NIP FOR SOME PAIR-LIKE THEORIES

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Abstract. Generalising work from [2] and [6], we give sufficient conditions for a theory T_P to inherit NIP from T, where T_P is an expansion of the theory T by a unary predicate P. We apply our result to theories, studied in [1], of the real field with a subgroup of the unit circle.

1. Introduction

We consider the situation where T is a complete one-sorted theory with infinite models, L is the language of T, P is a new unary predicate, $L_P = L \cup \{P\}$, $M \models T$ and T_P is the complete theory of some expansion of M to L_P. Our main result provides sufficient conditions for T_P to inherit NIP from T. It is a common generalisation of two other recent results, one of Berenstein, Dolich and Onshuus in [2] and one of Günaydın and Hieronymi in [6]. With respect to the result from [2], our generalisation removes the assumption that $P(M)$ be algebraically closed. With respect to the result from [6], it has the advantage that it works outside the setting where T is o-minimal. We apply our result to theories of the real field with a predicate for a subgroup of the unit circle which were studied by Belegradek and Zilber in [1]. Similar theories were studied by van den Dries and Günaydın in [5] and by Berenstein, Ealy and Günaydın in [3] and shown to have NIP in [6].

This short paper has evolved over a period of time and benefited from many useful conversations, comments and corrections for which I thank Hans Adler, John Baldwin, Oleg Belegradek, Alexander Berenstein, Dugald Macpherson, Boris Zilber and the referee. Above all I would like to thank my PhD supervisor Anand Pillay for much generous assistance with this material, including suggesting the main approach used.

2. A general result

First order logic is used throughout. The expansion of M to L_P is written as $(M, P(M))$. We work in L_P (or T_P) except where we specifically indicate L (or T). For example, acl denotes algebraic closure in the sense of T_P while acl_L denotes algebraic closure in the sense of T. Similarly $tp(\bar{a}/B)$ is a complete type in the sense of T_P while $tp_L(\bar{a}/B)$ is a complete type in the

Date: July 21, 2010.

sense of T. Otherwise our notation is fairly standard. We abbreviate $A \cup B$ to AB and sometimes sets are treated as tuples or vice versa. For a tuple of variables $\bar{x} = x_1 ... x_n$ we abbreviate $P(x_1) \wedge ... \wedge P(x_n)$ to $P(\bar{x})$. Our main result is the following.

Theorem 2.1. *T_P has NIP if, for any $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ such that all three models are sufficiently saturated, the following conditions are satisfied:*
(i) acl_L is a pregeometry on $\overline{M}$,
(ii) if $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$ then $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$,
(iii) for every finite $n \geq 1$ and $\bar{f} \in P(\overline{M})^n$, there exist $\kappa \leq |M|$ and $\widehat{f} \in P(\overline{M})^\kappa$ such that $\bar{f} \subseteq \widehat{f}$ and, for every $\widehat{f}' \models tp(\widehat{f}/M)$, $tp(\widehat{f}'/N)$ is implied by $tp_L(\widehat{f}'/N)$ in conjunction with $tp(\widehat{f}'/M)$,
(iv) T has NIP.

The thinking behind condition (iii) should become clear in the light of Sections 3 and 4. However it would probably be helpful at this stage to mention a neater version of it which is sufficient for some interesting applications:

(iii)$'$ for every finite $n \geq 1$ and $\bar{f} \in P(\overline{M})^n$, $tp(\bar{f}/N)$ is implied by $tp_L(\bar{f}/N)$ in conjunction with the information that $\bar{f} \in P(\overline{M})^n$.

Clearly when (iii)$'$ replaces (iii) in the assumptions of Theorem 2.1, these assumptions are if anything strengthened. We shall comment further on (iii)$'$ in Section 3.

The independence property (the negation of NIP) was introduced by Shelah in [8]. Our proof of Theorem 2.1 uses the following fact due to a combination of Shelah and Poizat. Details are given in chapter 12 of [7].

Fact 2.2. *The following conditions are equivalent and T has NIP if and only if they are true:*
(1) for every $M \prec N \models T$ such that both models are sufficiently saturated, there are no more than $2^{|M|}$ complete one-types $q(x)$ over N such that $q(x)$ is finitely realisable in M,
(2) for every finite $n \geq 1$ and every $M \prec N \models T$ such that both models are sufficiently saturated, there are no more than $2^{|M|}$ complete n-types $q(\bar{x})$ over N such that $q(\bar{x})$ is finitely realisable in M.

We conclude this section with a proof of Theorem 2.1. Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Let $b \in \overline{M}$ and suppose $tp(b/N)$ is finitely realisable in M. We show that there are no more than $2^{|M|}$ choices for $tp(b/N)$.

Case 1: Suppose $b \notin acl_L(NP(\overline{M}))$. By condition (ii), $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. Clearly $tp_L(b/N)$ is finitely realisable in M. By condition (iv) and Fact 2.2(1), there are no more than $2^{|M|}$ choices for $tp_L(b/N)$. Therefore there are, in this case, no more than $2^{|M|}$ choices for $tp(b/N)$.

Case 2: Suppose $b \in acl_L(NP(\overline{M}))$. Let $\bar{a}\bar{c}\bar{f}$ be a tuple such that $\bar{a} \in M^k$, $\bar{c} \in N^l$ and $\bar{f} \in P(\overline{M})^n$, for some $k, l, n < \omega$, and $b \in acl_L(\bar{a}\bar{c}\bar{f})$. We may assume $\bar{c}$ is of minimal length. Suppose $\bar{c}$ is not empty (that is to say $l \neq 0$). It follows by condition (i) that $\bar{c}$ is not acl_L-independent over $\bar{a}b\bar{f}$. Let $\varphi(\bar{x}, y, \bar{z}, \bar{w})$ be an L-formula which is realised by $\bar{a}, b, \bar{c}, \bar{f}$ and which witnesses the fact that $\bar{c}$ is not acl_L-independent over $\bar{a}b\bar{f}$. Since $tp(b/\bar{a}\bar{c})$ is finitely realisable in M, the formula $\psi(\bar{a}, y, \bar{c}) \equiv (\exists \bar{w})[P(\bar{w}) \wedge \varphi(\bar{a}, y, \bar{c}, \bar{w})]$ is realisable in M and so $\bar{c}$ is not acl_L-independent over $MP(\overline{M})$. This contradicts the minimality of the length of $\bar{c}$. Therefore $\bar{c}$ is empty. Therefore $b \in acl_L(M\bar{f})$.

Let κ and $\widehat{f}$ be as in condition (iii). Let $p(y\widehat{w}) = r(y) \wedge s(y\widehat{w})$ where $r(y) = tp(b/N)$ and $s(y\widehat{w}) = tp(b\widehat{f}/M)$. Then $p(y\widehat{w})$ is finitely realisable in M. By a well known argument (you extend a filter-base to an ultrafilter and extract what you want from that), $p(y\widehat{w})$ extends to a complete type $p'(y\widehat{w})$ over N which is finitely realisable in M. Let $b'\widehat{f'} \models p'(y\widehat{w})$. Then $tp(b'/N) = tp(b/N)$, $tp(\widehat{f'}/M) = tp(\widehat{f}/M)$ and $b' \in acl_L(M\widehat{f'})$.

By condition (iii), $tp(\widehat{f'}/N)$ is implied by $tp_L(\widehat{f'}/N)$ in conjunction with $tp(\widehat{f'}/M)$. Clearly $tp_L(\widehat{f'}/N)$ is finitely realisable in M. By condition (iv) and Fact 2.2(2), there are no more than $(2^{|M|})^{|M|}$ choices for $tp_L(\widehat{f'}/N)$. Therefore there are no more than $(2^{|M|})^{|M|} \times (2^{|M|})^{|M|}$ choices for $tp(\widehat{f'}/N)$. Therefore there are, in this case, no more than $(2^{|M|})^{|M|} \times (2^{|M|})^{|M|} \times |M| \times |M| = 2^{|M|}$ choices for $tp(b/N)$.

Adding the two cases together, there are no more than $2^{|M|}$ choices for $tp(b/N)$. Therefore T_P has NIP, by Fact 2.2(1).

3. Comparison with results in [2] and [6]

Theorem 2.7 of [2] makes use of the notion of P-independence which is defined as follows and makes sense provided acl_L is a pregeometry.

Definition 3.1. A set $A \subseteq M$ is said to be P-independent if A is acl_L-independent from $P(M)$ over $A \cap P(M)$.

We shall also want to speak of P-$tp(\bar{a})$ by which we mean the information that tells us which members of the tuple $\bar{a}$ belong to $P(M)$. The following is Theorem 2.7 from [2].

Theorem 3.2. *T_P has NIP if, for all $(M, P(M)) \models T_P$, the following conditions are satisfied:*
(a) acl_L is a pregeometry on M,
(b) for every finite $n \geq 1$, if $\bar{a}, \bar{b} \in M^n$ are such that both $\bar{a}$ and $\bar{b}$ are P-independent then $tp(\bar{a}) = tp(\bar{b})$ provided both $tp_L(\bar{a}) = tp_L(\bar{b})$ and P-$tp(\bar{a}) = P$-$tp(\bar{b})$,
(c) $acl_L(P(M)) = P(M)$,
(d) T has NIP.

The following result establishes that Theorem 2.1 is a generalisation of Theorem 3.2. The proof is a standard argument in this area. We include it for the sake of completeness. Note that when we speak of P-independence we mean with respect to the larger model $(\overline{M}, P(\overline{M}))$.

Proposition 3.3. *The assumptions of Theorem 3.2 excluding (c) imply the assumptions of Theorem 2.1 even when (iii)′ replaces (iii).*

Proof. Suppose the assumptions of Theorem 3.2 are satisfied with the possible exception of (c). Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Conditions (i) and (iv) follow immediately. Since $(N, P(N)) \prec (\overline{M}, P(\overline{M}))$ it is clear that, for every finite tuple $\bar{a}$ from N, there is a finite tuple $\bar{g}$ from $P(N)$ such that $\bar{a}\bar{g}$ is P-independent. Let $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$. For every P-independent tuple $\bar{a}$ from N, it is clear that $b\bar{a}$ is also P-independent. It follows by condition (b) that $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. So condition (ii) is satisfied. Let $n \geq 1$ be finite and $\bar{f} \in P(\overline{M})^n$. For every P-independent tuple $\bar{a}$ from N it is clear that $\bar{f}\bar{a}$ is also P-independent. It follows by condition (b) that $tp(\bar{f}/N)$ is implied by $tp_L(\bar{f}/N)$ in conjunction with the information that $\bar{f} \in P(\overline{M})^n$. So condition (iii)′ is satisfied. □

Theorem 3.2 is used in [2] to show that if T has NIP and is a geometric theory and T_P is the theory of lovely pairs of models of T, as defined in [4], then T_P has NIP. This provides an interesting class of examples to which Theorem 2.1 applies even when condition (iii) is replaced by (iii)′.

The following is Theorem 3.1 from [6].

Theorem 3.4. *Suppose T is o-minimal. T_P has NIP if, for all $(M, P(M)) \models T_P$, the following conditions are satisfied:*
(e) for every finite $n \geq 1$, every definable subset of $P(M)^n$ is a boolean combination of sets of the form $X \cap Y$ where X is $\emptyset$-definable and Y is L-definable,

(f) every definable $X \subseteq M$ is definable by a boolean combination of formulas of the form $(\exists \bar{z})[P(\bar{z}) \wedge \varphi(x, \bar{z})]$ where $\varphi(x, \bar{z})$ is an L-formula possibly with parameters,
(g) every open definable $X \subseteq M$ is a finite union of intervals.

The following result establishes that Theorem 2.1 is a generalisation of Theorem 3.4. The proof borrows from the argument used in [6] to prove Theorem 3.4.

Proposition 3.5. *Suppose T is o-minimal. The assumptions of Theorem 3.4 imply the assumptions of Theorem 2.1.*

Proof. Suppose the assumptions of Theorem 3.4 are satisfied. Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Conditions (i) and (iv) are well known consequences of T being o-minimal. Let $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$. By condition (f) and the fact that $(N, P(N)) \prec (\overline{M}, P(\overline{M}))$, to know $tp(b/N)$ it is enough to know which formulas of the form $\psi(y, \bar{a}) \equiv (\exists \bar{z})[P(\bar{z}) \wedge \varphi(y, \bar{a}, \bar{z})]$ belong to $tp(b/N)$, where $\varphi(\bar{y}, \bar{x}, \bar{z})$ is an L-formula with no parameters and $\bar{a} \in N^{|\bar{x}|}$. Knowing that $b \notin acl_L(NP(\overline{M}))$, it is enough to consider the case where, for each $\bar{f} \in P(\overline{M})^{|\bar{z}|}$, $\varphi(y, \bar{a}, \bar{f})$ defines a possibly empty open interval in $\overline{M}$. In this case the set defined by $\psi(y, \bar{a})$ is an open subset of $\overline{M}$. By condition (g) this set is L-definable and so, since $(N, P(N)) \prec (\overline{M}, P(\overline{M}))$, it is L-definable over N. Therefore $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. So condition (ii) is satisfied.

Let $n \geq 1$ be finite and $\bar{f} \in P(\overline{M})^n$. Let $Z \subseteq P(M)^n$ be definable over N such that $\bar{f} \in Z$. By condition (e), Z is a boolean combination of sets of the form $X \cap Y$ where X is $\emptyset$-definable and Y is L-definable. Since $(N, P(N)) \prec (\overline{M}, P(\overline{M}))$, we may assume Y is L-definable over N. It follows that $tp(\bar{f}/N)$ is implied by $tp_L(\bar{f}/N)$ in conjunction with $tp(\bar{f}/\emptyset)$. So condition (iii) is satisfied with $\widehat{f} = \bar{f}$. □

4. An example

We now consider a class of theories studied by Belegradek and Zilber in [1]. Let $\mathbb{R}$ be the real field and $\mathbb{S}$ the unit circle thought of as a subgroup of the multiplicative group of the complex field $\mathbb{C}$. Let $\Gamma(\mathbb{R}) \leq \mathbb{S}$ be a subgroup. Let $\Gamma(\mathbb{R})^{[n]} = \{g^n : g \in \Gamma(\mathbb{R})\}$. Assume $\Gamma(\mathbb{R})$ satisfies the following three conditions:

- $|\Gamma(\mathbb{R})| = \aleph_0$,
- $|\Gamma(\mathbb{R})/\Gamma(\mathbb{R})^{[n]}| < \aleph_0$ for every finite $n \geq 1$,
- for every finite $n \geq 1$, if $X = Y \cap \Gamma(\mathbb{R})^n$ for some algebraic set $Y \subseteq \mathbb{C}^n$

then X is definable over parameters in the group $(\Gamma(\mathbb{R}), \cdot)$ by a positive quantifier-free formula.

Let $L = \{<, +, \cdot, 0, 1, Re(g), Im(g)\}_{g \in \Gamma(\mathbb{R})}$ where $Re(g)$ and $Im(g)$ are suggestively named constant symbols for the real part and the imaginary part of each member of $\Gamma(\mathbb{R})$. Let T be the resulting L-theory of $\mathbb{R}$. Let $Re : \mathbb{S} \to \mathbb{R}$ be the function which assigns to each member of $\mathbb{S}$ its real part. Let P be a new unary predicate and $L_P = L \cup \{P\}$. Interpret P such that $P(\mathbb{R}) = Re(\Gamma(\mathbb{R}))$. Let T_P be the resulting L_P-theory of $\mathbb{R}$. Let Γ be a new binary predicate and $L_\Gamma = L \cup \{\Gamma\}$. Let the suggestively named $\Gamma(\mathbb{R})$ be the interpretation of Γ in $\mathbb{R}$. Let T_Γ be the resulting L_Γ-theory of $\mathbb{R}$. As is noted in [1], $\Gamma(\mathbb{R}) = Re^{-1}(Re(\Gamma(\mathbb{R})))$ and so T_P and T_Γ are definitionally equivalent.

T_Γ was studied by Belegradek and Zilber in [1]. They gave axioms for it and proved a near model completeness result. Expecting a positive answer, they asked if T_Γ has NIP. We use Theorem 2.1 to obtain a positive answer.

We check that T_P satisfies the assumptions of Theorem 2.1. Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Since T is an expansion by constants of the theory of real-closed ordered fields, it is well known that conditions (i) and (iv) are satisfied. We deduce conditions (ii) and (iii) from the results in [1]. The argument overlaps with the reasoning in [1]. Since the predicate Γ is $\emptyset$-definable in T_P we shall feel free to use it. Re will now denote the real part map from the unit circle in the big model $(\overline{M}, P(\overline{M}))$. So $\Gamma(\overline{M}) = Re^{-1}(P(\overline{M}))$. Since L contains constant symbols for all real parts and all imaginary parts of members of $\Gamma(\mathbb{R})$, we may assume $\Gamma(\mathbb{R}) \leq \Gamma(M)$. Let $\Gamma(\overline{M})_d$ be the largest divisible subgroup of $\Gamma(\overline{M})$. As is observed in [1], $\Gamma(\overline{M})_d$ has a direct complement H in $\Gamma(\overline{M})$ such that $|H| \leq 2^{\aleph_0}$. Let $\Gamma(\overline{M})_d^T$ be the torsion subgroup of $\Gamma(\overline{M})_d$. Clearly $|\Gamma(\overline{M})_d^T| \leq \aleph_0$. Let $K = \{h \cdot g : h \in H \text{ and } g \in \Gamma(\overline{M})_d^T\}$. Given the sufficient saturation of M, we may assume $K \leq \Gamma(M)$.

The following is proved in [1] by means of a back-and-forth argument.

Theorem 4.1. *Let $F \leq \Gamma(\overline{M})$ be a subgroup with the following properties:*
(α) $F = \{k \cdot d : k \in K$ and $d \in D\}$ for some divisible subgroup $D \leq \Gamma(\overline{M})$,
(β) $|F| \leq 2^{\aleph_0}$,
(γ) $\Gamma(\mathbb{R}) \leq F$.

Let $n \geq 1$ be finite and let $\bar{a} \in \overline{M}^n$ be acl_L-independent over $P(\overline{M})$. Then $tp(F\bar{a})$ is implied by the information so far mentioned in conjunction with $tp_L(F\bar{a})$.

Let $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$. Let $F \leq \Gamma(N)$ satisfy conditions (α), (β) and (γ) and let $\bar{a}$ be a finite tuple from N such that $\bar{a}$ is acl_L-independent over

$P(\overline{M})$. It follows from Theorem 4.1 that $tp(b/F\bar{a})$ is implied by $tp_L(b/F\bar{a})$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. It is then clear that $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. So condition (ii) is satisfied.

Let $n \geq 1$ be finite and $\bar{f} \in P(\overline{M})^n$. Let $F \leq \Gamma(\overline{M})$ satisfy conditions (α), (β) and (γ) and be such that $\bar{f} \subseteq Re(F)$. Let $\tilde{F} \leq \Gamma(N)$ satisfy conditions (α), (β) and (γ) and let $\bar{a}$ be a finite tuple from N such that $\bar{a}$ is acl_L-independent over $P(\overline{M})$. It is clear that the subgroup $\{c \cdot d : c \in F \text{ and } d \in \tilde{F}\}$ also satisfies conditions (α), (β) and (γ) and that this is a consequence of $tp(F/M)$ in conjuction with $tp(\tilde{F}/M)$. It follows from Theorem 4.1 that $tp(F/\tilde{F}\bar{a})$ is implied by $tp_L(F/\tilde{F}\bar{a})$ in conjunction with $tp(F/M)$. It is then clear that $tp(F/N)$ is implied by $tp_L(F/N)$ in conjunction with $tp(F/M)$. Clearly this remains true when F is replaced by any $F' \models tp(F/M)$. So condition (iii) is satisfied with $\kappa \leq 2^{\aleph_0}$ and $\widehat{f} = Re(F)$.

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