NIP FOR SOME PAIR-LIKE THEORIES

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ABSTRACT. Generalising work from [2] and [5], we give sufficient conditions for a theory T_P to inherit NIP from T, where T_P is an expansion of the theory T by a unary predicate P. We apply our result to a theory, studied in [1], of the real field with a subgroup of the unit circle.

1. Introduction

We consider the situation where T is a complete one-sorted theory with infinite models, L is the language of T, P is a new unary predicate, $L_P = L \cup \{P\}$, $M \models T$ and T_P is the complete theory of some expansion of M to L_P. Our main result provides sufficient conditions for T_P to inherit NIP from T. It is a common generalisation of two other recent results, one of Berenstein, Dolich and Onshuus in [2] and one of Günaydin and Hieronymi in [5]. With respect to the result from [2], our generalisation removes the assumption that $P(M)$ be algebraically closed. With respect to the result from [5], it has the advantage that it works outside the setting where T is o-minimal. We apply our result to a theory of the real field with a subgroup of the unit circle which was studied by Belegradek and Zilber in [1].

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2. A general result

First order logic is used throughout. The expansion of M to L_P is written as $(M, P(M))$. We work in L_P (or T_P) except where we specifically indicate L (or T). For example, acl denotes algebraic closure in the sense of T_P while acl_L denotes algebraic closure in the sense of T. Similarly $tp(\bar{a}/B)$ is a complete type in the sense of T_P while $tp_L(\bar{a}/B)$ is a complete type in the sense of T. Otherwise our notation is fairly standard. We abbreviate $A \cup B$ to AB and sometimes sets are treated as tuples or vice versa. For a tuple of variables $\bar{x} = x_1...x_n$ we abbreviate $P(x_1) \wedge ... \wedge P(x_n)$ to $P(\bar{x})$. Our main result is the following.

Theorem 2.1. *T_P has NIP if, for any $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ such that all three models are sufficiently saturated, the following conditions are satisfied:*
(i) acl_L is a pregeometry on $\overline{M}$,
(ii) if $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$ then $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$,
(iii) there is some sufficiently saturated $(N', P(N'))$ such that $(N, P(N)) \prec (N', P(N')) \prec (\overline{M}, P(\overline{M}))$ and, for any finite $n \geq 1$ and $\bar{f} \in P(\overline{M})^n$, there is some $\kappa \leq 2^{|M|}$ and $\widehat{f} \in P(\overline{M})^\kappa$ such that $\bar{f} \in acl(\widehat{f})$ and, for any extension of $tp(\widehat{f}/M)$ to a complete type $q(\widehat{z})$ over N such

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that $q(\widehat{z})$ is finitely realisable in M, there is an extension of $tp_L(\widehat{f}/M)$ to a complete L-type $q'(\widehat{z})$ over N' such that $q'(\widehat{z})$ is finitely realisable in M and $q(\widehat{z})$ is implied by $q'(\widehat{z})$ in conjunction with $tp(\widehat{f}/M)$,
(iv) T has NIP.

Some of the generality of Theorem 2.1 is obtained at the expense of elegance. The thinking behind condition (iii) should become clear in the light of Sections 3 and 4. However it would probably be helpful at this stage to mention a neater version of it which is sufficient for some interesting applications:

(iii)$'$ for any finite $n \geq 1$ and $\bar{f} \in P(\overline{M})^n$, $tp(\bar{f}/N)$ is implied by $tp_L(\bar{f}/N)$ in conjunction with the information that $\bar{f} \in P(\overline{M})^n$.

Clearly when (iii)$'$ replaces (iii) in the assumptions of Theorem 2.1, these assumptions are if anything strengthened. We shall comment further on (iii)$'$ in Section 3.

The independence property (the negation of NIP) was introduced by Shelah in [8]. Our proof of Theorem 2.1 uses the following fact due to a combination of Shelah and Poizat. Details are given in chapter 12 of [7].

Fact 2.2. *The following conditions are equivalent and T has NIP if and only if they are true:*
(1) for any $M \prec N \models T$ such that both models are sufficiently saturated, there are at most $2^{|M|}$ complete one-types $q(x)$ over N such that $q(x)$ is finitely realisable in M,
(2) for any finite $n \geq 1$ and any $M \prec N \models T$ such that both models are sufficiently saturated, there are at most $2^{|M|}$ complete n-types $q(\bar{x})$ over N such that $q(\bar{x})$ is finitely realisable in M.

We conclude this section with a proof of Theorem 2.1. Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Let $b \in \overline{M}$ and suppose $tp(b/N)$ is finitely realisable in M. We show that there are at most $2^{|M|}$ choices for $tp(b/N)$.

Case 1: Suppose $b \notin acl_L(NP(\overline{M}))$. By condition (ii), $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. Clearly $tp_L(b/N)$ is finitely realisable in M. By condition (iv) and Fact 2.2(1), there are at most $2^{|M|}$ choices for $tp_L(b/N)$. Therefore there are, in this case, at most $2^{|M|}$ choices for $tp(b/N)$.

Case 2: Suppose $b \in acl_L(NP(\overline{M}))$. Let $\bar{a}\bar{c}\bar{f}$ be a tuple such that $\bar{a} \in M^k$, $\bar{c} \in N^l$ and $\bar{f} \in P(\overline{M})^n$, for some $k, l, n < \omega$, and $b \in acl_L(\bar{a}\bar{c}\bar{f})$. We may assume $\bar{c}$ is of minimal length. Suppose $\bar{c}$ is not empty (that is to say $l \neq 0$). It follows by condition (i) that $\bar{c}$ is not acl_L-independent over $\bar{a}b\bar{f}$. Let $\varphi(\bar{x}, y, \bar{z}, \bar{w})$ be an L-formula which is realised by $\bar{a}, b, \bar{c}, \bar{f}$ and which witnesses the fact that $\bar{c}$ is not acl_L-independent over $\bar{a}b\bar{f}$. Since $tp(b/\bar{a}\bar{c})$ is finitely realisable in M, the formula $\psi(\bar{a}, y, \bar{c}) \equiv (\exists \bar{w})[P(\bar{w}) \wedge \varphi(\bar{a}, y, \bar{c}, \bar{w})]$ is realisable in M and so $\bar{c}$ is not acl_L-independent over $MP(\overline{M})$. This contradicts the minimality of the length of $\bar{c}$. Therefore $\bar{c}$ is empty. Therefore $b \in acl_L(M\bar{f})$.

Let $(N', P(N'))$, κ and $\widehat{f}$ be as in condition (iii). Let $p(y\widehat{w}) = r(y) \wedge s(y\widehat{w})$ where $r(y) = tp(b/N)$ and $s(y\widehat{w}) = tp(b\widehat{f}/M)$. Then $p(y\widehat{w})$ is finitely realisable in M. By a well known argument (you extend a filter-base to an ultrafilter and extract what you want from that), $p(y\widehat{w})$ extends to a complete type $p'(y\widehat{w})$ over N which is finitely realisable in M. Let $b'\widehat{f}' \models p'(y\widehat{w})$. Then $tp(b'/N) = tp(b/N)$, $tp(\widehat{f}'/M) = tp(\widehat{f}/M)$ and $b' \in acl(M\widehat{f}')$.

By condition (iii), $tp(\widehat{f}'/N)$ is implied by $tp(\widehat{f}/M)$ in conjunction with some complete L-type $q'(\widehat{w})$ over N' which extends $tp_L(\widehat{f}/M)$ and is finitely realisable in M. By condition (iv) and Fact 2.2(2), there are at most $2^{|M|}\times\kappa$ choices for $q'(\widehat{w})$. Therefore there are at most $2^{|M|}\times\kappa\times 2^{|M|}\times\kappa$ choices for $tp(\widehat{f}'/N)$. Therefore there are, in this case, at most $2^{|M|}\times\kappa\times 2^{|M|}\times\kappa\times|M|\times\kappa = 2^{|M|}$ choices for $tp(b/N)$.

Adding the two cases together, there are at most $2^{|M|}$ choices for $tp(b/N)$. Therefore T_P has NIP, by Fact 2.2(1).

3. Comparison with results in [2] and [5]

Theorem 2.7 of [2] makes use of the notion of P-independence which is defined as follows and makes sense provided acl_L is a pregeometry.

Definition 3.1. A set $A \subseteq M$ is said to be P-independent if A is acl_L-independent from $P(M)$ over $A \cap P(M)$.

We shall also want to speak of P-$tp(\bar{a})$ by which we mean the information that tells us which members of the tuple $\bar{a}$ belong to $P(M)$. The following is Theorem 2.7 from [2].

Theorem 3.2. *T_P has NIP if, for any sufficiently saturated $(M, P(M)) \models T_P$, the following conditions are satisfied:*
(a) acl_L is a pregeometry on M,
(b) for any finite $n \geq 1$, if $\bar{a}, \bar{b} \in M^n$ are such that both $\bar{a}$ and $\bar{b}$ are P-independent and both $tp_L(\bar{a}) = tp_L(\bar{b})$ and P-$tp(\bar{a}) = P$-$tp(\bar{b})$ then $tp(\bar{a}) = tp(\bar{b})$,
(c) $acl_L(P(M)) = P(M)$,
(d) T has NIP.

The following result establishes that Theorem 2.1 is a generalisation of Theorem 3.2. The proof is standard and trivial. Note that when we speak of P-independence we mean with respect to the larger model $(\overline{M}, P(\overline{M}))$.

Proposition 3.3. *The assumptions of Theorem 3.2 excluding (c) imply the assumptions of Theorem 2.1 even when (iii)$'$ replaces (iii).*

Proof. Suppose the assumptions of Theorem 3.2 are satisfied with the possible exception of (c). Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Conditions (i) and (iv) follow immediately. Since $(N, P(N))$ is a model it is clear that, for any finite tuple $\bar{a}$ from N, there is a finite tuple $\bar{g}$ from $P(N)$ such that $\bar{a}\bar{g}$ is P-independent. Let $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$. For any P-independent tuple $\bar{a}$ from N, it is clear that $b\bar{a}$ is also P-independent. It follows from condition (b) that $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. So condition (ii) is satisfied. Let $n \geq 1$ be finite and $\bar{f} \in P(\overline{M})^n$. For any P-independent tuple $\bar{a}$ from N it is clear that $\bar{f}\bar{a}$ is also P-independent. It follows from condition (b) that $tp(\bar{f}/N)$ is implied by $tp_L(\bar{f}/N)$ in conjunction with the information that $\bar{f} \in P(\overline{M})^n$. So condition (iii)$'$ is satisfied. ☐

Theorem 3.2 is used in [2] to show that if T has NIP and is a geometric theory and T_P is the theory of lovely pairs of models of T, as defined in [3], then T_P has NIP. This provides an interesting class of examples to which Theorem 2.1 applies even when condition (iii) is replaced by (iii)$'$.

There is a typo in the version of [5] currently available from the MODNET Preprint server, but through private correspondence with one of the authors we understand that their Theorem 1.3 should be as follows.

Theorem 3.4. *Suppose T is dense o-minimal. T_P has NIP if, for any $(M, P(M)) \models T_P$, the following conditions are satisfied:*
(e) for any finite $n \geq 1$, any definable subset of $P(M)^n$ is a boolean combination of sets of the form $X \cap Y$ where X is $\emptyset$-definable and Y is L-definable,
(f) any formula $\psi(\bar{x})$ without parameters is equivalent modulo T_P to a boolean combination of formulas of the form $(\exists \bar{z})[P(\bar{z}) \wedge \varphi(\bar{x}, \bar{z})]$ where $\varphi(\bar{x}, \bar{z})$ is an L-formula which also has no parameters,
(g) $(M, P(M))$ has o-minimal open core.

The following result establishes that Theorem 2.1 is a generalisation of Theorem 3.4. The proof overlaps with the argument used in [5] to prove Theorem 3.4.

Proposition 3.5. *Suppose T is dense o-minimal. The assumptions of Theorem 3.4 imply the assumptions of Theorem 2.1.*

Proof. Suppose the assumptions of Theorem 3.4 are satisfied. Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Conditions (i) and (iv) are well known consequences of T being a dense o-minimal theory. Let $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$. It follows from condition (f) that to know $tp(b/N)$ it is enough to know which formulas of the form $\psi(y, \bar{a}) \equiv (\exists \bar{z})[P(\bar{z}) \wedge \varphi(y, \bar{a}, \bar{z})]$ belong to $tp(b/N)$, where $\varphi(\bar{y}, \bar{x}, \bar{z})$ is an L-formula with no parameters and $\bar{a} \in N^{|x|}$. Knowing that $b \notin acl_L(NP(\overline{M}))$, it is enough to consider the case where, for each $\bar{f} \in P(\overline{M})^{|z|}$, $\varphi(y, \bar{a}, \bar{f})$ defines an open interval in $\overline{M}$. In this case the set defined by $\psi(y, \bar{a})$ is an open subset of $\overline{M}$. By condition (g) this set is L-definable and so, since $(N, P(N))$ is a model, it is L-definable over N. Therefore $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. So condition (ii) is satisfied.

Let $(N', P(N'))$ be such that $(N, P(N)) \prec (N', P(N')) \prec (\overline{M}, P(\overline{M}))$, $(N', P(N'))$ is sufficiently saturated and N' contains enough parameters so that, whenever Z is definable over N and Z is a boolean combination of sets of the form $X \cap Y$ where X is $\emptyset$-definable and Y is L-definable, it is possible to choose the sets Y to be L-definable over N'. Let $n \geq 1$ be finite and $\bar{f} \in P(\overline{M})^n$. Let $\bar{f}' \models tp(\bar{f}/M)$ be such that $tp(\bar{f}'/N)$ is finitely realisable in M. Let $\bar{f}'' \models tp(\bar{f}'/N)$ be such that $tp_L(\bar{f}''/N')$ is finitely realisable in M. It follows from condition (e) that $tp(\bar{f}'/N)$ is implied by $tp_L(\bar{f}''/N')$ in conjunction with $tp(\bar{f}/M)$. So condition (iii) is satisfied with $\kappa = n$ and $\widehat{f} = \bar{f}$. □

4. An example

We now consider a theory studied by Belegradek and Zilber in [1]. Let $\mathbb{R}$ be the real field and $\mathbb{S}$ the unit circle thought of as a subgroup of the multiplicative group of the complex field $\mathbb{C}$. Let $\Gamma(\mathbb{R}) \leq \mathbb{S}$ be a subgroup. Let $\Gamma(\mathbb{R})^{[n]} = \{g^n : g \in \Gamma(\mathbb{R})\}$. With reference to [6], $\Gamma(\mathbb{R})$ is said in [1] to have the Lang property if, for any finite $n \geq 1$ and algebraic set $X \subseteq \mathbb{C}^n$, $X \cap \Gamma(\mathbb{R})^n$ is a finite union of cosets of subgroups of $\Gamma(\mathbb{R})^n$. Assume $\Gamma(\mathbb{R})$ satisfies the following three conditions:

- $|\Gamma(\mathbb{R})| = \aleph_0$,
- $|\Gamma(\mathbb{R})/\Gamma(\mathbb{R})^{[n]}| < \aleph_0$ for every finite $n \geq 1$,

• $\Gamma(\mathbb{R})$ has the Lang property.

Let $L = \{<, +, \cdot, 0, 1, Re(g), Im(g)\}_{g\in\Gamma(\mathbb{R})}$ where $Re(g)$ and $Im(g)$ are suggestively named constant symbols for the real part and the imaginary part of each member of $\Gamma(\mathbb{R})$. Let T be the resulting L-theory of $\mathbb{R}$. Let $Re : \mathbb{S} \to \mathbb{R}$ be the function which assigns to each member of $\mathbb{S}$ its real part. Let P be a new unary predicate and $L_P = L \cup \{P\}$. Interpret P such that $P(\mathbb{R}) = Re(\Gamma(\mathbb{R}))$. Let T_P be the resulting L_P-theory of $\mathbb{R}$. Let Γ be a new binary predicate and $L_\Gamma = L \cup \{\Gamma\}$. Let the suggestively named $\Gamma(\mathbb{R})$ be the interpretation of Γ in $\mathbb{R}$. Let T_Γ be the resulting L_Γ-theory of $\mathbb{R}$. As is noted in [1], $\Gamma(\mathbb{R}) = Re^{-1}(Re(\Gamma(\mathbb{R})))$ and so T_P and T_Γ are definitionally equivalent.

T_Γ was introduced and studied by Belegradek and Zilber in [1]. They gave axioms for it and proved a near model completeness result. Expecting a positive answer, they asked if T_Γ has NIP. We use Theorem 2.1 to obtain a positive answer. A similar result is given in [5] for a theory of the real field with a multiplicative subgroup which has the Mann property, this theory having been studied by van den Dries and Günaydin in [4].

We check that T_P satisfies the assumptions of Theorem 2.1. Let $(M, P(M)) \prec (N, P(N)) \prec (\overline{M}, P(\overline{M})) \models T_P$ be such that all three models are sufficiently saturated. Since T is an expansion by constants of the theory of real closed ordered fields, it is clear that conditions (i) and (iv) are satisfied. We deduce conditions (ii) and (iii) from the results in [1]. The argument overlaps with the reasoning in [1]. Since the predicate Γ is $\emptyset$-definable in T_P we shall feel free to use it. Re will now denote the real part map from the unit circle in the big model $(\overline{M}, P(\overline{M}))$. So $\Gamma(\overline{M}) = Re^{-1}(P(\overline{M}))$. Since L contains constant symbols for all real parts and all imaginary parts of members of $\Gamma(\mathbb{R})$, we may assume $\Gamma(\mathbb{R}) \leq \Gamma(M)$. Let $\Gamma(\overline{M})_d$ be the largest divisible subgroup of $\Gamma(\overline{M})$. As is observed in [1], $\Gamma(\overline{M})_d$ has a direct complement H in $\Gamma(\overline{M})$ such that $|H| \leq 2^{\aleph_0}$. Let $\Gamma(\overline{M})^T_d$ be the torsion subgroup of $\Gamma(\overline{M})_d$. Clearly $|\Gamma(\overline{M})^T_d| \leq \aleph_0$. Let $K = \{h \times g : h \in H \text{ and } g \in \Gamma(\overline{M})^T_d\}$. Given the sufficient saturation of M, we may assume $K \leq \Gamma(M)$.

The following is proved in [1] by means of a back-and-forth argument.

Theorem 4.1. *Let $F \leq \Gamma(\overline{M})$ be a subgroup with the following properties:*
(α) $F = \{k \times d : k \in K \text{ and } d \in D\}$ for some divisible subgroup $D \leq \Gamma(\overline{M})$,
(β) $|F| \leq 2^{\aleph_0}$,
(γ) $\Gamma(\mathbb{R}) \leq F$.

Let $n \geq 1$ be finite and let $\bar{a} \in \overline{M}^n$ be acl_L-independent over $P(\overline{M})$. Then $tp(F\bar{a})$ is implied by the information so far mentioned in conjunction with $tp_L(F\bar{a})$.

Let $b \in \overline{M} \setminus acl_L(NP(\overline{M}))$. Let $\bar{g}$ be a finite tuple from $P(N)$ and $\bar{c}$ a finite tuple from N such that $\bar{c}$ is acl_L-independent over $P(N)$ and hence over $P(\overline{M})$. Let $\widetilde{F} \leq \Gamma(N)$ satisfy conditions (α), (β) and (γ) and be such that $\bar{g} \in dcl_L(\widetilde{F})$. It follows from Theorem 4.1 that $tp(b/\bar{g}\bar{c})$ is implied by $tp_L(b/\widetilde{F}\bar{c})$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. Therefore $tp(b/N)$ is implied by $tp_L(b/N)$ in conjunction with the information that $b \notin acl_L(NP(\overline{M}))$. So condition (ii) is satisfied.

Let $n \geq 1$ be finite and $\bar{f} \in P(\overline{M})^n$. Let $F \leq \Gamma(\overline{M})$ satisfy conditions (α), (β) and (γ) and be such that $\bar{f} \in acl(F)$. Let $F' \models tp(F/M)$ be such that $tp(F'/N)$ is finitely realisable in M. Let $\bar{g}$ be a finite tuple from $P(N)$ and $\bar{c}$ a finite tuple from N such that $\bar{c}$ is acl_L-independent over

$P(N)$ and hence over $P(\overline{M})$. Let $\widetilde{F} \leq \Gamma(N)$ satisfy conditions (α), (β) and (γ) and be such that $\bar{g} \in dcl_L(\widetilde{F})$. It is clear that the subgroup $\{a \times b : a \in F' \text{ and } b \in \widetilde{F}\}$ also satisfies conditions (α), (β) and (γ). It is also clear that this is a consequence of $tp(F'/M)$ in conjuction with $tp(\widetilde{F}/M)$. It follows from Theorem 4.1 that $tp(F'/\bar{g}\bar{c})$ is implied by $tp_L(F'/\widetilde{F}\bar{c})$ in conjunction with $tp(F'/M)$. Therefore $tp(F'/N)$ is implied by $tp_L(F'/N)$ in conjunction with $tp(F'/M)$. Setting $\widehat{f} = Re(F)$, condition (iii) is satisfied with $(N', P(N')) = (N, P(N))$ and $\kappa \leq 2^{\aleph_0}$.

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