto $E(M_R)$ parallel to N_R. Then $g \in \mathrm{End_R\, Q_R = Q}$ is an idempotent element $g = g^2$ with $E(M_R) = \mathrm{Im}\, g = gQ$ and $N_R = (1-g)Q$. If $M_R \subseteq K_R \subseteq Q_R$ is an essential extension of M_R in Q_R, then $(1-g)(K_R) = 0$, because K/M is singular, while Q_R is nonsingular. Thus $K_R \subseteq gQ$. A similar argument shows that $\mathrm{l.ann}_Q\, M_R = \mathrm{l.ann}_Q\, gQ$. □

Lemma 3.1 implies that the map $E\colon \mathrm{Latt(Q_R)} \to \mathbb{S}\mathrm{(Q)}$, $M_R \mapsto E(M_R) := gQ = \mathrm{r.ann}_Q(\mathrm{l.ann}_Q\, M_R)$, is well defined. The main result of this section is that the operations of Lemma 3.1 respect definability.

Theorem 3.2. *Let R be a right nonsingular ring with maximal R-ring of quotients $q\colon R \to Q$. If $\varphi \in \mathbb{L}(R,1)$, then*

(1) $\varphi({}_RQ) = E(\varphi({}_RR))$, *and*
(2) $\varphi^{\otimes}(Q_R) = \mathrm{l.ann}_Q(\varphi({}_RR)) = \mathrm{l.ann}_Q\, \varphi({}_RQ))$,

whence the inclusion $\Theta_1({}_RR) \subseteq \Theta_1({}_RQ)$ of congruences.

Proof. Because Q_R is injective, it is right FP-injective, which implies that $\varphi^{\otimes}(Q_R) = \mathrm{l.ann}_Q\, \varphi({}_RR)$, the first equality of (2). As Q is regular, Proposition 2.14 implies (1),

$$\varphi({}_RQ) = \varphi^f({}_QQ) = \mathrm{r.ann}_Q\,(\varphi^f)^{\otimes}(Q_Q) = \mathrm{r.ann}_Q\, \varphi^{\otimes}(Q_R) = \mathrm{r.ann}_Q(\mathrm{l.ann}_Q \varphi({}_RR)) = E(\varphi({}_RR)).$$

The second equality of (2) follows from (1) and Lemma 3.1.(2). The last statement follows from the implication that if $\varphi({}_RR) = \psi({}_RR)$, then $\varphi({}_RQ) = E(\varphi({}_RR)) = E(\psi({}_RR)) = \psi({}_RQ)$. □

The inclusion of congruences in Theorem 3.2 induces a morphism of localizations $\mathbb{L}(R,1)_R \to \mathbb{L}(R,1)_Q$, $\varphi_R \mapsto \varphi_Q$. To characterize the situation when this is an isomorphism, recall that a submodule $M_R \subseteq N_R$ is *closed* in N_R if there exists no proper essential extension of M_R in N_R.

Corollary 3.3. *The following are equivalent:*

(1) $\Theta_1({}_RR) = \Theta_1({}_RQ)$;
(2) *for every $\psi \in \mathbb{L}(R,1)_R$, the right ideal $\psi({}_RR) \subseteq R_R$ is closed in R_R; and*
(3) *the inclusion ${}_RR \subseteq {}_RQ$ is a pure monomorphism.*

Proof. (1) ⇒ (2). Suppose that $r \in R_R$ is such that $\psi({}_RR) \subseteq \psi({}_RR) + rR$ is essential, and consider the formula $\varphi(u) = \psi(u) + r|u$; then $\varphi({}_RR) = \psi({}_RR) + rR$ and

$$\varphi({}_RQ) = E(\varphi({}_RR)) = E(\varphi({}_RR) + rR) = E(\psi({}_RR)) = \varphi({}_RQ) + rQ.$$

By hypothesis, $\psi({}_RR) = \varphi({}_RR)$, which implies that $\psi({}_RR)$ is closed in R_R.
(2) ⇒ (3). If $\psi({}_RR)$ is closed in R_R, then $\psi({}_RQ) \cap R = E(\psi({}_RR)) \cap R = \psi({}_RR)$. If this holds for every $\psi \in \mathbb{L}(R,1)$, then ${}_RR$ is a pure submodule of ${}_RQ$, by [18, Proposition 2.1.6].
(3) ⇒ (1). If ${}_RR \subseteq {}_RQ$ is a pure monomorphism, then $\Theta_1({}_RQ) \subseteq \Theta_1({}_RR)$. □

If the inclusion $\mathbb{S}(R) \subseteq \mathbb{L}(R,1)$ is composed with the localization at ${}_RR$, it remains an embedding, $\mathbb{S}(R) \to \mathbb{L}(R,1)_R$, $eR \mapsto (e|u)_R$. This is clear, because $\mathrm{Ev}({}_RR)(e|u)_R = eR$, but Theorem 3.2 implies that it remains an embedding even if it is composed with the localization at ${}_RQ$. For, the summands of R_R are closed submodules (because R is right nonsingular), and if e and g are idempotents of R such that $eR \neq gR$, then $eQ = E(eR) \neq E(gR) = gQ$. This proves the first part of the following.

Corollary 3.4. *The restriction to* $\mathbb{S}(R)$ *of the map* $\mathbb{L}(R,1)_R \to \mathbb{L}(R,1)_Q$ *is an embedding. It is an isomorphism if and only if for every* $\varphi \in \mathbb{L}(R,1)$, *there exists an idempotent* $e \in R$ *such that* $\varphi({}_RR) \subseteq eR$ *is an essential extension.*

Proof. By Theorem 3.2, the map $eR \mapsto (e|u)_Q$ is onto if and only if for every $\varphi \in \mathbb{L}(R,1)$, there exists an idempotent $e \in R$ such that $\varphi({}_RQ) = eQ$. If there exists an idempotent $e \in R$ such that $\varphi({}_RR) \subseteq eR$ is essential, then this is clearly the case. Conversely, suppose that $\varphi({}_RQ) = E(\varphi({}_RR)) = eQ$. Then $\varphi({}_RR) \subseteq \varphi({}_RQ) \cap R_R = eQ_R \cap R_R = eR$, and this inclusion is essential. □

The condition on R that arises in Corollary 3.4 is an elementary version of the *extending,* or *CS* condition [15] for the module ${}_RR$, considered as a module over its endomorphism ring. By an argument similar to that used in the proofs of Corollaries 3.3 and 3.4 one obtains the following.

Corollary 3.5. *The localization* $\mathbb{L}(R,1)_Q$ *is a complemented lattice if and only if for every* $\varphi \in \mathbb{L}(R,1)$, *there exists a* $\psi \in \mathbb{L}(R,1)$ *such that* 1) $\varphi({}_RR) \cap \psi({}_RR) = 0$; *and* 2) $\varphi({}_RR) + \psi({}_RR)$ *is essential in* R_R.

3.2. **Conditions on Q.** Theorem 3.2 can be used to characterize standard conditions on the maximal R-ring of quotients in terms of the structure of definable subgroups of ${}_RR$, when R is a right nonsingular ring. The condition that ${}_RQ$ is flat is considered in [20, §XII.6]. Denote the image of the map $\mu\colon \varphi({}_RR) \otimes Q \to Q$, $\sum_i r_i \otimes q_i \mapsto \sum_i r_iq_i$, by $\varphi({}_RR)Q$; it is clearly contained in $\varphi({}_RQ)$. Recall from [18, Theorem 2.3.9] that ${}_RQ$ is flat if and only if for every $\varphi \in \mathbb{L}(R,1)$, $\varphi({}_RR)Q = \varphi({}_RQ)$.

Corollary 3.6. *The left R-module* ${}_RQ$ *is flat if and only if for every* $\varphi \in \mathbb{L}(R,1)_R$, *there exists a finitely generated right ideal* $I \in \mathbb{L}(R)$ *such that* $I_R \subseteq \varphi({}_RR)$ *is essential.*

Proof. If ${}_RQ$ is flat, then for every $\varphi \in \mathbb{L}(R,1)_R$, $\varphi({}_RR)Q = \varphi({}_RQ) = gQ \in \mathbb{S}(Q)$, so that we can express the idempotent $g \in Q$ by the form $g = \sum_i r_iq_i$, where $r_i \in \varphi({}_RR)$. Let $I_R = \sum_i r_iR$ be the finitely generated right ideal of R_R generated by the r_i. Then $(I|u)_R \leq \varphi_R \in \mathbb{L}(R,1)_R$, But $(I|u)_Q = \varphi_Q$ in $\mathbb{L}(R,1)_Q$, because $\varphi({}_RQ) = gQ = IQ$, so that the inclusion $I_R \subseteq \varphi({}_RR)$ is essential, by Theorem 3.2. Conversely, if $I_R \subseteq \varphi({}_RR)$ is essential, then Theorem 3.2 implies that $\varphi({}_RQ) = IQ \subseteq \varphi({}_RR)Q$. □

The condition of Corollary 3.6 may be regarded as a weakening of *right coherence,* which is characterized by the property that every $\varphi({}_RR)$, $\varphi \in \mathbb{L}(R,1)$ is a finitely generated right ideal of R. The proof of the following is dual to the proof of Corollary 3.6.

Corollary 3.7. *The right R-module Q_R is flat if and only if for every $\varphi \in \mathbb{L}(R,1)_R$, there exists a finitely generated left ideal $J \subseteq \varphi^{\otimes}(R_R)$ such that $\varphi({}_RR) \subseteq \mathrm{r.ann}_R(J)$ is essential.*

Proof. The condition is clearly sufficient, for if such a left ideal ${}_RJ$ exists, then apply Theorem 3.2 to the formula $Ju \doteq 0$ to get that

$$\varphi^{\otimes}(Q_R) = \mathrm{l.ann}_Q\, \varphi({}_RR) = \mathrm{l.ann}_Q(\mathrm{r.ann}_R(J)) = QJ \subseteq Q\varphi^{\otimes}(R_R).$$

For the converse, suppose Q_R to be flat, so that $Q\varphi^{\otimes}(R_R) = \varphi^{\otimes}(Q_R)$, for every $\varphi \in \mathbb{L}(R,1)_R$. By Theorem 3.2, there is an idempotent $h \in Q$ such that $Qh = \varphi^{\otimes}(Q_R) = Q\varphi^{\otimes}(R_R)$, which we can express as a finite linear combination $h = \sum_j q_j r_j$, with $r_j \in \varphi^{\otimes}(R_R)$. Let $J = \sum_j Rr_j \subseteq \varphi^{\otimes}(R_R)$, which implies that $\varphi({}_RR) \subseteq \mathrm{r.ann}_R(J)$ and that $Qh = Q\varphi^{\otimes}(R_R) \supseteq QJ \supseteq Qh$. Now Theorem 3.2 implies that $\mathrm{l.ann}_Q\, \varphi({}_RR) = \varphi^{\otimes}(Q_R) = QJ = \mathrm{l.ann}_Q(\mathrm{r.ann}_R(J))$, whence $E(\mathrm{r.ann}_R(J)) = E(\varphi({}_RR))$ and the conclusion that $\varphi({}_RR) \subseteq \mathrm{r.ann}_R(J)$ is an essential extension. □

Recall from above that a left module ${}_RM$ is FP-injective if for every $\varphi \in \mathbb{L}(R,1)$, $\varphi({}_RM) = \mathrm{ann}_M\, \varphi^{\otimes}(R_R)$. Together with Corollary 3.7, the following implies that if Q_R is flat, then ${}_RQ$ is FP-injective.

Corollary 3.8. *The left R-module ${}_RQ$ is FP-injective if and only if for every $\varphi \in \mathbb{L}(R,1)_R$, the inclusion $\varphi({}_RR) \subseteq \mathrm{r.ann}_R(\varphi^{\otimes}(R_R))$ of right R-modules is essential.*

Proof. Note that the inclusion $\mathrm{r.ann}_R\, \varphi^{\otimes}(R_R) = R \cap \mathrm{r.ann}_Q\, \varphi^{\otimes}(R_R) \subseteq \mathrm{r.ann}_Q\, \varphi^{\otimes}(R_R)$ is essential, because $\mathrm{r.ann}_Q\, \varphi^{\otimes}(R_R)$ is a summand of Q_R and $R_R \subseteq Q_R$ is essential. The inclusion $\varphi({}_RR) \subseteq \mathrm{r.ann}_Q(\varphi^{\otimes}(R_R))$ is therefore essential if and only if $\varphi({}_RQ) = E(\varphi({}_RR)) = \mathrm{r.ann}_Q(\varphi^{\otimes}(R_R))$. □

3.3. **Elementary classes of rings.** Let us take up the question of whether the conditions on a ring that arise in Corollaries 3.3-3.8 are elementary with respect to the language $\mathcal{L}(\mathbf{Ring}) = (+, \cdot, -, 0, 1)$ of rings. The class of right nonsingular rings is an elementary class, axiomatized by the sentence

$$\forall r (\forall s (\forall u\, ((ru \doteq 0 \wedge \exists v\, (sv \doteq u)) \to u \doteq 0) \to s \doteq 0) \to r \doteq 0).$$

It will be easier to read such sentences of $\mathcal{L}(\mathbf{Ring})$ if we introduce a logical connective for (right) *essential implication:* if $\psi(u)$ and $\varphi(u)$ are formulae in $\mathcal{L}(\mathbf{Ring})$ in one free variable, define

$$\psi(u) \to_e \varphi(u) := (\psi(u) \to \varphi(u)) \wedge \forall v \exists r\, (\varphi(v) \wedge v \neq 0) \to (\psi(rv) \wedge rv \neq 0).$$

This implication will only be used for formulae that define in a ring S a right ideal. In that case, $S \models \forall u\, (\psi(u) \to_e \varphi(u))$ holds if and only if the subset $\psi(S) \subseteq \varphi(S)$ is an essential extension of submodules of S_S. With this convention available, the axiom for a right nonsingular ring is abbreviated to the more colloquial expression $\forall r\, (\forall u\, (ru \doteq 0 \to_e u \doteq u)) \to r \doteq 0$.

Proposition 3.9. *The class of right nonsingular rings R for which the inclusion ${}_RR \subseteq {}_RQ$ is a pure monomorphism is elementary is axiomatized by the schema*

$$\mathrm{Cl}(m,k) := \forall (A,B)\, \forall r\, ((\varphi(u) \to_e (\varphi(u) + r|u)) \to \varphi(r)),$$

where $\varphi(u)$ *is the positive primitive formula* (1) *and* $r|u := \exists v\,(rv \doteq u)$, *with* $m,\ k \in \mathbb{N}$.

Proof. The axiom $\mathrm{Cl}(m,k)$ expresses that for every positive primitive formula $\varphi(u)$ given by an $m \times (1+k)$ matrix (A,B), the finite matrix subgroup $\varphi({}_RR)$ is a closed right ideal in R_R; the equivalence follows from Corollary 3.3. □

Proposition 3.10. *The class of right nonsingular rings for which the embedding* $\mathbb{S}(R) \to \mathbb{L}(R,1)_Q$, $eR \mapsto (e|u)_Q$, *is an isomorphism is elementary, axiomatized by the schema*

$$C_1(m,k) = \forall (A,B)\,\exists e\,((e \doteq e^2) \wedge \forall u\,(\varphi(u) \to_e e|u)),$$

where $\varphi(u)$ *is the positive primitive formula* (1), *with* $m,\ k \in \mathbb{N}$.

Proof. The axiom $C_1(m,k)$ expresses that for every positive primitive formula $\varphi(u)$ given by an $m \times (1+k)$ matrix (A,B), the finite matrix subgroup $\varphi({}_RR)$ is essential in a summand eR of R_R; the equivalence follows from Corollary 3.4. □

The statement of Corollary 3.5 asserts the existence of a positive primitive formula (in one variable) presented by a matrix of unspecified dimension, while the statements of Corollary 3.6 and 3.7 assert the existence of an unspecified finitely generated ideal; we do not know if the classes characterized by these statements are elementary. By contrast, the class of left FP-injective rings is elementary, axiomatized by the axiom schema

$$\mathrm{AP}(m,k) = \forall (A,B)\,\forall u\,(\varphi(u) \leftrightarrow \mathrm{r.ann}(\varphi^{\otimes}(v))(u)),\ m,k \in \mathbb{N},$$

where $\varphi(u)$ is given by (1), $\varphi^{\otimes}(v)$ by (5) and $\mathrm{r.ann}(\sigma(v))(u) := \forall v(\sigma(v) \to vu \doteq 0)$. The implication $\varphi(u) \to \mathrm{r.ann}(\varphi^{\otimes}(v))(u)$ is true for all rings and pp-formulae $\varphi(u)$, but if we replace the implication by essential implication, we obtain a schema that axiomatizes the rings of Corollary 3.8.

Proposition 3.11. *The class of right nonsingular rings for which the left* R*-module* ${}_RQ$ *is FP-injective is elementary, axiomatized by the schema*

$$\mathrm{AP}_e(m,k) = \forall (A,B)\,\forall u\,(\varphi(u) \to_e \mathrm{r.ann}(\varphi^{\otimes}(v))(u)),\ m,k \in \mathbb{N}.$$

4. Decoordinatization

Let us now give a detailed analysis of $\mathbb{L}(R,2)$, the algebra of definable relations on a left R-module. A positive primitive formula $\sigma(u,v)$ in *two* variables defines a relation $\sigma(M) \subseteq M^2$ on a module ${}_RM$. This induces on $\mathbb{L}(R,2)$ the structure of a relational algebra that includes the ring R as a subalgebra, via the map $r \mapsto (v \doteq ru)$ that associates to $r \in R$ the graph of its action.

4.1. **The relational operations.** A positive primitive formula $\sigma(u,v) \in \mathbb{L}(R,2)$ can be expressed as

$$(10) \qquad \exists \mathbf{w}\ (A_0, A_*, A_\infty) \begin{pmatrix} u \\ \mathbf{w}^t \\ v \end{pmatrix} \doteq 0,$$

where A_0 and A_∞ are column matrices, and A_* is an $m \times k$ matrix, with as many rows as A_0 and A_∞, associated to the existentially bound variables. We can suppress the variables and use the synthetic notation $(A_0|A_*|A_\infty)$ suggested by P.M. Cohn in [5, proof of Theorem VII.4.8]. If $A_* = 0$, we write $(A_0|A_\infty)$.

Definition 4.1. *Define* relational *operations on* $\mathbb{L}(R,2)$ *so that if* $\alpha(u,v)$, $\beta(u,v) \in \mathbb{L}(R,2)$, *then*

(1) $(\alpha \cdot \beta)(u,v) := \exists w\ (\beta(u,w) \wedge \alpha(w,v))$; *and*

(2) $(\alpha + \beta)(u,v) := \exists v_1, v_2\ (\beta(u,v_1) \wedge \alpha(u,v_2) \wedge (v \doteq v_1 + v_2))$.

There are two distinguished relational *elements* $1(u,v) := u \doteq v = (1 \mid -1)$ *and* $0(u,v) := v \doteq 0 = (0 \mid -1)$.

Definition 4.1 introduces an unfortunate notational ambiguity. The relational operations are not the same as their lattice theoretic counterparts: the relational $+$ is not the same as the supremum on the lattice $\mathbb{L}(R,2)$, and the relational constants 0 and 1 are not the minimum and maximum elements of the lattice. In an effort to disambiguate the situation, we give priority to the relational symbols and refer to the lattice $\mathbb{L}(R,2)$, from now on, in terms of its partial order. The full structure of $\mathbb{L}(R,2)$ will therefore be given by the signature of a partially ordered relational algebra $(\mathbb{L}(R,2), \leq, +, \cdot, 0, 1)$.

The relational operations of $(\mathbb{L}(R,2), \leq, +, \cdot, 0, 1)$ are given synthetically by

$$(A_0|A_*|A_\infty) \cdot (B_0|B_*|B_\infty) = \left(\begin{array}{c|ccc|c} B_0 & B_* & B_\infty & 0 & 0 \\ 0 & 0 & A_0 & A_* & A_\infty \end{array}\right), \tag{11}$$

and

$$(A_0|A_*|A_\infty) + (B_0|B_*|B_\infty) = \left(\begin{array}{c|ccc|c} A_0 & A_* & A_\infty & 0 & 0 \\ B_0 & 0 & -B_\infty & B_* & B_\infty \end{array}\right). \tag{12}$$

The second equality follows from $R\text{-Mod} \models (\alpha+\beta)(u,v) \leftrightarrow \exists w\ (\alpha(u,w) \wedge \beta(u, v-w))$.

Proposition 4.2. *The partially ordered (universal) algebra* $(\mathbb{L}(R,2), \leq, +, \cdot, 0, 1)$ *of definable relations consists of a partially ordered monoid* $(\mathbb{L}(R,2), \leq, \cdot, 1)$ *and a partially ordered commutative monoid* $(\mathbb{L}(R,2), \leq, +, 0)$, *related by the distributive inequalities*

$$(\alpha+\beta)\gamma \leq \alpha\gamma + \beta\gamma \ \ \textit{and} \ \ \alpha(\beta+\gamma) \geq \alpha\beta + \alpha\gamma.$$

The subalgebra of definable functions (scalars) is given by the image of the map

$$i_R \colon (R, +, \cdot, 0, 1) \to (\mathbb{L}(R,2), \leq, +, \cdot, 0, 1), \ \ r \mapsto (ru - v \doteq 0) = (r \mid -1).$$

Proof. Verification of the partially ordered monoid structures is routine. The right distributive inequality $(\alpha+\beta)\gamma \leq \alpha\gamma + \alpha\gamma$ is nothing more than the implication

$$\begin{aligned} R\text{-Mod} \ \ \models \ \ & \exists w\, (\gamma(u,w) \wedge \exists z_1, z_2\, (\alpha(w,z_1) \wedge \beta(w,z_2) \wedge (z \doteq z_1 + z_2))) \\ \leftrightarrow \ \ & \exists w, z_1, z_2\, ((\gamma(u,w) \wedge \alpha(w,z_1)) \wedge (\gamma(u,w) \wedge \beta(w,z_2)) \wedge (z \doteq z_1 + z_2)) \\ \rightarrow \ \ & \exists z_1, z_2\, (\alpha\gamma(u,z_1) \wedge \gamma\beta(u,z_2) \wedge (z \doteq z_1 + z_2)). \end{aligned}$$

The left distributive inequality is verified similarly. The result that a definable function is necessarily of the form $v \doteq ru$ is found in [18, Lemma 6.1.4]. □

Since the definable functions in $\mathbb{L}(R,2)$ form an antichain in the partial order, the partial order is not included in the signature of the ring. The definition of the partially ordered algebra $\mathbb{L}(R,2)$ is functorial, for if $f\colon R \to S$ is a morphism of rings, one obtains the following commutative diagram of partially ordered algebras

$$\begin{array}{ccc} R & \xhookrightarrow{\;i_R\;} & \mathbb{L}(R,2) \\ \Big\downarrow{\scriptstyle f} & & \Big\downarrow{\scriptstyle \mathbb{L}(f,2)} \\ S & \xhookrightarrow{\;i_S\;} & \mathbb{L}(S,2). \end{array}$$

4.2. **The relational dual.** In the language $\mathcal{L}(R^{\mathrm{op}})$ of right R-modules, the action of $r \in R$ is written on the right, so that a positive primitive formula $\sigma(u,v)$ in two variables will have the general form

$$\exists \mathbf{w}\ (u, \mathbf{w}, v) \begin{pmatrix} A_0 \\ A_* \\ A_\infty \end{pmatrix} \doteq 0,$$

represented synthetically as $\left(\begin{array}{c} A_0 \\ \hline A_* \\ \hline A_\infty \end{array}\right)$, where A_0 and A_* are row matrices with the same number of columns as A_*. The relational algebra $\mathbb{L}(R^{\mathrm{op}},2)$ contains the ring R^{op} via the map $r \mapsto (v \doteq ur) = \left(\begin{array}{c} r \\ \hline -1 \end{array}\right)$, and multiplication in $\mathbb{L}(R^{\mathrm{op}},2)$ is represented synthetically as

$$\left(\begin{array}{c} A_0 \\ \hline A_* \\ \hline A_\infty \end{array}\right) \cdot \left(\begin{array}{c} B_0 \\ \hline B_* \\ \hline B_\infty \end{array}\right) = \left(\begin{array}{cc} B_0 & 0 \\ \hline B_* & 0 \\ B_\infty & A_0 \\ 0 & A_* \\ \hline 0 & A_\infty \end{array}\right). \tag{13}$$

Definition 4.3. *The* relational dual *is the function* $\mathrm{opp}\colon \mathbb{L}(R,2) \to \mathbb{L}(R^{\mathrm{op}},2)$ *given by*

$$(A_0|A_*|A_\infty) \longmapsto \left(\begin{array}{ccc} 0 & 0 & 1 \\ \hline A_0 & A_* & A_\infty \\ \hline -1 & 0 & 0 \end{array}\right).$$

The relational dual of a formula $\sigma(u,v) \in \mathbb{L}(R,2)$ *is also called its* opposite, $\sigma^{\mathrm{opp}}(u,v) \in \mathbb{L}(R^{\mathrm{op}},2)$. *The inverse* $\mathrm{opp}\colon \mathbb{L}(R^{\mathrm{op}},2) \to \mathbb{L}(R,2)$ *uses the same notation and is defined in a like manner:*

$$\left(\begin{array}{c} A_0 \\ \hline A_* \\ \hline A_\infty \end{array}\right) \longmapsto \left(\begin{array}{c|c|c} 0 & A_0 & -1 \\ 0 & A_* & 0 \\ 1 & A_\infty & 0 \end{array}\right).$$

The relational dual is a twisted version of the Prest dual, $\sigma^{\mathrm{opp}}(u,v) = \sigma^{\otimes}(v,-u)$. The criterion [18, Theorem 1.3.7] implies that if M_R and ${}_RN$ are R-modules, then $a \otimes y = b \otimes x$ in $M \otimes_R N$ if and only if there is a definable relation $\sigma \in \mathbb{L}(R,2)$ such that $N \models \sigma(x,y)$ and $M \models \sigma^{\mathrm{opp}}(a,b)$. If we informally write this as $y = \sigma(x)$ and $b = \sigma^{\mathrm{opp}}(a)$, then the equality of elementary tensors becomes $a \otimes \sigma(x) = \sigma^{\mathrm{opp}}(a) \otimes x$.

Proposition 4.4. *The map* opp: $\mathbb{L}(R,2)^{\mathrm{op}} \to \mathbb{L}(R^{\mathrm{op}},2)$ *is part of a commutative diagram*

$$\begin{array}{ccc} R^{\mathrm{op}} & \xhookrightarrow{(i_R)^{\mathrm{op}}} & \mathbb{L}(R,2)^{\mathrm{op}} \\ \Big\| & & \Big\downarrow \text{opp} \\ R^{\mathrm{op}} & \xhookrightarrow{i_{R^{\mathrm{op}}}} & \mathbb{L}(R^{\mathrm{op}},2) \end{array}$$

of morphisms of partially ordered relational algebras.

Proof. The equation $\sigma^{\mathrm{opp}}(u,v) = \sigma^{\otimes}(v,-u)$ implies that $\sigma \mapsto \sigma^{\mathrm{opp}}$ is order reversing. Commutativity follows from opp: $(r \mid -1) \longmapsto \left(\begin{array}{cc} 0 & 1 \\ \hline r & -1 \\ \hline -1 & 0 \end{array}\right) = \left(\begin{array}{c} r \\ \hline -1 \end{array}\right)$, since Mod-$R \models \exists w\ (wr \doteq v \wedge u \doteq w) \leftrightarrow ur \doteq v$. Let us verify that opp: $(\mathbb{L}(R,2),\cdot,1)^{\mathrm{op}} \to (\mathbb{L}(R^{\mathrm{op}}),\cdot,1)$ is an anti-isomorphism. This means that we need to confirm the equality got by applying opp to both sides of (11):

$$\left(\begin{array}{ccc} 0 & 0 & 1 \\ \hline B_0 & B_* & B_\infty \\ \hline -1 & 0 & 0 \end{array}\right) \left(\begin{array}{ccc} 0 & 0 & 1 \\ \hline A_0 & A_* & A_\infty \\ \hline -1 & 0 & 0 \end{array}\right) = \left(\begin{array}{ccccc} 0 & 0 & 0 & 0 & 1 \\ \hline B_0 & B_* & B_\infty & 0 & 0 \\ 0 & 0 & A_0 & A_* & A_\infty \\ \hline -1 & 0 & 0 & 0 & 0 \end{array}\right).$$

By (13), the left side is given by

$$\left(\begin{array}{cccccc} 0 & 0 & 1 & 0 & 0 & 0 \\ \hline A_0 & A_* & A_\infty & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & B_0 & B_* & B_\infty \\ \hline 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right) = \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & A_0 & A_* & A_\infty \\ 0 & 0 & 1 & -1 & 0 & 0 \\ B_0 & B_* & B_\infty & 0 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$
$$= \left(\begin{array}{cccccc} 0 & 0 & 0 & 0 & 0 & 1 \\ \hline B_0 & B_* & B_\infty & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & A_0 & A_* & A_\infty \\ \hline -1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

where the right matrix in the first equality is obtained from the left by permuting some columns; this leaves the formula invariant. The second equality is obtained by transposing the second and fourth rows; because these rows are associated to the existentially bound variables, their permutation also leaves the formula invariant. Now the third and fourth columns of this last matrix express the condition

$$\text{Mod-}R \models \exists w\,((\mathbf{x}B_\infty + w \doteq 0) \wedge (w \doteq \mathbf{y}A_0)) \leftrightarrow (\mathbf{x}B_\infty + \mathbf{y}A_0 \doteq 0),$$

as required. The proof that opp: $(\mathbb{L}(R,2),+,0)^{\mathrm{op}} \to (\mathbb{L}(R^{\mathrm{op}},2),+,0)$ is an anti-isomorphism is similar. $\square$

4.3. **Localization.** If ${}_RM$ is a left R-module, considered as an R-T-bimodule, for some ring T, then the lattice of $\mathrm{Latt}(\mathrm{M}^2_\mathrm{T})$ is also an algebra of relations, in the sense of Definition 4.1. The quotient of $\mathbb{L}(R,2)$ modulo the congruence $\Theta_2(M)$ induces on the localization $\mathbb{L}(R,2) \twoheadrightarrow \mathbb{L}(R,2)/\Theta_2(M) = \mathbb{L}(R,2)_M$ the structure of a partially ordered relational algebra which may be identified, along $\sigma(u,v)_M \mapsto \sigma(M) \subseteq M^2$, as a subalgebra of $\mathrm{Latt}(\mathrm{M}^2_\mathrm{T})$. The following lemma ensures that if ${}_RM$ and ${}_RN$ are R-modules for which $\Theta_1(M) = \Theta_1(N)$, then there is an isomorphism $\mathbb{L}(R,2)_M \cong \mathbb{L}(R,2)_N$ of the localized partially ordered relational algebras.

Lemma 4.5. *If ${}_RM$ and ${}_RN$ are R-modules, then $\Theta_1(M) = \Theta_1(N)$ if and only if $\Theta_2(M) = \Theta_2(N)$.*

Proof. For the right to left implication, just use the fact that $\varphi(u) \equiv \psi(u)$ $(\Theta_1(M))$ if and only if $\varphi(u) \wedge (v \doteq 0) \equiv \psi(u) \wedge (v \doteq 0)$ $(\Theta_2(M))$. For the converse, suppose that $\sigma(u,v) \equiv \tau(u,v)$ $(\Theta_2(M))$, and let us show the same for $\Theta_2(N)$. We may assume, without loss of generality, that $\sigma(u,v) \leq \tau(u,v) \in \mathbb{L}(R,2)$. Then $\exists v\, \sigma(u,v) \equiv \exists\, \tau(u,v)$ $(\Theta_1(M))$, which implies, by hypothesis, the same for $\Theta_1(N)$. This means that if $(a,b) \in \tau(N) \subseteq N^2$, then there exists a $b' \in N$ such that $(a,b') \in \sigma(N) \subseteq \tau(N)$, whence $(0, b-b') = (a,b)-(a,b') \in \tau(N)$. Now $\sigma(u,v) \equiv \tau(u,v)$ $(\Theta_2(M))$ also implies that $\sigma(0,v) \equiv \tau(0,v)$ $(\Theta_1(M))$, and the hypothesis ensures the same for $\Theta_1(N)$. We infer that $(0, b-b') \in \sigma(N)$, and therefore that $(a,b) = (a,b') + (0, b-b') \in \sigma(N)$. Thus $\Theta_2(M) \subseteq \Theta_2(N)$, and the conclusion follows. ☐

If the localization at N is restricted to the ring, one obtains the commutative diagram

$$\begin{array}{ccc} R & \hookrightarrow & \mathbb{L}(R,2) \\ \downarrow{\scriptstyle \delta_N} & & \downarrow \\ R_N & \hookrightarrow & \mathbb{L}(R,2)_N \end{array}$$

where $\delta_N \colon R \to R_N$ is the R-ring of N-*definable scalars.* The localized formula $\sigma(u,v)_N$ belongs to R_N if it defines in N^2 the graph of a function with domain variable u and codomain variable v. Equivalently, the relation $\sigma(N) \subseteq N^2$ is

(1) *entire:* $\exists v\, \sigma(u,v) \equiv (u \doteq u)$ $(\Theta_1(N))$; and
(2) *functional:* $\sigma(0,v) \equiv (v \doteq 0)$ $(\Theta_1(N))$.

4.4. **The canonical map.** The module ${}_RN$ may now be considered as a left R_N-module; Diagram (7) takes the form

$$\begin{array}{ccc} \mathbb{L}(R,n) & \xrightarrow{\mathbb{L}(\delta_N,n)} & \mathbb{L}(R_N,n) \\ \downarrow & & \downarrow \\ \mathbb{L}(R,n)_N & \xrightarrow{\mathbb{L}(\delta_N,n)_N} & \mathbb{L}(R_N,n)_N. \end{array} \tag{14}$$

Proposition 4.6. *The maps $\mathbb{L}(\delta_N,n)_N \colon \mathbb{L}(R,n)_N \longrightarrow \mathbb{L}(R_N,n)_N$, $n \geq 1$, are isomorphisms of lattices.*

Proof. It suffices to show that the image of $\mathbb{L}(\delta_N,n)_N$ contains all the quantifier free formulae in $\mathcal{L}(R_N)$. So let $A = (\sigma_{ij})$ be an $m \times n$ matrix over R_N, and consider the system $\alpha(\mathbf{u},\mathbf{v}) = A\mathbf{u}^t \doteq \mathbf{v}^t$. The solutions to this system in N are the same as that of the formula $\beta(\mathbf{u},\mathbf{v}) = \exists w_{ij}\,(\bigwedge_i (v_i \doteq \sum_j w_{ij} \wedge \bigwedge_j \sigma_{ij}(u_j, w_{ij})))$, whence $(A\mathbf{u}^t \doteq \mathbf{v}^t)_N = \beta^{\delta_N}(\mathbf{u},\mathbf{v})_N$. □

Definition 4.7. *Given a left R-module, define the* canonical map $\mathbb{L}(R_N, 1) \to \mathbb{L}(R, 1)_N$ *from the lattice associated to the ring of definable scalars to the localization by $\varphi^{\delta_N} \mapsto \varphi_N$. We will say that the R-ring of N-definable scalars coordinatizes the localization at N if the canonical map is an isomorphism.*

A glance at Diagram (14) proves the first part of the following.

Theorem 4.8. *The canonical map $\mathbb{L}(R_N, 1) \to \mathbb{L}(R, 1)_N$ is an isomorphism if and only if N is a congruence faithful R_N-module. In that case, the R-ring $\delta_N : R \to R_N$ is epic.*

Proof. To prove the second statement, we work in the language $\mathcal{L}(R_N)$ of modules over the ring of definable scalars. Let $\sigma \in R_N$ be represented by the defining positive primitive formula $\sigma(u, v) \in \mathbb{L}(R, 2)$. The congruences $\sigma^{\delta_N}(0, v) \equiv v \doteq 0\,(\Theta_1(N))$ and $\sigma^{\delta_N}(u, \sigma u) \equiv u \doteq u\,(\Theta_1(N))$ hold, so the condition that N is a congruence faithful R_N-module implies that these are equalities in $\mathbb{L}(R_N, 1) : \sigma^{\delta_N}(0, v) = (v \doteq 0)$ and $\sigma^{\delta_N}(u, \sigma u) = (u \doteq u)$. Together, they imply that the formula $\sigma^{\delta_N}(u, v)$ in the language $\mathcal{L}(R_N)$ defines a "global" scalar (on every R_N-module), and that this scalar agrees with the action of σ. Suppose now that $f, g, : R_N \rightrightarrows S$ are parallel arrows with the same restriction to R, $f\delta_N = g\delta_N$. The morphisms f and g endow S with two R_N-module structures, that agree on the action of R. But the foregoing considerations imply that the action of $\sigma \in R_N$ is uniquely determined by that action. □

4.5. **Regular rings of definable scalars.** One of the interesting cases of Theorem 4.8 is when the R-ring of definable scalars is regular.

Proposition 4.9. *Let ${}_RN$ be a left R-module. The localization $\mathbb{L}(R, 1) \to \mathbb{L}(R, 1)_N$ is complemented if and only if the R-ring $\delta_N : R \to R_N$ of definable scalars is regular. If $\delta_N : R \to R_N$ is a regular R-ring, then it is epic and coordinatizes the localization $\mathbb{L}(R, 1)_N$.*

Proof. Let us first prove the second statement. The module ${}_RN$ is faithful as an R_N-module, so if R_N is regular, it is congruence faithful, by the dicussion following Proposition 2.13. The ring of definable scalars coordinatizes the localization by Theorem 4.8. The right to left implication of the first statement then follows, for if R_N is regular, then the isomorphism $\mathbb{L}(R_N, 1) \to \mathbb{L}(R, 1)_N$ implies that $\mathbb{L}(R, 1)_N$ is complemented.

Suppose that the localization $\mathbb{L}(R, 1)_N$ is complemented, and that $\sigma(u, v)_N \in R_N$. Then there exist $\varphi,\ \psi \in \mathbb{L}(R, 1)$ such that

$$\sigma(u, 0)_N \oplus \varphi(u)_N = 1 = \psi(v)_N \oplus (\exists u\, \sigma(u, v))_N. \tag{15}$$

The formula $\tau(u, v) = (\psi(u) \wedge v \doteq 0) + (\sigma(v, u) \wedge \varphi(v))$ defines a scalar on N satisfying $\sigma\tau\sigma = \sigma \in R_N$. □

4.6. **The dominion of an R-ring.** Let $f : R \to S$ be an R-ring. An element $s \in S$ is *dominated by* f if for every pair of parallel ring morphisms $g_1, g_2 : S \rightrightarrows T$ satisfying $g_1 f = g_2 f$, $g_1(s) = g_2(s)$. The elements of S dominated by f form a subring D_f, the *dominion of* f, that clearly contains the image of f, and may therefore be considered as an R-subring

$$\begin{array}{ccc} R & \xrightarrow{d_f} & D_f \\ {\scriptstyle f}\big\downarrow & \swarrow & \\ S. & & \end{array}$$

The elements of the dominion of f are those that satisfy the equation $s \otimes 1 = 1 \otimes s$ in $S \otimes_R S$ ([19, p. 122] or [20, §XI.1]).

If $s \in D_f$, the criterion of Proposition 2.9 applied the the equation $s \otimes 1 - 1 \otimes s = 0$ implies the existence of $\rho \in \mathbb{L}(R, 2)$ such that ${}_RS \models \rho(1, s)$ and $S_R \models \rho^{\otimes}(s, -1)$, or, equivalently, $S_R \models \rho^{\mathrm{opp}}(1, s)$. An argument of Prest [18, proof of Theorem 6.1.8] implies that $\rho_S \in \mathbb{L}(R, 2)_S$ is an S-definable scalar and that it defines on ${}_RS$ the action of s. Symmetrically, $\rho_S^{\mathrm{opp}} \in \mathbb{L}(R^{\mathrm{op}}, 2)_S$ defines on S_R the scalar given by multiplication by s on the right. In this way, we obtain an inclusion of R-rings $f\colon D_f \subseteq R_S \subseteq S$, where $R_S \subseteq \mathbb{L}(R, 2)_S$ is the ring of S-definable scalars, regarded as a subring of S, via the map $\rho \mapsto \rho(1)$. By symmetry, there is another such inclusion of R-rings, $f^{\mathrm{op}}\colon D_f^{\mathrm{op}} \subseteq (R^{\mathrm{op}})_S \subseteq S^{\mathrm{op}}$, where $(R^{\mathrm{op}})_S$ is the subring of scalars defined on S_R.

Theorem 4.10. *Let $f\colon R \to S$ be an R-ring. If the dominion D_f or the ring R_S of definable scalars is regular, then $D_f = R_S$. Therefore, if $\mathbb{L}(R, 1)_S$ is complemented, it is coordinatized by the dominion of f.*

Proof. If the R-ring of definable scalars is regular, then it epic, by Proposition 4.9. All the elements of S are therefore dominated by f, whence the equality $D_f = R_S$. If the localization $\mathbb{L}(R, 1)_S$ is complemented, another application of Proposition 4.9 yields that the ring of definable scalars coordinatizes it, which proves the second statement.

If, on the other hand, the dominion of f is regular, then we may think of S as a left module over D_f. This action of D_f on S is the definable so, arguing as in the proof of Proposition 4.6, we get an isomorphism $\mathbb{L}(D_f, 1)_S \cong \mathbb{L}(R, 1)_S$. This implies that $\mathbb{L}(R, 1)_S$ is complemented, and that R_S is regular, by Proposition 4.9, which puts us in the first case. □

Example 4.11. *Let R be any associative ring, and consider the R-ring given by the product $f\colon R \to S = \Pi\,\Delta$ of all the epic R-fields $R \to \Delta$ (not necessarily commutative). By* [12, Theorem 2.2]*, the R-ring of definable scalars is the universal abelian R-ring. Theorem 4.10 implies that $R_S = D_f$ is the dominion of f.*

Corollary 3.5 gives a characterization of the right nonsingular rings for which the localization $\mathbb{L}(R, 1) \to \mathbb{L}(R, 1)_Q$ is complemented. By Theorem 4.10 these are the nonsingular rings whose maximal R-ring of quotients has a regular dominion. The special case when the localization $\mathbb{L}(R, 1)_Q \cong \mathbb{S}(R)$ is isomorphic to the space of right summands of R was characterized in Corollary 3.4 and shown to be elementary in Proposition 3.10.

Corollary 4.12. *The class of right nonsingular rings R whose space $\mathbb{S}(R)$ of right summands is coordinatized by the dominion of the right maximal R-ring of fractions is an elementary class.*

4.7. **Coordinatization by the maximal ring of quotients.** Let us finally consider the case when the maximal R-ring of a right nonsingular ring is epic, that is, when $q\colon R \to Q$ is its own dominion. Theorem 4.10 implies that the localization $\mathbb{L}(R, 1)_Q \cong \mathbb{L}(Q, 1) = \mathbb{S}(Q)$ is coordinatized by Q. The following is then an immediate consequence of Corollary 3.4.

Corollary 4.13. *Let R be a right nonsingular ring with maximal R-ring $q\colon R \to Q$ of fractions. The induced embedding $\mathbb{S}(q)\colon \mathbb{S}(R) \to \mathbb{S}(Q)$ of the respective spaces of right summands is an isomorphism if and only if 1) $q\colon R \to Q$ is epic and 2) every finite matrix subgroup $\varphi({}_RR)$ is essential in a point $eR \in \mathbb{S}(R)$.*

The (right) *Goldie dimension* $\dim(R_R)$ of a ring, regarded as a right module over itself, is an elementary property. Precisely, the statement $\dim(R_R) \geq n$ is expressible in $\mathcal{L}(\mathbf{Ring})$ by the sentence

$$\exists x_1, x_2, \dots, x_n \, (\bigwedge_i x_i \neq 0 \wedge \forall r_1, \dots, r_n \, (\sum_i x_i r_i \doteq 0 \to x_i r_i \doteq 0)).$$

This implies that if R and S are elementarily equivalent rings, then $\dim(R_R)$ and $\dim(S_S)$ are both finite and equal or both infinite.

Lemma 4.14. *If R is a right nonsingular ring with* $\dim(R_R) = \infty$, *then* $|\mathbb{S}(Q)| \geq 2^{\aleph_0}$ *and* $|D| \leq |R| + \aleph_0$.

Proof. For the first inequality, observe that $\dim(Q_R) = \dim(R_R) = \infty$ implies that Q_R contains an infinite direct sum. The injective envelopes of the various subsums give rise to a continuum of summands of Q_Q. The second inequality follows from the proof of [18, Theorem 6.1.8], which implies that every element of the dominion is a definable scalar, represented by a formula in $\mathbb{L}(R\text{-Mod})$. □

The cardinality considerations of Lemma 4.14 show that the class of nonsingular rings whose space of summands is coordinatized by the maximal ring of quotients is not elementary.

Theorem 4.15. *Let R be a right nonsingular ring of infinite (right) Goldie dimension,* $\dim(R_R) = \infty$, *whose space $\mathbb{S}(R)$ of right summands is coordinatized by the maximal ring of quotients. If $R_0 \prec R$ is a countable elementary subring, then it is also right nonsingular of infinite right Goldie dimension - with maximal R_0-ring of quotients $q_0\colon R_0 \to Q_0$ - and the space $\mathbb{S}(R_0)$ of right summands is coordinatized by D_{q_0}, but $D_{q_0} \neq Q_0$.*

Proof. Because the class of right nonsingular rings of infinite right Goldie dimension is elementary, the elementary subring R_0 also belongs to the class. But because R_0 is countable, Lemma 4.14 implies $|D_{q_0}| \leq \aleph_0 < |\mathbb{S}(Q_0)| \leq |Q_0|$. □

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