A THEORY OF BRANCHES FOR ALGEBRAIC CURVES

TRISTRAM DE PIRO

Abstract. This paper develops some of the methods of the "Italian School" of algebraic geometry in the context of infinitesimals. The results of this paper have no claim to originality, they can be found in [10], we have only made the arguments acceptable by modern standards. However, as the question of rigor was the main criticism of their approach, this is still a useful project. The results are limited to algebraic curves. As well as being interesting in their own right, it is hoped that these may also help the reader to appreciate their sophisticated approach to algebraic surfaces and an understanding of singularities. The constructions are also relevant to current research in Zariski structures, which have played a major role both in model theoretic applications to diophantine geometry and in recent work on non-commutative geometry.

1. Introduction, Preliminary Definitions, Lemmas and Notation

We begin this section with the preliminary reminder to the reader that the following results are concerned with *algebraic* curves. However, the constructions involved are geometric and rely heavily on the techniques of Zariski structures, originally developed in [12] and [3]. One might, therefore, speculate that the results could, in themselves, be used to develop further the general theory of such structures. Our starting point is the main Theorem 17.1 of [12], also formulated for Zariski geometries in [3];

Theorem 1.1. *Main Theorem 17.1 of* [12]

Let M be a Zariski structure and C a presmooth Zariski curve in M. If C is non-linear, then there exists a nonconstant continuous map;

$f : C \rightarrow P^1(K)$

The author was supported by a Marie Curie Research Fellowship.

Moreover, f is a finite map ($f^{-1}(x)$ is finite for every $x \in C$), and for any n, for any definable $S \subseteq C^n$, the image $f(S)$ is a constructible subset (in the sense of algebraic geometry) of $[P^1(K)]^n$.

Remarks 1.2. *Here, K denotes an algebraically closed field, we refer the reader to the original paper for the remaining terminology of the Theorem.*

Using this theorem, one can already see that there exists a close connection between a geometric theory of *algebraic* curves and Zariski curves. We begin this section by pointing out some of the main obstacles to developing the results of this paper in the context of Zariski curves, leaving the resolution of the main technical problems for another occasion. This discussion continues up to (††), when we introduce the main notation and preliminary lemmas of the paper.

In Section 2, the first major obstacle that we encounter is a suitable generalisation of the notion of a linear system. Using Theorem 1.1, any linear system Σ of algebraic hypersurfaces on $[P^1(K)]^n$ will define a linear system of Zariski hypersurfaces on C^n , by composing with the finite cover f $(*)$. One would then expect to be able to develop much of the theory of g_n^r given in Section 2 for such systems, applied to a Zariski curve $S \subseteq C^n$. This follows from the following observations. First, there exists a generalised Bezout's theorem, holding in Zariski structures, see the paper [12], hence one might hope to obtain an analogous result to Theorem 2.3. Secondly, one would expect that the local calculations, by algebraic power series, which we used in Lemma 2.10, would transfer to intersections on S. This uses the fact that, at a point where f is unramified, S and the algebraic curve $f(S)$ are *locally* isomorphic, in the sense of infinitesimal neighborhoods, and, at a point p where f is ramified, with multiplicity r, we have the straightforward relation;

$$I^{(\Sigma)}_{italian}(p, S, f^*(\phi_\lambda)) = rI^{(\Sigma)}_{italian}(f(p), f(S), \phi_\lambda)$$

where I have used the notation of Theorem 2.3, ϕ_λ denotes a hypersurface in $[P^1(K)]^n$ and $f^*(\phi_\lambda)$ denotes its inverse image in S^n.

In Section 3, the notion of a multiple point is introduced. As this is defined locally, one would expect this definition to be generalisable to Zariski curves. Also, the geometric notion of 2 algebraic curves being biunivocal has a natural generalisation to Zariski curves. However, the main result of Section 3, that any algebraic curve is birational(biunivocal) to an algebraic curve without multiple points, is not so easily transferred. This follows from the simple observation that there exist Zariski curves S, which *cannot* be embedded in a projective space $P^n(K)$, see Section 10 of [3]. This failure of Theorem 3.3 could be explained alternatively, by noting that the combinatorics involved must fail for multiplicity calculations using Zariski curves. The solution to this problem seems to be quite difficult and one must presumably attempt to resolve it by introducing a larger class of hypersurfaces than that defined by a linear system Σ, as in $(*)$ above. See also the remarks below.

In Section 4, the method of Conic projections is not immediately transferable to Zariski structures, as one needs to define what is meant by a *line* in this context. However, we note the following geometric property of a line in relation to other algebraic curves;

Theorem 1.3. *(Luroth) Let $f : l \rightarrow C$ be a finite morphism of a line l onto a projective algebraic curve C. Then f is biunivocal.*

Proof. See [2]. □

This theorem in fact characterises a line l up to birationality. Its proof requires a global topological property of the line, namely that the genus of l is zero. Although the above property may be formulated for Zariski curves, it does not guarantee the *existence* of such a curve $S \subseteq (C^n)^{eq}$, which is not trivially biunivocal to an *algebraic* line. One would expect the solution of this problem to require more advanced techniques, such as a geometric definition of the genus of a Zariski curve. Severi, in fact, gives such a definition for algebraic curves in [10] and one might hope that his definition would generalise to Zariski curves. One could then hope to extend the methods of Conic projections in this context.

The results of Section 5 rely centrally on the main result of Section 3, hence their generalisation to Zariski curves require a resolution of the problems noted above. One should observe that the Italian geometers definition of a branch is very different to a local definition using algebraic power series, see also the remarks later in this section, hence cannot be straightforwardly generalised using Theorem 1.1. Given a Zariski curve $S \subset C^n$ and $p \in S$, let $\{\gamma_p^1, \ldots, \gamma_p^n\}$ enumerate the branches of $f(p) \in f(S)$. We can then define;

$$\overline{\gamma}_p^j = \{x \in S \cap \mathcal{V}_p : f(x) \in \gamma_p^j\}$$

In the case when f is unramified (in the sense of Zariski structures) at p, this would give an adequate definition of a branch $\overline{\gamma}_p^j$ for the Zariski curve C. However, the definition is clearly inadequate when p is a point of ramification and requires a generalisation of the methods of Section 5.

The results of Section 6 depend mainly on Cayley's classification of singularities. As the main technical tool used in the proof of this result is the method of algebraic power series, by using Theorem 1.1, one would expect the results given in this Section to generalise more easily to Zariski curves.

(††) We will work in the language $\mathcal{L}_{spec}$, as defined in [7]. $P(L)$ will denote $\bigcup_{n\geq 1} P^n(L)$ where L is an algebraically closed field. Unless otherwise stated, we will assume that the field has characteristic 0, this is to avoid problems concerning Frobenius. The results that we prove in this paper hold in arbitrary characteristic, if we avoid exceptional cases, however we need to make certain modifications to the proof. We will discuss these modifications in the final section of the paper.

We assume the existence of a universal specialisation $P(K) \rightarrow_\pi P(L)$ where K is algebraically closed and $L \subset K$. Given $l \in L$, we will denote its infinitesimal neighborhood by $\mathcal{V}_l$, that is $\pi^{-1}(l)$. As we noted in the paper [7], it is not strictly necessary to consider a universal specialisation when defining the non-standard intersection multiplicity of curves, one need only consider a prime model of the theory T_{spec}. However, some of our proofs will require more refined infinitesimal arguments which are not first order in the language $\mathcal{L}_{spec}$ and therefore cannot immediately be transferred to a prime model. We will only refer to the non-standard model when using infinitesimal arguments.We assume the reader is familiar with the arguments employed in the papers [5], [6] and [12]. Of particular importance are the following notions;

(i). The technique of Zariski structures. We assume that $P^n(L)$ may be considered as a Zariski structure in the topology induced by algebraically closed subvarieties. When referring to the dimension of an algebraically closed subvariety V, we will use the model theoretic definition as given in the paper [5]. We will assume the reader is acquainted with the notions surrounding the meaning of "generic" in this context.

(ii). The method of algebraic power series and their relation to infinitesimals. This technique was explored extensively in the papers [5] and [6]. In the following paper, we will use power series methods to parametrise the branches of algebraic curves without being overly rigorous. By a branch in this context, we refer to the "Newtonian definition" rather than the one used by the Italian school of algebraic geometry, which is the subject of this paper.

(iii). The non-standard statement and proof of Bezout's theorem for projective algebraic curves in $P^2(L)$. This was given in the paper [6].

We will assume that L has infinite transcendence degree and therefore has the property;

Given any subfield $L_0 \prec L$ of finite transcendence degree and an integer $n \geq 1$, we can find $\bar{a}_n \in P^n(L)$ which is generic over L_0. (*)

We refer the reader to (i) above for the relevant definition of generic. In general, when referring to a generic point, we will mean generic with respect to some algebraically closed field of finite transcendence degree. This field will be the algebraic closure of the parameters defining any algebraic object given in the specific context.

By a projective algebraic curve, we will mean a closed irreducible algebraic subvariety C of $P^n(L)$ for some $n \geq 1$ having dimension 1. Occasionally, we will consider the case when C has distinct irreducible components $\{C_1, \ldots, C_r\}$, which will be made clear in a given situation. In both cases, we need only consider the usual Zariski topology on $P^n(L)$ as given in (i) above. By a plane projective algebraic curve, we will mean a projective algebraic curve contained in $P^2(L)$. By a projective line l in P^n, we will mean a projective algebraic curve isomorphic to $P^1(L)$. Any distinct points $\{p_1, p_2\}$ in $P^n(L)$ determine a unique projective line denoted by $l_{p_1p_2}$. We will call the line generic if there exists a generic pair $\{p_1p_2\}$ determining it. We occasionally assume the existence of the closed algebraic variety $I \subset (P^n \times P^n) \setminus \Delta \times P^n$, which parametrises the family of lines in P^n, defined by;

$I(a,b,y)\equiv y\in l_{ab}$

In order to see that this does define a closed algebraic variety, take the standard open cover $\{U_i:=(X_i\neq 0):0\leq i\leq n\}$ of P^n. Then observe that $I\cap((U_i\times U_j)\setminus\Delta\times P^n)$ is locally trivialisable by the maps;

$\Theta_{ij}:(U_i\times U_j)\setminus\Delta\times P^1\rightarrow I\cap(U_i\times U_j\times P^n)$

$\Theta_{ij}(a,b,[t_0:t_1])=$

$[t_0\frac{a_0}{a_i}+t_1\frac{b_0}{b_j}:\ldots:t_0+t_1\frac{b_i}{b_j}:\ldots:t_0\frac{a_j}{a_i}+t_1:\ldots:t_0\frac{a_n}{a_i}+t_1\frac{b_n}{b_j}]$

and the transition functions $\Theta_{ijkl}=\Theta_{ij}\circ\Theta_{kl}^{-1}$ are algebraic. In general, we will leave the reader to check in the course of the paper that certain naturally defined algebraic varieties are in fact algebraic. By a projective plane P in P^n, we will mean a closed irreducible projective subvariety of P^n, isomorphic to $P^r(L)$, for some $0\leq r\leq n$. We did, however, use a special notation for a plane projective curve, as defined above. Any sequence of points $\bar{a}$ determines a unique projective plane $P_{\bar{a}}$, defined as the intersection of all planes containing $\bar{a}$. We will call a sequence $\{p_0,\ldots,p_r\}$ linearly independent if $P_{p_0,\ldots,p_r}$ is isomorphic to $P^r(L)$. As before, if $U\subset(P^n)^{r+1}$ defines the open subset of linearly independent elements, we can define the cover $I\subset U\times P^n$, which parametrises the family of r-dimensional planes in P^n;

$I(p_0,\ldots,p_r,y)\equiv y\in P_{p_1,\ldots,p_r}$

As before, one can see that this is a closed projective algebraic subvariety of $U\times P^n$.

By a non-singular projective algebraic curve, we mean a projective algebraic curve $C\subset P^n$ which is non-singular in the sense of [2] (p32). Given any point $p\in C$, we then define its tangent line l_p as follows;

By Theorem 8.17 of [2], there exist homogeneous polynomials $\{G_1,\ldots,G_{n-1}\}$ such that C is defined in an affine open neighborhood U of p in P^n by the homogenous ideal $J=<G_1,\ldots,G_{n-1}>(*)$. Let $dG_j=\frac{\partial G_j}{\partial X_0}X_0+\ldots+\frac{\partial G_j}{\partial X_n}X_n$ be the differential of G_j. Then dG_j defines a family of hyperplanes, parametrised by an open neighborhood of p in C. If $x_i=\frac{X_i}{X_0}$ is a choice of affine coordinate system containing p and

$G_j^{res}(x_1,\ldots,x_n)=\frac{G_j}{X_0^{deg(G_j)}}$, we define $dG_j^{res}=\frac{\partial G_j^{res}}{\partial x_1}dx_1+\ldots+\frac{\partial G_j^{res}}{\partial x_n}dx_n$ and $J^{res}=<G_1^{res},\ldots,G_{n-1}^{res}>$. Then dG_j^{res} defines a family of *affine* hyperplanes, parametrised by an open neighborhood of p in C. We claim that $dG_j^{res}(p)=dG_j(p)^{res}$, this follows by an easy algebraic calculation, using the fact that;

$\frac{\partial G_j}{\partial X_0}(p)p_0+\ldots+\frac{\partial G_j}{\partial X_n}(p)p_n=0$

The differentials $\{dG_1^{res},\ldots,dG_{n-1}^{res}\}$ are independent at p, by the same Theorem 8.17 of [2], hence $\bigcap_{1\leq j\leq n-1}dG_j(p)$, defines a line $l_p\subset P^n$, which we call the tangent line. We need to show that the definition is independent of the choice of $\{G_1,\ldots,G_{n-1}\}$. Suppose that we are given another choice $\{H_1,\ldots,H_{n-1}\}$. Again, using Theorem 8.17 of [2], we can find a matrix $(f_{ij})_{1\leq i,j\leq n-1}$ of polynomials in $R(U)$, for some affine open neighborhood U of p in P^n $(**)$, such that the matrix of values $(f_{ij}(p))_{1\leq i,j\leq n-1}$ is invertible and;

$H_i^{res}=\sum_{1\leq j\leq n-1}f_{ij}G_j^{res} \pmod{(J^{res}(U))^2}$

Then, by properties of differentials, and the fact that, for $g\in(J^{res}(U))^2$, $dg(p)\equiv 0$, we have;

$dH_i^{res}(p)=\sum_{1\leq j\leq n-1}f_{ij}(p)dG_j^{res}(p)$

This implies that $\bigcap_{1\leq j\leq n-1}dG_j(p)=\bigcap_{1\leq j\leq n-1}dH_j(p)$ as required.

We define the tangent variety $Tang(C)$ to be $\bigcup_{x\in C}l_x$. We claim that this is a closed projective subvariety of P^n. In order to see this, using the notation of the above argument, let $\{U_i\}$ be an affine cover of P^n and let $\{G_{i1},\ldots,G_{ij},\ldots,G_{i,n-1}\}$ be homogeneous polynomials with the properties $(*)$ and $(**)$ given above. Let $W\subset P^n\times P^n$ be the closed projective variety, defined on $P^n\times U_i$ by;

$W_i(\bar{X},\bar{Y})\equiv U_i(\bar{Y})\wedge\bigwedge_{1\leq j\leq n-1}G_{ij}(\bar{Y})=0\wedge\bigwedge_{1\leq j\leq n-1}dG_{ij}(\bar{Y}).\bar{X}=0$

Then, by completeness, the variety $V(\bar{X})\equiv(\exists\bar{Y})W(\bar{X},\bar{Y})$ is a closed projective variety of P^n. The above argument shows that $V=Tang(C)$.

We will require a more sophisticated notation when considering hypersurfaces H of $P^n(L)$ for $n\geq 1$. Namely, by a hypersurface of degree

m, we will mean a homogenous polynomial $F(X_0, X_1, \ldots, X_n)$ of degree m in the variables $[X_0 : X_1 : \ldots : X_n]$. In the case when F is irreducible or has distinct irreducible factors $\{F_1, \ldots, F_r\}$, such a hypersurface may be considered as the union of r distinct irreducible closed algebraic subvarieties of codimension 1 in $P^n(L)$. We will then refer to the hypersurface as having distinct irreducible components. Again, we can understand such hypersurfaces using the usual Zariski topology as given in (i) above. By a hyperplane, we will mean an irreducible hypersurface isomorphic to $P^{n-1}(L)$. Equivalently, a hyperplane is defined by a homogeneous polynomial of degree 1. In general, let $F = F_1^{n_1} \centerdot \ldots F_j^{n_j} \centerdot \ldots F_r^{n_r}$ be the factorisation of F into irreducibles for $1 \leq j \leq r$. We will want to take into account the "non-reduced" character of F if some $n_j \geq 2$. Therefore, given an irreducible homogeneous polynomial G of degree m and an integer $s \geq 1$, we will refer to G^s as an s-fold component of degree ms. Geometrically, we interpret G^s as follows;

Consider the space of *all* homogenous polynomials of degree ms parametrised by $P^N(L)$. Let $W \subset P^N \times P^n$ be the irreducible projective variety defined by $W(\bar{x}, \bar{y})$ iff $\bar{y} \in Zero(F_{\bar{x}})$, where $F_{\bar{x}}$ is the homogenous polynomial (defined uniquely up to scalars) by the parameter $\bar{x}$. The coefficients of the homogeneous polynomial G^s determine uniquely an element $\bar{a}$ in P^N and we can consider G^s as the fibre $W(\bar{a}) \subset P^n$. In the Zariski topology, this consists of the variety defined by the irreducible polynomial G. However, in the language $\mathcal{L}_{spec}$, we can realise its non-reduced nature using the following lemmas;

Lemma 1.4. *Let $S \subset P^n$ be an irreducible hypersurface and C a projective algebraic curve, then, if C is not contained in S, $S \cap C$ consists of a finite non-empty set of points.*

Proof. The proof follows immediately from the Projective Dimension Theorem, see for example [2]. □

Lemma 1.5. *Let $G = 0$ define an irreducible hypersurface of degree m. Let l be a generic line, then l intersects $G = 0$ in precisely m points.*

Proof. Let $Par_l = P^n \times P^n \setminus \Delta$ be the parameter space for lines in P^n as defined above. Let Par_m be the projective parameter space for all homogeneous forms of degree m. Then we can form the variety $W \subset Par_l \times Par_m \times P^n$ given by;

$W(l, x, y) \;\; iff \;\; y \in l \cap Zero(G_x),$

where G_x is the homogenous polynomial corresponding to the parameter x. If l is chosen to be generic over the parameters defining G_x, then $l \cap Zero(G_x)$ is finite. It follows that there exists an open subset $U \subset Par_l \times Par_m$ consisting of parameters $\{l, x\}$ such that $l \cap G_x$ has finite intersection. As U is smooth, the finite cover W restricted to U is equidimensional and we may apply Zariski structure arguments, as was done in [6]. Considering W as a finite cover of U, if $y \in l \cap Zero(G_x)$, we define $Mult_y(l, G_x)$ to be $Mult_{(l,x,y)}(W/U)$ in the sense of Zariski structures. Using the notation in [6], we can also define $LeftMult_y(l, G_x)$ and $RightMult_y(l, G_x)$. In more geometric language;

$LeftMult_y(l, G_x) = Card(\mathcal{V}_y \cap l' \cap Zero(G_x))$

where l' is a generic infinitesimal variation of l in the nonstandard model $P(K)$.

$RightMult_y(l, G_x) = Card(\mathcal{V}_y \cap l \cap Zero(G_{x'}))$

where $x' \in \mathcal{V}_x$ is generic in Par_m, considered as a variety in the nonstandard model $P(K)$.

We now claim that there *exists* a line l with exactly m points of intersection with $G_{x_0} = G$. If G_{x_0} defines a projective algebraic curve in $P^2(L)$, the result follows immediately from an application of Bezout's theorem and the fact that there exists a line l having only (algebraically) transverse intersections with G_{x_0}. This result was shown in [6]. Otherwise, we obtain the case that G_{x_0} defines a plane projective curve by repeated application of Bertini's Theorem for Hyperplane Sections, see [4]. Namely, we can find a 2-dimensional plane P such that $P \cap G_{x_0}$ defines a plane projective algebraic curve C_{x_0}. We may assume that such a plane P is determined by a generic triple $\{a, b, c\}$. We have an isomorphism $\theta : P^2(L) \to P$ given by;

$\theta([Y_0, Y_1, Y_2]) = [\frac{a_0}{a_i}Y_0 + \frac{b_0}{b_j}Y_1 + \frac{c_0}{c_k}Y_2 : \ldots : \frac{a_n}{a_i}Y_0 + \frac{b_n}{b_j}Y_1 + \frac{c_n}{c_k}Y_2]$

If the form G_{x_0} is given by $\sum_{i_0+\ldots+i_n=m} x_{i_0\ldots i_n} X_0^{i_0} \ldots X_n^{i_n}$, then the corresponding curve C_{x_0} is given in coordinates $\{Y_0, Y_1, Y_2\}$ by;

$C_{x_0}(Y_0, Y_1, Y_2) = \sum_{j_0+j_1+j_2=m} F_{j_0 j_1 j_2}(a, b, c, \bar{x}_0) Y_0^{j_0} Y_1^{j_1} Y_2^{j_2}$

where $F_{j_0 j_1 j_2}$ is an algebraic function of $\{a, b, c, \bar{x}\}$ and, in particular, linear in the variables $\{\bar{x}\}$. By the assumption that C_{x_0} does not vanish identically on P, we must have that $F_{j_0 j_1 j_2}$ is not identically zero for some $(j_0 j_1 j_2)$ with $j_0 + j_1 + j_2 = m$. It follows that C_{x_0} defines a plane projective curve of degree m. Now, by the above argument, we may find a line $l_0 \subset P$ having exactly m intersections with C_{x_0}. Therefore, l_0 has exactly m intersections with G_{x_0} as well.

Now consider the fibre $U(x_0) = \{l : l \cap G_{x_0}\ is\ finite\}$ and restrict the cover W to $U(x_0)$. By the above calculation, we have that;

$\sum_{y \in l_0 \cap G_{x_0}} Mult_{l_0, y}(W/U(x_0)) = \sum_{y \in l_0 \cap G_{x_0}} LeftMult_y(l_0, G_{x_0}) \geq m\ (*)$

By elementary properties of Zariski structures, if l is chosen to be generic over the parameters defining G_{x_0}, then $(*)$ holds, with l replacing l_0, and, moreover, $LeftMult_y(l, G_{x_0}) = 1$ for each intersection $y \in l \cap G_{x_0}$. This implies that l has *at least* m points of intersection with $G_{x_0} = G$. In order to obtain equality, we now consider the fibre $U(l) = \{x : l \cap G_x\ is\ finite\}$ and restrict the cover W to $U(l)$. It will follow from the result given in the next section, the Hyperspatial Bezout Theorem, that, for *any* $x \in U(l)$;

$\sum_{y \in l \cap G_x} Mult_{x,y}(W/U(l)) = \sum_{y \in l \cap G_x} RightMult_y(l, G_x) = deg(l)deg(G_x)$

where, for a projective algebraic curve, degree is given by Definition 1.12. We clearly have that $deg(l) = 1$ and, by assumption, that $deg(G_{x_0}) = m$. Hence, we must have that l intersects G_{x_0} in exactly m points and, moreover, $RightMult_y(l, G_{x_0}) = 1$ for each intersection $y \in l \cap G_{x_0}$ as well.

The lemma is now proved but we can give an algebraic formulation of the result. Namely, if l is determined by the generic pair $\{a, b\}$, then we have an isomorphism $\theta : P^1(L) \to l$ given by;

$\theta([Y_0, Y_1]) = [\frac{a_0}{a_i} Y_0 + \frac{b_0}{b_j} Y_1 : \ldots : \frac{a_n}{a_i} Y_0 + \frac{b_n}{b_j} Y_1]$

If the form G_{x_0} is given by $\sum_{i_0+\ldots+i_n=m} x_{i_0 \ldots i_n} X_0^{i_0} \ldots X_n^{i_n}$, then the equation of G_{x_0} on l is given in coordinates $\{Y_0, Y_1\}$ by the homogenous polynomial;

$$P_{x_0}(Y_0,Y_1)=\sum_{j_0+j_1=m}F_{j_0j_1}(a,b,\bar{x}_0)Y_0^{j_0}Y_1^{j_1}$$

We clearly have, by the same reasoning as above, that P_{x_0} has degree m. The result of the lemma gives that P_{x_0} has exactly m roots in $P^1(L)$. Hence, these roots are all distinct. It follows that the scheme theoretic intersection of l and $G=0$ consists of m distinct reduced points. That is l intersects $G=0$ algebraically transversely.

□

We now extend the lemma to include reducible varieties. As before G is an irreducible algebraic form of degree m and we consider the power G^s for some $s\geq 1$. We will say that $y\in l\cap G^s=0$ is counted r-times if $RightMult_y(l,G^s)=r$ in the sense defined above.

Lemma 1.6. *Let $G=0$ define an irreducible hypersurface of degree m and let $s\geq 1$. Let l be a generic line, then l intersects $G^s=0$ in m points each counted s-times.*

Proof. We first show that l is generic with respect to $G=0$. Let $\lambda=\{\lambda_i\}$ be the parameters defining $G=0$ and let $\mu=\{\mu_j\}$ be the parameters defining $G^s=0$. We claim that λ and μ are interdefinable, considered as elements of the projective spaces Par_m and Par_{ms}, in the structure $P(L)$, which was considered in [5]. The fact that $\mu\in dcl(\lambda)$ is clear. Conversely, let α be an automorphism fixing μ and let $G_{\alpha(\lambda)}$ be the algebraic form of degree m obtained from $G=G_\lambda$. As $(G_\lambda)^s=G_\mu$, we have that $(G_{\alpha(\lambda)})^s=G_\mu$. Hence, $Zero(G_{\alpha(\lambda)})=Zero(G_\lambda)$. By the projective Nullstellenstatz, and the fact that both algebraic forms are irreducible, we must have that $<G_\lambda>=<G_{\alpha(\lambda)}>$. Hence, there must exist a unit U in the ring $L[X_0,\ldots,X_n]$ such that $G_\lambda=UG_{\alpha(\lambda)}$. As the only such units are scalars, we obtain immediately that $\alpha(\lambda)=\lambda$ in Par_{ms}. As $P(L)$ is sufficiently saturated, we obtain that $\lambda\in dcl(\mu)$. By the previous lemma, we obtain that l intersects $G=0$ in exactly m points, hence l intersects $G^s=0$ in exactly m points as well. Therefore, the first part of the lemma is shown. We now apply the Hyperspatial Bezout Theorem, see below, to obtain that;

$$\sum_{y\in l\cap G^s=0}RightMult_y(l,G^s)=deg(l)deg(G^s)=ms$$

Hence, the lemma is shown by proving that for *any* $y \in l \cap G^s = 0$, $RightMult_y(l, G^s) \geq s$. As before, we choose a generic plane P containing l, such that $C_{x_0} = P \cap G_{x_0}$ defines a projective algebraic curve of degree m. Using an explicit presentation of an isomorphism $\theta : P^2(L) \rightarrow L$, as was done above, we clearly have that the intersection $(P \cap (G_{x_0})^s = 0)$ consists of the non-reduced curve $C^s_{x_0} = 0$, which has degree ms. We now apply a result from the paper [6],(Lemma 4.16), which gives that, for $y \in l \cap C_{x_0}$;

$$RightMult_y(l, C^s_{x_0}, Par_{Q_{ms}}) = s \,.\, RightMult_y(l, C_{x_0}, Par_{Q_{ms}}) \geq s$$

where we have use the fact that the parameter defining $C^s_{x_0}$ moves in the projective parameter space Q_{ms} defining plane projective curves of degree ms in $P^2(L)$. We now claim that, for $y \in l \cap G_{x_0} = 0$;

$$RightMult_y(l, G^s_{x_0}) \geq RightMult_y(l, C^s_{x_0}, Par_{Q_{ms}}) \; (*)$$

Recall that, given G_x an algebraic form of degree m, the restriction of G_x to the plane P_{abc} is given by the formula;

$$C_x(Y_0, Y_1, Y_2) = \sum_{j_0+j_1+j_2=m} F_{j_0 j_1 j_2}(a, b, c, \bar{x}) Y_0^{j_0} Y_1^{j_1} Y_2^{j_2}$$

We first claim that each $F_{j_0 j_1 j_2}$ is *not* identically zero $(**)$. Let H_x be a hyperplane given in coordinates by;

$$H_x = \sum_{r=0}^{n} x_i X_i = 0$$

Then the restriction of H_x to P_{abc} is given by the plane;

$$P_x = \left(\sum_{r=0}^{n} \frac{x_r a_r}{a_i}\right) Y_0 + \left(\sum_{r=0}^{n} \frac{x_r b_r}{b_j}\right) Y_1 + \left(\sum_{r=0}^{n} \frac{x_r c_r}{c_k}\right) Y_2$$

By elementary linear algebra, we can find hyperplanes $\{H_0, H_1, H_2\}$ whose restriction to P_{abc} define the planes $\{Y_0 = 0, Y_1 = 0, Y_2 = 0\}$. It follows by direct calculation that the algebraic form of degree m defined by $H_0^{j_0} H_1^{j_1} H_2^{j_2}$ restricts to the curve of degree m defined by $Y_0^{j_0} Y_1^{j_1} Y_2^{j_2} = 0$. Hence, $(**)$ is shown. Now consider the function;

$$\overline{F} = \{F_{j_0 j_1 j_2}\} : Par_m \rightarrow Par_{Q_m}$$

By earlier remarks, the algebraic function $\overline{F}$ is linear in the variables $\{x_i\}$ and defined over $\{a, b, c\}$. Hence, its image defines a plane

$P \subset Par_{Q_m}$. By the above calculation, this plane P contains the linearly independent set $\{p_{j_0j_1j_2} : j_0 + j_1 + j_2 = m\}$ where $C_{p_{j_0j_1j_2}}$ defines the curve $Y_0^{j_0}Y_1^{j_1}Y_2^{j_2} = 0$. Hence, $\overline{F}$ is surjective. Moreover, by elementary facts about linear maps, the fibres of $\overline{F}$ are equidimensional. We can then show $(*)$. Suppose that;

$$RightMult_y(l, C_{x_0}^s, Par_{Q_{ms}}) = k$$

Then one can find $x' \in \mathcal{V}_{x_1} \cap Par_{Q_{ms}}$, generic over the parameter x_1 defining $C_{x_0}^s$, such that $C_{x'}$ intersects l in the distinct points $\{y_1, \ldots, y_k\} \subset \mathcal{V}_y$. As x_1 is regular for the cover $\overline{F}$, if x_2 defines $G_{x_0}^s$, we can find $x'' \in \mathcal{V}_{x_2} \cap Par_{ms}$, such that $\overline{F}(x'') = x'$. The algebraic form defined by $G_{x''}$ then intersects the plane P_{abc} in the curve $C_{x'}$, hence it must intersect the line l in the distinct points $\{y_1, \ldots, y_k\} \subset \mathcal{V}_y$ as well. This implies that;

$$RightMult_y(l, G_{x_0}^s) \geq k$$

Hence $(*)$ is shown. The lemma is then proved. In this lemma we have not shown anything interesting algebraically. Namely, if one considers the restriction of $G_{x_0}^s$ to l, we obtain the homogeneous polynomial $P_{x_0}^s$. By the previous lemma, P_{x_0} has m distinct roots in $P^1(L)$, hence $P_{x_0}^s$ has m distinct roots with multiplicity s. Therefore, the scheme theoretic intersection of l with $G_{x_0}^s$ consists of m distinct copies of the non-reduced scheme $L[t]/(t)^s$. The usefulness of the result will be shown in the following lemmas. $\square$

Remarks 1.7. *Note that the latter part of the argument in fact shows that, for* any *line intersecting $G_{x_0}^s$ in finitely many points, we must have that each point of intersection y is counted at least s-times.*

Lemma 1.8. *Let $F = 0$ define a hypersurface of degree k. Let $F = F_1^{n_1} \cdot \ldots \cdot F_j^{n_j} \cdot \ldots \cdot F_r^{n_r}$ be its factorisation into irreducibles, with $degree(F_j) = m_j$. Then there exists a line l, intersecting each component F_j in exactly m_j points, each counted n_j times, with the property that the sets $\{(F_j \cap l) : 1 \leq j \leq r\}$ are pairwise disjoint. Moreover, the set of lines having this property form a Zariski open subset of Par_l, defined over the parameters of $F = 0$.*

Proof. Let $(x_1, \ldots, x_r) \in Par_{m_1} \times \ldots \times Par_{m_r}$ be the tuple defining each reduced irreducible component of $F = 0$. By an elementary argument, extending the proof in Lemma 1.6, the tuple is interalgebraic with the tuple $x \in Par_k$ defining the hypersurface $F = 0$. Let

$\theta_{(x_j,m_j,n_j)}(y) \subset Par_l$ be the statement that a line l_y intersects F_j in exactly m_j points, each counted n_j times;

$\exists_{z_1\neq\ldots\neq z_{m_j}}[\bigwedge_{1\leq i\leq m_j} z_i \in l_y \cap F_{j,x_j} \wedge RightMult_{z_i}(l_y, F_{j,x_j}) = n_j \wedge$
. $\forall w(w \in l_y \cap F_{j,x_j} \rightarrow \bigvee_{1\leq i\leq m_j} z_i = w)]$

By definability of multiplicity in Zariski structures, Lemmas 1.5 and Lemmas 1.6, each $\theta_{(x_j,m_j,n_j)}(y)$ is definable over x_j and is a Zariski dense algebraic subset of Par_l. By the previous remark, the complement of $\theta_{(x_j,m_j,n_j)}(y)$ in the set of lines having finite intersection with F_j is given by;

$\exists w[w \in l_y \cap F_{j,x_j} \wedge RightMult_w(l_y, F_{j,x_j}) \geq n_j + 1]$

It follows that $\theta_{(x_j,m_j,n_j)}(y)$ defines a Zariski open subset of Par_l. Now let;

$\theta(y) = \bigwedge_{1\leq j\leq r} \theta_{(x_j,m_j,n_j)}(y)$

Then $\theta(y)$ defines a Zariski open subset of Par_l. Finally, let W be the the union of the pairwise intersections of the irreducible components F_j. Then, by elementary dimension theory, W is Zariski closed of dimension at most $n-2$. Hence, the condition on Par_l that a line passes through W defines a proper closed set over the parameters $(x_1, \ldots, x_r)$. Let $U(y)$ be the Zariski open complement of this set in Par_l. Then any line l satisfying $\theta(y) \wedge U(y)$ has the properties required of the lemma. □

Definition 1.9. *We will call a line l satisfying the conclusion of the lemma* transverse *to F.*

We will now give an alternative characterisation of transversality.

Lemma 1.10. *Let $F = 0$ define a hypersurface of degree k. Then a line l is transverse to F iff l intersects F in finitely many points and, for each $y \in l\cap(F = 0)$, $LeftMult_y(l, F) = 1$. Moreover, the notion of transversality may be formulated by a predicate in the language $\mathcal{L}_{spec}$, $Transverse_k \subset Par_k \times Par_l$;*

$Transverse_k(\lambda, l)$ *iff l is transverse to F_λ*

Proof. The first part of the proof is clear using previous results of this section. For the second part, use the results in Section 3 of [7]. □

We now have;

Lemma 1.11. *A Nullstellensatz for Non-Reduced Hypersurfaces*

Let $F = 0$ *define a hypersurface of degree* k *and let* $Zero(F) = Zero(F_{\lambda_1}) \cup \ldots \cup Zero(F_{\lambda_r})$ *be its geometric factorisation into irreducibles, using the Zariski topology. Let* $\sigma(\lambda_j, m_j, n_j) \subset Par_{m_j}$*, for* $1 \leq j \leq r$*, be the predicates defined in* $\mathcal{L}_{spec}$ *by;*

A transverse line to F_{λ_j} *intersects* F_{λ_j} *in exactly* m_j *points, each counted* n_j *times.*

Then the original homogeneous polynomial F *is characterised, up to scalars, by the sequence;*

$$(Zero(F_{\lambda_1}), \ldots, Zero(F_{\lambda_r}), \sigma(\lambda_1, m_1, n_1), \ldots, \sigma(\lambda_r, m_r, n_r))$$

Proof. By the proof of previous results from this section, the formulae $\sigma(\lambda_j, m_j, n_j)$, for $1 \leq j \leq r$, determine the multiplicity of each component F_{λ_j}. The result then follows by uniqueness of factorisation. $\square$

We will refer to a hypersurface as generic if the parameter defining it is generic in the parameter space of all hypersurfaces of the same degree.

Definition 1.12. *The degree of a projective algebraic curve* C *is the number of intersections with a generic hyperplane.*

We need to check this is a good definition. Let $(P^n)^*$ be the dual space of P^n. $(P^n)^*$ is the parameter space for all hyperplanes H in P^n. We have that;

$$\{a \in (P^n)^* : dim(C \cap H_a) \geq 1\}$$

is closed in $(P^n)^*$, hence, for generic $a \in (P^n)^*$, $C \cap H_a$ is finite (and non-empty). Choosing *some* generic a in $(P^n)^*$, let m be the number of intersections of H_a with C. Let $\theta(x) \subset (P^n)^*$ be the statement;

$$\exists_{x_1 \neq \ldots \neq x_m} (\bigwedge_{1 \leq i \leq m} x_i \in C \cap H_x \wedge \forall y (y \in C \cap H_x \rightarrow \bigvee_{1 \leq i \leq m} x_i = y))$$

$\theta(x)$ is algebraic and defined over $\emptyset$, hence, as it contains a, must be Zariski dense in $(P^n)^*$. In particular, it contains *any* generic a in $(P^n)^*$.

Definition 1.13. *The degree of a hypersurface F is the degree of the homogenous polynomial defining it. (See the above remarks)*

In the following paper, the notion of *birationality* between projective algebraic curves, will be central.

Definition 1.14. *We define a linear system Σ on P^r to be the collection of algebraic forms of degree k, for some $k \geq 1$, corresponding to a plane, which we will denote by Par_Σ, contained in Par_k, the parameter space of homogeneous polynomials of degree k. If Par_Σ has dimension n, we define a basis of Σ to be an ordered set of $n+1$ forms corresponding to a maximally independent set of parameters in Par_Σ. Equivalently, a basis of Σ is an ordered system;*

$$\{\phi_0(X_0,\ldots,X_r),\ldots,\phi_n(X_0,\ldots,X_r)\}$$

of homogeneous polynomials of degree k belonging to Σ which are independent, that is there do not exist parameters $\{\lambda_0,\ldots,\lambda_n\}$ such that;

$$\lambda_0\phi_0+\ldots+\lambda_n\phi_n\equiv 0$$

Definition 1.15. *Given a linear system Σ of dimension n on P^r, we define the base locus of the system Σ by;*

$$Base(\Sigma)=\{\bar{x}\in P^r:\phi_0(\bar{x})=\ldots=\phi_n(\bar{x})=0\}$$

for any basis of Σ. Given any 2 bases $\{\phi_0,\ldots,\phi_j,\ldots,\phi_n\}$ and $\{\psi_0,\ldots,\psi_i,\ldots,\psi_n\}$ of Σ, we can find an invertible matrix of scalars $(\lambda_{ij})_{0\leq i,j\leq n}$ such that;

$$\psi_i=\sum_{j=0}^n\lambda_{ij}\phi_j$$

Hence, the base locus of Σ is well defined. As a basis corresponds to a maximally independent sequence in Par_Σ, we could equivalently define;

$$Base(\Sigma)=\{\bar{x}\in P^r:\phi(\bar{x})=0\}$$

for every *algebraic form ϕ belonging to Σ.*

Lemma 1.16. *Let Σ be a linear system of dimension n and degree k on P^r. Then, a choice of basis B for Σ defines a morphism $\Phi_{\Sigma,B}$:*

$P^r \setminus Base(\Sigma) \rightarrow P^n$ with the property that $Image(\phi_{\Sigma,B})$ is not contained in any hyperplane section of P^n. Moreover, given any 2 bases $\{B, B'\}$ of Σ, there exists a homography $\theta_{B,B'} : P^n \rightarrow P^n$ such that;

$$\Phi_{\Sigma,B'} = \theta_{B,B'} \circ \Phi_{\Sigma,B}$$

Proof. Given a choice of basis $B = \{\phi_0, \ldots, \phi_n\}$ for Σ, one checks that the map defined by;

$$\Phi_{\Sigma,B}([X_0 : \ldots : X_r]) = [\phi_0(X_0, \ldots, X_r) : \ldots : \phi_n(X_0, \ldots, X_r)]$$

is a morphism with the required properties. Now suppose that $\{\phi_0, \ldots, \phi_j, \ldots, \phi_n\}$ and $\{\psi_0, \ldots, \psi_i, \ldots, \psi_n\}$ are 2 bases B and B' for Σ. Let $(\lambda_{ij})_{0 \leq i,j \leq n}$ be the matrix of scalars as given in Definition 1.15. Then one can define a homography $\theta_{B,B'}$ by;

$$\theta_{B,B'}([Y_0 : \ldots : Y_n]) = [\textstyle\sum_{j=0}^{n} \lambda_{0j} Y_j : \ldots : \sum_{j=0}^{n} \lambda_{ij} Y_j : \ldots : \sum_{j=0}^{n} \lambda_{nj} Y_j]$$

It is clear that this homography has the required property of the lemma.

□

Definition 1.17. *We define a rational map from P^r to P^n to be a morphism defined by a choice of basis for a linear system Σ.*

Remarks 1.18. *Given a linear system Σ, we will generally refer to a morphism given by Lemma 1.16 as simply Φ_Σ, leaving the reader to remember that a choice of basis is involved. As any 2 such choices differ by a homography, any properties of one morphism transfer directly to the other, so one hopes that this terminology will not cause confusion. More geometrically, observe that, if $x \in P^r \setminus Base(\Sigma)$, then the set of algebraic forms in Σ, vanishing at x, defines a hyperplane $H_x \subset \Sigma$. A choice of basis $\{\phi_0, \ldots, \phi_n\}$ for Σ identifies this hyperplane H_x with a point $[\phi_0(x), \ldots, \phi_n(x)]$ of the dual space P^{n*}.*

Definition 1.19. *We say that two projective algebraic curves C_1 and C_2 are birational if there exists $U \subset C_1$ and $V \subset C_2$, with U and V open in C_1 and C_2 respectively, such that U and V are isomorphic as algebraic varieties. We will use the notation $\Phi : C_1 \leftrightsquigarrow C_2$ for a birational map.*

We will require the following presentation of birational maps;

Lemma 1.20. *Let $C_1 \subset P^r$ and $C_2 \subset P^n$ be birational projective algebraic curves, as in Definition 1.16, with the property that no hyperplane section of P^r or P^n contains C_1 or C_2 respectively. Then we can find linear systems $\{\Sigma, \Sigma'\}$, rational maps;*

$$\phi_\Sigma : P^r \setminus Base(\Sigma) \rightarrow P^n \qquad \phi_{\Sigma'} : P^n \setminus Base(\Sigma') \rightarrow P^r$$

and open subsets $\{U', V'\}$ of $\{C_1, C_2\}$, which are disjoint from $\{Base(\Sigma), Base(\Sigma')\}$, such that the restrictions $\phi_\Sigma : U' \rightarrow V'$ and $\phi_{\Sigma'} : V' \rightarrow U'$ are (inverse) isomorphisms.

Proof. As usual, let $[X_0 : \ldots : X_r]$ and $[Y_0 : \ldots : Y_n]$ be homogeneous coordinates for P^r and P^n respectively. Taking the hyperplanes $X_0 = 0$ and $Y_0 = 0$, we can find affine presentations;

$$\frac{L[x_1,\ldots,x_r]}{J_1} = R(C_1 \setminus C_1 \cap (X_0 = 0)) = R(U'')$$

$$\frac{L[y_1,\ldots,y_n]}{J_2} = R(C_2 \setminus C_2 \cap (Y_0 = 0)) = R(V'')$$

where $\{U'', V''\}$ are open subsets of $\{C_1, C_2\}$, $\{J_1, J_2\}$ are prime ideals.

We can then find $U' \subset U'' \cap U$ and $V' \subset V'' \cap V$ such that U' and V' are isomorphic as algebraic subvarieties of C_1 and C_2 (consider the elements of U which are mapped to V'' by the original isomorphism). Now choose polynomials $F(\bar{x})$ and $G(\bar{y})$ such that;

$$\frac{L[x_1,\ldots,x_r]_F}{J_1'} = R(U') \qquad \frac{L[y_1,\ldots,y_n]_G}{J_2'} = R(V')$$

As $R(U') \cong R(V')$, we can find rational functions $\{\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})\}$ and $\{\psi_1(\bar{y}), \ldots, \psi_r(\bar{y})\}$ (with denominators powers of F and G respectively) defining morphisms;

$$\Phi : A^r \setminus \{F = 0\} \rightarrow A^n \qquad \Psi : A^n \setminus \{G = 0\} \rightarrow A^r$$

and representing the isomorphism $U' \cong V'$. We now show how to convert Φ into Φ_Σ. By equating denominators, we are able to write $\{\phi_1(\bar{x}), \ldots, \phi_n(\bar{x})\}$ as $\{\frac{p_1(\bar{x})}{q(\bar{x})}, \ldots, \frac{p_n(\bar{x})}{q(\bar{x})}\}$. Now make the substitutions $x_i = \frac{X_i}{X_0}$ in $\{p_1(\bar{x}), \ldots, p_n(\bar{x}), q(\bar{x})\}$ and multiply through by the highest power of X_0 to obtain homogeneous polynomials of the same degree $\{P_1(\bar{X}), \ldots, P_n(\bar{X}), Q(\bar{X})\}$. Let Σ be the linear system defined by the plane spanned by these homogeneous polynomials and define Φ_Σ by;

$$\rho Y_0 = Q(X_0, \ldots, X_r), \rho Y_1 = P_1(X_0, \ldots, X_r), \ldots, \rho Y_n = P_n(X_0, \ldots, X_r)$$

where ρ is a constant of proportionality. (This is an alternative notation for a map between projective spaces, used frequently in papers by the Italian geometers Castelnouvo, Enriques and Severi). By the assumption on $\{C_1, C_2\}$ concerning hyperplane sections, the homogeneous polynomials $\{Q, P_1, \ldots, P_n\}$ form a basis for Σ, hence this defines a rational map. Similarily, one can find a linear system Σ' and convert Ψ into a rational map $\phi_{\Sigma'}$. The rest of the properties of the lemma follow immediately from the construction. $\square$

We should also note the following equivalent criteria for birationality of projective algebraic curves;

Lemma 1.21. *Let C_1 and C_2 be projective algebraic curves. Then C_1 and C_2 are birational iff;*

(i). There is an isomorphism of function fields $L(C_1) \cong L(C_2)$.
(ii). (In characteristic 0) There exist a_1 generic in C_1, a_2 generic in C_2 and an algebraic relation $Rational(x, y)$ such that $a_1 \in dcl_{Rational}(a_2)$ and $a_2 \in dcl_{Rational}(a_1)$.
. (In charateristic p) One can use the same criteria but must pay attention to the presence of the Frobenius map, see the papers [5] *and* [8] *for details on how to resolve this issue.*

Definition 1.22. *Let $\Phi : C_1 \leftrightsquigarrow C_2$ be a birational map, as in Definition 1.19. We define the correspondence $\Gamma_\Phi \subset C_1 \times C_2$ associated to Φ to be;*

$$\overline{(Graph(\Phi) \subset U \times V)}$$

where, for W an algebraic subset of $C_1 \times C_2$, we let $\bar{W}$ denote its Zariski closure.

Definition 1.23. *Let C_1 and C_2 be projective algebraic curves. We will say that 2 birational maps $\Phi_1 : C_1 \leftrightsquigarrow C_2$ and $\Phi_2 : C_1 \leftrightsquigarrow C_2$ are equivalent if there exists $U \subset C_1$ such that Φ_1 and Φ_2 are both defined and agree on U. Clearly, equivalence of birational maps is an equivalence relation.*

Lemma 1.24. *Let $\Phi_1 : C_1 \leftrightsquigarrow C_2$ and $\Phi_2 : C_1 \leftrightsquigarrow C_2$ be equivalent birational maps, then $\Gamma_{\Phi_1} = \Gamma_{\Phi_2}$.*

Proof. Immediate from the definitions.

□

Definition 1.25. *We will denote the equivalence class of a birational map Φ by $[\Phi]$. By the above lemma, we can associate a correspondence $\Gamma_{[\Phi]}$ to an equivalence class of birational maps*

Lemma 1.26. *Obstruction to Birationality at Singular Points*

Let $\Gamma_{[\Phi]}$ be a birational correspondence between C_1 and C_2. If x is a non-singular point of C_1, there exists a unique corresponding point y of C_2 and vice-versa.

Proof. Let $U \subset C_1$ be the set of non-singular points of C_1. We can consider Γ_Φ as a cover of U. As U is smooth, we may apply the technique of Zariski structures for this cover. By birationality, if $x \in U$ is generic, there exists a unique $(xy) \in \Gamma_\Phi$ and, moreover, $Mult(y/x) = 1$. This last fact was shown, for example, in the paper [5] and given originally in [12]. By further properties of Zariski structures, again see either of the above, the total multiplicity of points in the cover over U is preserved, in particular, for *any* $x \in U$, there exists a unique corresponding $(xy) \in \Gamma_\Phi$. □

Remarks 1.27. *Note that non-singularity is not necessarily preserved when associating y to x in the above lemma. This motivates the following definition.*

Definition 1.28. *Given a birational correspondence $\Gamma_{[\Phi]}$, we define the canonical sets $U_{[\Phi]} \subset \Gamma_{[\Phi]}$,$V_{[\Phi]} \subset C_1$ and $W_{[\Phi]} \subset C_2$ to be the sets;*

$$U_{[\Phi]} = \{(x, y) \in \Gamma_{[\Phi]} : NonSing(x), NonSing(y)\}$$

$$V_{[\Phi]} = \pi_1(U_{[\Phi]})$$

$$W_{[\Phi]} = \pi_2(U_{[\Phi]})$$

Lemma 1.29. *Given a birational correspondence $\Gamma_{[\Phi]}$, there exists an isomorphism $\Phi_1 : V_{[\Phi]} \to W_{[\Phi]}$ such that $\Gamma_{[\Phi]} = \Gamma_{[\Phi_1]}$*

Proof. By an elementary result in algebraic geometry, see for example [2], a morphism $\Phi : U \subset C_1 \to P^n$, where U is an open subset of C_1, extends uniquely to the non-singular points of C_1. Combining this with Lemma 1.26, we obtain immediately the result. □

We now need to relate the canonical sets of a birational correspondence $\Gamma_{[\Phi]}$ with a *particular* presentation of $[\Phi]$ given by Lemma 1.20;

Definition 1.30. *Let ϕ_Σ be as in Lemma 1.20, we define the canonical open sets associated to ϕ_Σ to be;*

$$V_{\phi_\Sigma} = V_{[\Phi]} \setminus Base(\Sigma)$$

$$W_{\phi_\Sigma} = \phi_\Sigma(V_{\phi_\Sigma})$$

Lemma 1.31. *We have the following relations between canonical sets;*

$$V_{\phi_\Sigma} \subset V_{[\Phi]} \subset NonSing(C_1)$$

$$W_{\phi_\Sigma} \subset W_{[\Phi]} \subset NonSing(C_2)$$

$\phi_\Sigma : V_{\phi_\Sigma} \to W_{\phi_\Sigma}$ *is an isomorphism.*

V_{ϕ_Σ} *and* $Base(\Sigma)$ *are disjoint.*

Proof. The proof is an easy exercise. □

Remarks 1.32. *It would be desirable to find a particular presentation Φ_Σ of a birational class $[\Phi]$, for which $Base(\Sigma)$ is disjoint from the canonical set $V_{[\Phi]}$. In general, one can easily prove the weaker result;*

There exist 2 presentations Φ_{Σ_1} and Φ_{Σ_2} of a birational class $[\Phi]$, such that;

$$V_{[\Phi]} = V_{\Phi_{\Sigma_1}} \cup V_{\Phi_{\Sigma_2}}$$

$$W_{[\Phi]} = W_{\Phi_{\Sigma_1}} \cup W_{\Phi_{\Sigma_2}}$$

We also note that the choice of Σ presenting a birational class $[\Phi]$ is far from unique. For example, let $Id : P^1 \to P^1$ be the identity map. This isomorphism can be represented by any of the birational maps;

$$\phi_n[X_0 : X_1] = [X_0^n : X_0^{n-1}X_1] \ (n \geq 1)$$

Let Σ_n be the linear system of dimension 1 and degree n defined by the pair of homogeneous polynomials $\{X_0^n, X_0^{n-1}X_1\}$. Then $\phi_n = \Phi_{\Sigma_n}$ (with this choice of basis).

We finally note the following well known theorem, see for example [2];

Theorem 1.33. *Let C be a projective algebraic curve, then C is birational to a plane projective algebraic curve.*

2. A Basic Theory of g_n^r

We begin this section with the definition of intersection multiplicity used by the original Italian school of algebraic geometry.

Definition 2.1. *Let $C \subset P^w$ be a projective algebraic curve. Let Par_F be the projective parameter space for* all *hypersurfaces of a given degree e and let $U = \{\lambda \in Par_F : |C \cap F_\lambda| < \infty\}$ be the open subvariety of Par_F corresponding to hypersurfaces of degree e having finite intersection with C. For $\lambda \in U$, $p \in C \cap F_\lambda$, we define;*

$I_{italian}(p, C, F_\lambda) = Card(C \cap F_{\lambda'} \cap \mathcal{V}_p)$ *for* $\lambda' \in \mathcal{V}_\lambda$ *generic in* U.

Remarks 2.2. *That this* is *a rigorous definition follows from general properties of Zariski structures. The definition is the same as $RightMult_p(C, F_\lambda)$ which we considered in the previous section. We will often abbreviate the notation $I_{italian}(p, C, F_\lambda) = s$ by saying that p is* counted *s times for the intersection of C with F_λ.*

The basic theory of g_n^r relies principally on the following result;

Theorem 2.3. *Hyperspatial Bezout*

Let $C \subset P^w$ be a projective algebraic curve of degree d and F_λ a hypersurface of degree e having finite intersection with C. Then;

$\sum_{p \in C \cap F_\lambda} I_{italian}(p, C, F_\lambda) = de$

We first require the following lemma, preserving the notation from Definition 2.1;

Lemma 2.4. *Let H_λ be a generic hyperplane, then;*

$I_{italian}(p, C, H_\lambda) = 1$ *for* all $p \in C \cap H_\lambda$

and each $p \in C \cap H_\lambda$ is non-singular.

In the Italian terminology, each point p of intersection is counted once *or the intersection is transverse. Using the methods developed in Section 1, it is not difficult to prove that each point of intersection is transverse (using the scheme theoretic definition).*

Proof. Suppose, for contradiction, that $I_{italian}(p_1, C, H_\lambda) \geq 2$ for some $p_1 \in C \cap H_\lambda$. Let $\{p_1, \ldots, p_d\}$ be the total set of intersections, where d is the degree of C, see Definition 1.12. Then we can find $\lambda' \in \mathcal{V}_\lambda$ generic in U and a distinct pair $\{p_1^1, p_1^2\}$ in $\mathcal{V}_{p_1} \cap C \cap H_{\lambda'}$. By properties of Zariski structures, see [5] or [12], we can also find $\{p_2^1, \ldots, p_d^1\}$ such that $p_j^1 \in C \cap H_{\lambda'} \cap \mathcal{V}_{p_j}$ for $2 \leq j \leq d$. It follows from the definition of an infinitesimal neighborhood that, for $p \neq q$ with $\{p, q\} \subset P^w$, $\mathcal{V}_p$ and $\mathcal{V}_q$ are disjoint. Hence, $\{p_1^1, p_1^2, p_2^1, \ldots, p_d^1\}$ defines a distinct set of intersections of C with $H_{\lambda'}$. It follows that C and $H_{\lambda'}$ have at least $d+1$ intersections, contradicting the facts that λ' is generic in U and the degree of C is equal to d. For the second part of the lemma, observe that the set of nonsingular points $NonSing(C)$ is a dense open subset of C, defined over the field of definition of C. The condition that a hyperplane passes through at least one point of $(C \setminus NonSing(C))$ is therefore a union of finitely many proper hyperplanes $P_1 \cup \ldots \cup P_m$ contained in Par_H, also defined over the field of definition of C. As H_λ was chosen to be generic, its parameter λ cannot lie inside $P_1 \cup \ldots \cup P_m$. Hence, the result is shown.

□

We now complete the proof of Theorem 2.3;

Proof. Choose $\{\lambda_1, \ldots, \lambda_e\}$ independent generic tuples in $P^{w*} = Par_H$, the parameter space for hyperplanes on P^w. Let F_e be the form of degree e defined by;

$$F_e = H_{\lambda_1} \centerdot \ldots \centerdot H_{\lambda_e} = \Sigma_{i_0+\ldots+i_w=e} \lambda_{i_0\ldots i_n} Y_0^{i_0} \ldots Y_w^{i_w} = 0$$

We first claim that the intersections $C \cap H_{\lambda_j}$ are pairwise disjoint sets for $1 \leq j \leq e$. The condition that a hyperplane H_μ passes through at least one point of the intersection $\{(C \cap H_{\lambda_1}) \cup \ldots \cup (C \cap H_{\lambda_j})\}$ is a union of finitely many proper closed hyperplane conditions on Par_H, defined over the parameters of C and the tuple $\{\lambda_1, \ldots, \lambda_j\}$, for $1 \leq j \leq e-1$. As the tuples $\{\lambda_1, \ldots, \lambda_e\}$ were chosen to be generically independent in Par_H, the result follows. Now, by Lemma 2.4 and the definition of

F_e, we obtain a total number de of intersections between C and F_e. We claim that for each point p of intersection;

$I_{italian}(p, C, F_e) = 1$ $(*)$

This does not follow immediately from Lemma 2.4 as the parameter $\{\lambda_{i_0 \ldots i_n}\}$ defining F_e is allowed to vary in the parameter space Par_e of *all* forms of degree e. We prove the claim by reducing the problem to one about plane projective curves.

We use Lemma 1.20 and Theorem 1.33 to find a plane projective algebraic curve $C_1 \subset P^2$ and a linear system Σ such that $\Phi_\Sigma : C_1 \leftrightsquigarrow C$. Let $\{\phi_0, \ldots, \phi_w\}$ be a basis for Σ, defining the birational map Φ_Σ. We may suppose that each ϕ_i is homogenous of degree k in the variables $\{X_0, X_1, X_2\}$ for P^2. Let $\{V_{[\Phi]}, V_{\Phi_\Sigma}, W_{[\Phi]}, W_{\Phi_\Sigma}\}$ be the canonical sets associated to Γ_{Φ_Σ}, see Definitions 1.28 and 1.30. Note that the canonical sets are all definable over the data of Φ_Σ. Hence, we may, without loss of generality, assume that the point p given in $(*)$ above lies in W_{Φ_Σ} and its corresponding $p' \in C_1$ lies in V_{Φ_Σ}. In particular, p' defines a non-singular point of the curve C_1. Now, given an algebraic form F_μ of degree e;

$F_\mu = \Sigma_{i_0 + \ldots + i_w = e} \mu_{i_0 \ldots i_n} Y_0^{i_0} \ldots Y_w^{i_w} = 0$

we obtain a corresponding algebraic curve ψ_μ of degree ke on P^2 given by the equation;

$\psi_\mu = \Sigma_{i_0 + \ldots + i_w = e} \mu_{i_0 \ldots i_n} \phi_0^{i_0} \ldots \phi_w^{i_w} = 0$ $(\dagger)$

We claim that;

$I_{italian}(p, C, F_\mu) \leq I_{italian}(p', C_1, \psi_\mu)$ $(**)$

For suppose that p is counted s-times for the intersection of C with F_μ, then we can find $\mu' \in \mathcal{V}_\mu$ generic in U such that $C \cap F_{\mu'} \cap \mathcal{V}_p$ consists of the distinct points $\{p_1, \ldots, p_s\}$. By elementary properties of infinitesimals, $\{p_1, \ldots, p_s\} \subset W_{\Phi_\Sigma}$, hence we can find a corresponding distinct set $\{p'_1, \ldots, p'_s\}$ in V_{Φ_Σ}. By the defining property of a specialisation ,the fact that the correspondence $\Gamma_{[\Phi_\Sigma]}$ is closed and the definition of $\psi_{\mu'}$ we must have that $\{p'_1 \ldots, p'_s\} \subset C_1 \cap \mathcal{V}_p \cap \psi_{\mu'}$. As the map $\theta : Par_e \rightarrow Par_{ke}$, defined by $(\dagger)$, is algebraic, it follows that $\psi_{\mu'}$ defines an infinitesimal variation of ψ_μ in the space of all algebraic curves of degree ke on P^2. Hence, it follows that p' is counted at least s-times for the intersection of C_1 with ψ_μ. Therefore, $(**)$ is shown. Now, given a hyperplane H_μ;

$H_\mu = \mu_0 Y_0 + \ldots + \mu_w Y_w = 0$

we obtain a corresponding algebraic curve ϕ_μ of degree k on P^2, defined by the equation;

$\phi_\mu = \mu_0\phi_0 + \ldots + \mu_w\phi_w = 0$

Corresponding to the factorisation $F_\lambda = F_e = H_{\lambda_1} \centerdot \ldots \centerdot H_{\lambda_e}$, we obtain the factorisation $\psi_\lambda = \phi_{\lambda_1} \centerdot \ldots \centerdot \phi_{\lambda_e}$. Therefore, in order to show $(*)$, it will be sufficient to prove that;

$I_{italian}(p', C_1, \phi_{\lambda_1} \centerdot \ldots \centerdot \phi_{\lambda_e}) = 1$ $(***)$

Let p belong (uniquely) to the intersection $C \cap H_{\lambda_j}$. We claim first that;

$I_{italian}(p', C_1, \phi_{\lambda_j}) = 1$

We clearly have that the linear system Σ consists of $\{\phi_\lambda : \lambda \in P^{w*}\}$. Hence, as by construction p' does not belong to $Base(\Sigma)$ and is non-singular, the result in fact follows from Lemma 2.4 and a local result given later in this section, Lemma 2.10, which is independent of this theorem. By results of [6] on plane projective curves, $(***)$ follows. Hence, $(*)$ is shown as well.

We have now proved that there exists a form F_e of degree e which intersects C in exactly de points with multiplicity. The theorem now follows immediately from the corresponding result in Zariski structures that, for a finite equidimensional cover $G \subset Par_e \times P^w$;

$\Sigma_{x \in G(\lambda)} Mult_{(\lambda,x)}(G/Par_F)$ is preserved.

□

Using this theorem, we develop a basic theory of g_n^r on C, a projective algebraic curve in P^w. Suppose that we are given a linear system Σ of dimension r, consisting of algebraic forms ϕ_λ, parametrised by Par_Σ. We will assume that $C \cap \phi_\lambda$ has finite intersection for each $\lambda \in Par_\Sigma$, which we will abbreviate by saying that Σ has finite intersection with C. Then, for $\lambda \in Par_\Sigma$, we obtain the weighted set of points;

$$W_\lambda = \{n_{p_1}, \ldots, n_{p_m}\}$$

where

$$\{p_1, \ldots, p_m\} = C \cap \phi_\lambda$$

and

$$I_{italian}(p_j, C, \phi_\lambda) = n_{p_j} \text{ for } 1 \leq j \leq m.$$

By Theorem 2.3, the total weight $n_{p_1} + \ldots + n_{p_m}$ of these points is always equal to de.

It follows that, as λ varies in Par_Σ, we obtain a series of weighted sets $Series(\Sigma) = \{W_\lambda : \lambda \in Par(\Sigma)\}$. We now make the following definition;

Definition 2.5. *We define $order(Series(\Sigma))$ to be the total weight of any of the sets in $Series(\Sigma)$. We define $dim(Series(\Sigma))$ to be $dim(\Sigma)$. We define $g_n^r(\Sigma)$ to be the series of weighted sets parametrised by Par_Σ where;*

$$n = order(Series(\Sigma)) \text{ and } r = dim(Series(\Sigma)).$$

We now make the following local analysis of $g_n^r(\Sigma)$.

Definition 2.6. *Let Σ be a linear system having finite intersection with $C \subset P^w$. If ϕ_λ belongs to Σ and $p \in C \cap \phi_\lambda$, we define;*

$$I_{italian}^{\Sigma}(p, C, \phi_\lambda) = Card(C \cap \phi_{\lambda'} \cap \mathcal{V}_p) \text{ for } \lambda' \in \mathcal{V}_\lambda \text{ generic in } Par_\Sigma.$$

Remarks 2.7. *This is a good definition as Par_Σ is smooth. The difference between $I_{italian}$ and $I_{italian}^{\Sigma}$ is that in the first case we can vary the parameter λ over* all *forms of degree e, while, in the second case, we restrict the parameter to forms of the linear system defined by Σ.*

Definition 2.8. *We will refer to $C \setminus Base(\Sigma)$ as the set of mobile points for the system Σ.*

We now make the following preliminary definition;

Definition 2.9. *We will define a coincident mobile point for Σ to be a point $p \in C \cap \phi_\lambda$, for some $\lambda \in Par_\Sigma$, such that p lies outside $Base(\Sigma)$ and with the further property that;*

$$I^{\Sigma}_{italian}(p, C, \phi_\lambda) \lneqq I_{italian}(p, C, \phi_\lambda)$$

It is an important property of *linear* systems in characteristic 0 that there do not exist coincident mobile points. (See the final section for the corresponding result in arbitrary characteristic.) We will prove this in the following lemmas, the notation of Definition 2.6 will be maintained until Lemma 2.17.

Lemma 2.10. *Non-Existence of Coincident Mobile Points*

Let $p \in (C \setminus Base(\Sigma)) \cap \phi_\lambda$ be a non-singular point. Then;

$$I_{italian}(p, C, \phi_\lambda) = I^{\Sigma}_{italian}(p, C, \phi_\lambda)$$

Proof. We prove this by induction on $m = I_{italian}(p, C, \phi_\lambda)$. The case $m = 1$ is clear as we *always* have that;

$$I^{\Sigma}_{italian}(p, C, \phi_\lambda) \leq I_{italian}(p, C, \phi_\lambda)$$

Suppose that $I_{italian}(p, C, \phi_\lambda) = m+1$. Let $\lambda' \in Par_\Sigma \cap \mathcal{V}_\lambda$ be generic and let $\{p_1, \ldots, p_r\}$ enumerate $\mathcal{V}_p \cap C \cap \phi_{\lambda'}$. Suppose $r \geq 2$, then, by results of [6], (summability of specialisation), we have that;

$$I_{italian}(p_j, C, \phi_{\lambda'}) \leq m \text{ for each } 1 \leq j \leq r.$$

By properties of infinitesimal neighborhoods, each p_j is non-singular and lies in $C \setminus Base(\Sigma)$. Hence, by the induction hypothesis, it follows that;

$$I^{\Sigma}_{italian}(p_j, C, \phi_{\lambda'}) = I_{italian}(p_j, C, \phi_{\lambda'}) \text{ for each } 1 \leq j \leq r$$

Again, using the same result from [6], (summability of specialisation), we have that;

$I^{(\Sigma)}_{italian}(p,C,\phi_\lambda)=\sum_{1\leq j\leq r}I^{(\Sigma)}_{italian}(p_j,C,\phi_{\lambda'})$

where the (Σ) notation is used to show that the result holds for either of the above defined multiplicities. Hence;

$I_{italian}(p,C,\phi_\lambda)=I^{\Sigma}_{italian}(p,C,\phi_\lambda)$

We may, therefore, assume that;

For *any* generic $\lambda'\in\mathcal{V}_\lambda\cap Par_\Sigma$, there exists a *unique* $p'\in\mathcal{V}_p\cap C\cap\phi_{\lambda'}$, with $I_{italian}(p',C,\phi_{\lambda'})=I_{italian}(p,C,\phi_\lambda)$ $(*)$

Now, as $p\notin Base(\Sigma)$, we can find ϕ_μ with $p\notin C\cap\phi_\mu$. We consider the pencil of algebraic forms defined by $\{\phi_\lambda,\phi_\mu\}$. We may assume p is in finite position on the curve C. For ease of notation, we will assume that $w=2$ and make the generalisation to arbitrary dimension w at the end of the lemma. Let $f(X,Y)=0$ define the curve C in affine coordinates such that p corresponds to the point $(0,0)$. We rewrite the pencil of curves (ϕ_λ,ϕ_μ) in affine coordinates, which gives the 1-parameter family;

$g(X,Y;t)=\sum_{i+j\leq deg(\Sigma)}(\lambda_{ij}+t\mu_{ij})X^iY^j=0$

The following calculation is somewhat informal, see remark (ii) of Section 1;

As $p\notin Base(\Sigma)$, the function $\frac{\phi_\lambda}{\phi_\mu}$ is defined at $p=(0,0)$ and gives an algebraic morphism;

$\frac{\phi_\lambda}{\phi_\mu}:W\subset C\rightarrow Par_t$

on some open $W\subset C$, with $\frac{\phi_\lambda}{\phi_\mu}(0,0)=0$.

The cover $graph(\frac{\phi_\lambda}{\phi_\mu})\subset W\times Par_t$ of Par_t is Zariski unramified at $(0,0,0)$ as, given generic $t\in\mathcal{V}_0$, there exists a unique (x,y) such that $(x,y)\in\mathcal{V}_{(0,0)}\cap C\cap(\phi_\lambda+t\phi_\mu)$, by $(*)$ and the fact that Par_t is smooth. By results of [8], (Theorem 6.10), if we assume the ground field L has characteristic 0, the cover is etale at $(0,0,0)$. We will make the modification for non-zero characteristic in the final section of this paper. (1)

Now, as $f(X,Y)=0$ is non-singular at $p=(0,0)$, we can apply the implicit function theorem to obtain a parametrisation in algebraic power series $(x(t),y(t))$ of the branch at $(0,0)$ such that $f(x(t),y(t))=0$. (See the remark (ii) in Section 1 and [6] for the correct interpretation of these power series on appropriate etale covers and the corresponding definition of a branch.) We then obtain a map defined by algebraic power series $\theta : A^{1,et}\rightarrow Par_t$ given by;

$\theta(t)=\frac{\phi_\lambda}{\phi_\mu}(x(t),y(t))$

where $A^{1,et}$ is an etale cover of A^1 over the distinguished point (0). We have that θ is etale at $(0^{lift},0)$, as the composition of etale maps is etale. By the inverse function theorem, we may find an etale isomorphism $\rho : A^{1,et}\rightarrow A^{1,et}$ such that $\theta(\rho(t))=t$ and $(x_1(t),y_1(t))=(x(\rho(t)),y(\rho(t)))$ also parametrises the branch at $(0,0)$.

We therefore have that;

$g(x_1(t),y_1(t);t)=0$ (**) (2)

Now, by assumption, $I_{italian}(p,C,\phi_\lambda)\geq 2$. Hence, by results of [6];

$g(X,Y;0)$ is algebraically tangent to $f(X,Y)=0$ at $(0,0)$ (3).

By the chain rule and $(**)$, we have that;

$\frac{\partial g_t}{\partial X}_{(x_1(t),y_1(t))}x_1'(t)+\frac{\partial g_t}{\partial Y}_{(x_1(t),y_1(t))}y_1'(t)+\frac{\partial g}{\partial t}_{(x_1(t),y_1(t))}=0\ (***)\ (4)$

Hence, at $t=0$, we have that $\frac{\partial g}{\partial t}_{(0,0)}=0$, that is $p=(0,0)$ belongs to $\phi_\mu=0$, which is a contradiction. The calculation $(***)$ holds for formal power series in $L[[t]]$. In particular, it holds for algebraic power series. We now give a brief justification for this calculation;

Let $v=ord_t$ be the standard valuation on the power series ring $L[[t]]$. Given a power series $f\in L[[t]]$ and a sequence of power series $\{f_n : n\in\mathcal{Z}_{\geq 0}\}$, we will say that $\{f_n\}$ converges to f, abbreviated by $\{f_n\}\rightarrow f$, if;

$(\forall m\in\mathcal{Z}_{\geq 0})(\exists n(m)\in\mathcal{Z}_{\geq 0})(\forall k\geq n(m))[v(f-f_k)\geq m]$

Now choose sequences $\{x_1^n(t), y_1^n(t)\}$ of *polynomials* in $L[t]$ such that $\{x_1^n(t)\} \rightarrow x_1(t)$ and $\{y_1^n(t)\} \rightarrow y_1(t)$. We claim that;

$$\{g_n(t) = g(x_1^n(t), y_1^n(t); t)\} \rightarrow g(x_1(t), y_1(t); t)$$

This follows by standard continuity arguments for polynomials in the non-archimidean topology induced on $L[[t]]$ by v. We then have that;

$$g_n'(t) = \frac{\partial g}{\partial X}_{(x_1^n(t),y_1^n(t),t)} x_1^n(t)' + \frac{\partial g}{\partial Y}_{(x_1^n(t),y_1^n(t),t)} y_1^n(t)' + \frac{\partial g}{\partial t}_{(x_1^n(t),y_1^n(t),t)}$$

This follows from the fact that the chain rule and product rule hold in the polynomial ring $L[t]$, even in non-zero characteristic. We now claim that $\{x_1^n(t)'\} \rightarrow x_1(t)'$ and $\{y_1^n(t)'\} \rightarrow y_1(t)'$. This holds by the definition of convergence and the fact that, for a power series $f \in L[[t]]$, if $ord_t(f) = r$, then $ord_t(f') \geq r - 1$. Using standard continuity arguments and uniqueness of limits, one obtains the result (4). One can also give a geometric interpretation of the calculation (4) using duality arguments. We will discuss this problem on another occasion.

In order to finish the argument, we claim that;

$$\frac{\partial g_0}{\partial X}_{(0,0)} x_1'(0) + \frac{\partial g_0}{\partial Y}_{(0,0)} x_2'(0) = 0 \ (5)$$

This follows from (3) and the fact that algebraic tangency can be characterised by the property that Dg_0 at $(0,0)$ contains the tangent line l_p of C. This is clear if g_0 is non-singular at p, in particular if g_0 has a non-reduced component at p. Otherwise, it follows easily from [6] or [2]. Hence, at $t = 0$, we have that $\frac{\partial g}{\partial t}_{(0,0)} = 0$, that is $p = (0,0)$ belongs to $\phi_\mu = 0$, which is a contradiction.

We now consider the case for arbitrary dimension w. We will use Theorem 1.33 to find a plane projective curve $C_1 \subset P^2$ birational to C. Using Lemma 1.20, we can find a linearly independent system Σ' and a birational presentation $\Psi_{\Sigma'} : C_1 \leftrightsquigarrow C$. We will assume that the point p under consideration lies inside the canonical set $W_{\Psi_{\Sigma'}}$ with corresponding $p' \in V_{\Psi_{\Sigma'}}$. This can in fact always be arranged, see the section on Conic Projections. However, for the moment, we can, if necessary, replace the set $NonSing(C)$ by $W_{\Psi_{\Sigma'}}$. Now, we follow through the calculation given above for $w = 2$. The argument up to (1) is unaffected. We first justify the calculation (2). Let $f(X,Y) = 0$ be an affine representation of C_1, such that the point p' corresponds to $(0,0)$. Then, we may obtain a local power series representation $(x(t), y(t))$ of $f(X,Y)$ at $(0,0)$ and, applying $\Psi_{\Sigma'}$, a local power series representation $\Psi_{\Sigma'}(x(t), y(t)) = (x_1(t), \ldots, x_w(t))$ of the corresponding $p \in C$. We may then, applying the same argument, obtain the relation $g(x_1(t), \ldots, x_w(t); t) = 0$, where;

$$g(X_1, \ldots, X_w; t) = \Sigma_{i_1+\ldots+i_w \leq deg(\Sigma)} \lambda_{i_1\ldots i_w} X_1^{i_1} \ldots X_w^{i_w} + t\mu_{i_1\ldots i_w} X_1^{i_1} \ldots X_w^{i_w}$$

is an affine representation of the pencil $\phi_\lambda + t\phi_\mu$.

We now need to justify the calculation in (3). Write ϕ_λ in the form;

$$\phi_\lambda = \Sigma_{i_0+\ldots i_w = deg(\Sigma)} \lambda_{i_0\ldots i_w} X_0^{i_0} \ldots X_n^{i_w} = 0$$

Let $\Sigma' = \{\psi_0, \ldots, \psi_w\}$, then the assumption that ϕ_λ passes through p implies that the curve;

$$D = \Sigma_{i_0+\ldots+i_w = deg(\Sigma)} \lambda_{i_0\ldots i_w} \psi_0^{i_0} \ldots \psi_{i_w}^{i_w} = 0$$

passes through the corresponding p' of C_1. By the fact that $I_{italian}(p, C, \phi_\lambda) \geq 2$, we can vary the coefficients $\{\lambda_{i_0\ldots i_w}\}$ of ϕ_λ to obtain distinct intersections $\{x'', x'''\}$ in $C \cap \phi_{\lambda'} \cap \mathcal{V}_p$. By properties of infinitesimals, these intersections lie in the fundamental set $W_{\Psi_{\Sigma'}}$. Hence, we can find corresponding intersections $\{x'''', x'''''\}$ in $\mathcal{V}_{p'} \cap V_{\psi_{\Sigma'}}$ with the corresponding variation of D. This implies that;

$$I_{italian}(p', C_1, D) \geq 2$$

By results of the paper [6], D must be algebraically tangent to the curve C_1 at p'. Hence, by the chain rule, and the characterisation of algebraic tangency given above, ϕ_λ is algebraically tangent to the curve C at p, in the sense that its differential $D\phi_\lambda$ at p, contains the tangent line l_p of C $(*)$. The reader should also look at the proof of Theorem 2.3, where a similar calculation was carried out. Finally, we need to justify (4). This is clear from the calculation done above. The final step (5) is also clear from the corresponding calculation and $(*)$.

$\square$

Remarks 2.11. *The lemma fails for* non-linear *systems. Let C be defined in affine coordinates (x, y) by $y = 0$ and let $\{\phi_t\}$ be the pencil of curves, defined in characteristic* 0*, by $y = (x-t)^2 = x^2 - 2tx + t^2$.*

By construction, each ϕ_t is tangent to $y=0$ at $(t,0)$. It follows that each $(t,0)\in C$ is a coincident mobile point for ϕ_t.

Lemma 2.12. *Suppose that $p\in C\setminus Base(\Sigma)\cap\phi_\lambda$ is a singular point, then;*

$I_{italian}(p,C,\phi_\lambda)=I^{\Sigma}_{italian}(p,C,\phi_\lambda)$.

Proof. Suppose that $I^{\Sigma}_{italian}(p,C,\phi_\lambda)=m$. As $p\notin Base(\Sigma)$, the condition that ϕ_λ passes through p defines a proper closed subset of the parameter space Par_Σ. Hence, we can find $\lambda'\in\mathcal{V}_\lambda$ generic in Par_Σ and $\{p_1,\ldots,p_m\}=C\cap\phi_{\lambda'}\cap\mathcal{V}_p$, distinct from p, witnessing this multiplicity. As both $NonSing(C)$ and $C\setminus Base(\Sigma)$ are open and defined over L, we have that $\{p_1,\ldots,p_m\}$ must lie in the intersection of these sets. Applying the result of the previous lemma, we must have that;

$I_{italian}(p_j,C,\phi_{\lambda'})=1$ for $1\leq j\leq m$

Hence, by summability of specialisation, again see the paper [6], we must have that $I_{italian}(p,C,\phi_\lambda)=m$ as required. □

In the following Lemma 2.13, Lemma 2.16 and Lemma 2.17, by a canonical set on C, we will mean either a set of the form $V_{\phi_{\Sigma_1}}$, for the domain of a birational map ϕ_{Σ_1}, or a set of the form $W_{\psi_{\Sigma_2}}$, for the image of a birational map ψ_{Σ_2}, see also Definition 1.30. For ease of notation, we will abbreviate either of these sets by W. In particular, W may include the canonical set V_{Φ_Σ} defined by (any) choice of basis for the linear system Σ.

Lemma 2.13. *Multiplicity at non-base points witnessed by transverse intersections in the canonical sets.*

Let $p\in C\setminus Base(\Sigma)$, then, if $m=I_{italian}(p,C,\phi_\lambda)$, we can find $\lambda'\in\mathcal{V}_\lambda$, generic in Par_Σ, and distinct $\{p_1,\ldots,p_m\}=C\cap\phi_{\lambda'}\cap\mathcal{V}_p$ such that $\{p_1,\ldots,p_m\}$ lies in the canonical set W ,with the point p removed, $W\setminus\{p\}$, and the intersection of C with $\phi_{\lambda'}$ at each p_j is transverse for $1\leq j\leq m$.

Proof. As $p\notin Base(\Sigma)$, the condition that ϕ_λ does not pass through p defines an open subset of Par_Σ. By the previous lemma, taking generic (over L), $\lambda'\in\mathcal{V}_\lambda$, we can find $\{p_1,\ldots,p_m\}=C\cap\phi_{\lambda'}\cap\mathcal{V}_p$, distinct from p. Finally, $C\setminus W$ defines a finite subset of C (over L). Clearly, $\{p_1,\ldots,p_m\}$ avoid this set, otherwise, by properties of specialisations, some p_j would equal p for $1\leq j\leq m$. Finally, the transversality result follows from the fact that $I_{italian}(p_j, C, \phi_{\lambda'}) = 1$ for $1 \leq j \leq m$, (using Lemmas 2.10 and 2.12 again). $\square$

We have analogous results to Lemmas 2.10, 2.12 and 2.13 for points in $Base(\Sigma)$;

We first require the following;

Lemma 2.14. *Let $p \in C \cap Base(\Sigma)$, then there exists an open subset $U_p \subset Par(\Sigma)$ and an integer $I_p \geq 1$ such that;*

$I_{italian}(p, C, \phi_\lambda) = I_p$ *for* $\lambda \in U_p$.

and

$I_{italian}(p, C, \phi_\lambda) \geq I_p$ *for* $\lambda \in Par_\Sigma$.

Proof. By properties of Zariski structures, we have that;

$$W_k = \{\lambda \in Par(\Sigma) : I_{italian}(p, C, \phi_\lambda) \geq k\}$$

are definable and Zariski closed in Par_Σ. The result then follows by taking $I_p = min_{\lambda \in Par_\Sigma} I_{italian}(p, C, \phi_\lambda)$ and the fact that Par_Σ is irreducible.

$\square$

We can now formulate the corresponding version of Lemmas 2.12 and 2.13 for base points;

Lemma 2.15. *Let $p \in C \cap Base(\Sigma) \cap \phi_\lambda$. Then;*

$$I_{italian}(p, C, \phi_\lambda) = I_p + I^{\Sigma}_{italian}(p, C, \phi_\lambda) - 1$$

Proof. Let $m = I^{\Sigma}_{italian}(p, C, \phi_\lambda)$. Choosing $\lambda' \in \mathcal{V}_\lambda$ generic in Par_Σ, we can find $\{p_1, \ldots, p_{m-1}\} = C \cap \mathcal{V}_p \cap \phi_{\lambda'}$, distinct from p, witnessing this multiplicity. Therefore, for $1 \leq j \leq m-1$, $p_j \notin Base(\Sigma)$, by properties of specialisations and the fact that $Base(\Sigma)$ is finite and defined over L. Hence, we can apply the results of Lemmas 2.10 and 2.12 to conclude that $I_{italian}(p_j, C, \phi_{\lambda'}) = 1$ for $1 \leq j \leq m-1$. As λ' was generic in Par_Σ, using the previous Lemma 2.14, we have that $I_{italian}(p, C, \phi_{\lambda'}) = I_p$. Now, it follows easily, using summability of specialisation (see the paper [6]), that $I_{italian}(p, C, \phi_\lambda) = I_p + (m-1)$. The lemma is proved. $\square$

Lemma 2.16. *Let $p \in C \cap Base(\Sigma) \cap \phi_\lambda$, then, if $m = I^{\Sigma}_{italian}(p, C, \phi_\lambda)$, we can find $\lambda' \in \mathcal{V}_\lambda$, generic in Par_Σ, and $\{p, p_1, \ldots, p_{m-1}\} = C \cap \phi_{\lambda'} \cap \mathcal{V}_p$ witnessing this multiplicity such that $\{p_1, \ldots, p_{m-1}\}$ lie in the canonical set W, see the explanation before Lemma 2.13, and the intersections $C \cap \phi_{\lambda'}$ at p_j are transverse for $1 \leq j \leq m-1$.*

Proof. Use the proof of Lemma 2.13, basic properties of infinitesimals and the fact that W is open in C and definable over L. $\square$

Lemma 2.17. *Generic Intersections*

Fix a canonical set W and let ϕ_λ be generic in Σ, then each point of intersection of C with ϕ_λ outside $Base(\Sigma)$ lies inside the canonical set W and is transverse.

Proof. The finitely many points of $(C \setminus W)$ are defined over the data of $\{W, C\}$. Hence, the condition on Par_Σ that ϕ_λ intersects a point of $(C \setminus W)$ outside $Base(\Sigma)$ consists of a finite union of proper hyperplanes defined over the data of $\{W, C\}$. Therefore, for generic ϕ_λ, each point of intersection of ϕ_λ with C, outside $Base(\Sigma)$, lies inside W. Now observe that the condition of transversality between C and ϕ_λ, inside W, defines a constructible condition on Par_Σ, over the data of $\{C, W\}$. Namely;

$$\theta(\lambda) \equiv \forall y[(y \in \phi_\lambda \cap W) \rightarrow NonSing(y) \wedge RightMult_y(C, \phi_\lambda) = 1]$$

By Lemmas 2.13 and 2.16, the condition is Zariski dense in Par_Σ. Hence, the result follows.

$\square$

We now make the following definitions;

Definition 2.18. *Let Σ be a linear system defining a $g^r_n(\Sigma)$ on a projective algebraic curve $C \subset P^w$. Let $\{W_\lambda\} = Series(\Sigma)$. If $p \in W_\lambda$, we say that;*

p is s-fold (s-plo in Italian) for the $g^r_n(\Sigma)$ if $I_{italian}(p, C, \phi_\lambda) \geq s$.

p is counted (contato) s-times for the $g^r_n(\Sigma)$ if p has multiplicity s in W_λ, equivalently $I_{italian}(p, C, \phi_\lambda) = s$.

p has s-fold contact (contatto) with ϕ_λ if $I^{\Sigma}_{italian}(p, C, \phi_\lambda) = s$.

Remarks 2.19. *The Italian terminology is generally quite confusing. The above lemmas show that the discrepancy between contatto and contato occurs only at fixed points of the system Σ. The philosophy behind their approach is that algebraic calculations may be reduced to visual arguments using the ideas that a s-fold contact at p is a limit of s-points converging along the curve from intersections with forms in the system Σ and that these points are preserved by birationality. In the case of a fixed point, p may be counted* more *times than its actual contact with ϕ_λ and this excess intersection is never actually visually manifested by a variation. The Italian approach is to ignore this excess intersection. This motivates the following definition;*

Definition 2.20. *Let a $g^r_n(\Sigma)$ be given on C. For $p \in C \cap \phi_\lambda$, we define;*

$$I^{mobile}_{italian}(p, C, \phi_\lambda) = Card(C \cap \phi_{\lambda'} \cap \{\mathcal{V}_p \setminus p\}),\ \lambda' \in \mathcal{V}_\lambda \text{ generic in } Par_\Sigma.$$

Remarks 2.21. *One needs to check, as usual, that this is a good definition. This follows, for example, by Lemma 2.14, Remarks 2.7 and Lemma 2.15.*

Lemma 2.22. *For $p \notin Base(\Sigma)$, $I^{mobile}_{italian}(p, C, \phi_\lambda) = I_{italian}(p, C, \phi_\lambda)$*

For $p \in Base(\Sigma)$, we have that;

$$I_p + I^{mobile}_{italian}(p, C, \phi_\lambda) = I_{italian}(p, C, \phi_\lambda)$$

Proof. The proof follows immediately from the same results cited in the previous remark. □

We now make the following definition;

Definition 2.23. *By a g^r_n, we mean the series obtained from a given $g^r_{n'}(\Sigma)$ by removing some (possibly all) of the fixed point contributions I_x in $Base(\Sigma)$. That is, we subtract some part of I_x from each weighted set W_λ. We define n to be the total multiplicity of each W_λ after subtracting some of the fixed point contribution, so $n \leq n'$. We say that the g^r_n has no fixed points if all the fixed point contributions are removed. We refine the Italian terminology for a g^r_n by saying that x is s-fold for W_λ if it appears in the weighted set with multiplicity at least s and x*

is counted s-times for W_λ if it appears with multiplicity exactly s. We define $Base(g_n^r)$ to be $\{x \in C : \forall\lambda(x \in W_\lambda)\}$.

We now have the following;

Lemma 2.24. *For a given g_n^r, we always have that $r \leq n$.*

Proof. Let the g_n^r be defined by Σ, a linear system of dimension r, having finite intersection with C. Pick $\{p_1, \ldots, p_r\}$ independent generic points of C, not contained in $Base(\Sigma)$. The condition that ϕ_λ passes through p_j defines a proper hyperplane condition H_{p_j} on Par_Σ. The base points of the subsystem defined by $H_{p_1} \cap \ldots \cap H_{p_j}$ are defined over $p_1, \ldots, p_j$ and finite, for $1 \leq j \leq r-1$, as, by assumption, no form in Σ contains C. We must, therefore, have that $dim(H_{p_1} \cap \ldots \cap H_{p_r}) = 0$. That is there exists a unique ϕ_λ in Σ passing through $\{p_1, \ldots, p_r\}$. We must have that the total intersection multiplicity of C with ϕ_λ outside the fixed contribution is at least r, by construction, hence $r \leq n$ as required.

□

Definition 2.25. *Subordinate Systems*

We say that

$g_{n_1}^{r_1} \subseteq g_{n_2}^{r_2}$

if there exist linear systems $\Sigma' \subseteq \Sigma$ (having finite intersection with C) of dimension r_1 and dimension r_2 respectively such that $g_{n_1}^{r_1}$ is obtained from Σ', $g_{n_2}^{r_2}$ is obtained from Σ and, for each $\lambda \in Par_{\Sigma'}$, $W_\lambda \subseteq V_\lambda$. Here, by $\{W_\lambda, V_\lambda\}$, we mean the weighted sets parametrised by $\{g_{n_1}^{r_1}, g_{n_2}^{r_2}\}$ and, by $W_\lambda \subseteq V_\lambda$, we mean that $n_{p_i} \leq m_{p_i}$ where $W_\lambda = \{n_{p_1}, \ldots, n_{p_r}\}$ and $V_\lambda = \{m_{p_1}, \ldots, m_{p_r}\}$.

Remarks 2.26. *Note that the relationship of subordination is clearly transitive. That is, if $g_{n_1}^{r_1} \subseteq g_{n_2}^{r_2} \subseteq g_{n_3}^{r_3}$, then $g_{n_1}^{r_1} \subseteq g_{n_3}^{r_3}$.*

Definition 2.27. *Composite Systems*

We say that a g_n^r, defined by Σ, is composite, if, for generic $p \in C$, every weighted set W_λ containing p also contains a distinct $p'(p)$ with $p' \notin Base(\Sigma)$.

Remarks 2.28. *Note that the definition of composite is well defined, for, if the given g_n^r is defined by Σ, the statement;*

$\theta(x) \equiv \exists x' \notin Base(\Sigma)\forall\lambda \in Par_\Sigma(x \in C \cap \phi_\lambda \rightarrow x' \in C \cap \phi_\lambda)$

defines a constructible subset of C. Hence, if it holds for some generic p, it holds for any *generic p in C. In modern terminology, we would say that the given g_n^r seperates points (generically), see Proposition 7.3 of* [2].

Definition 2.29. *Simple Systems*

We say that a g_n^r is simple if it is not composite.

The importance of simple g_n^r is due to the following;

Lemma 2.30. *Construction of a Birational Model*

A simple g_n^r on C defines a projective image $C' \subset P^r$, birational to C.

Proof. Let the g_n^r be defined by a linear system Σ, with a choice of basis $\{\phi_0, \ldots, \phi_r\}$, (having finite intersection with C), possibly after removing some fixed point contribution. Let Φ_Σ be the morphism defined as in Lemma 1.16. This morphism is defined on an open subset $U = C \setminus Base(\Sigma)$ of C. By continuity, the image of Φ_Σ on U is irreducible, hence either defines a constructible $V \subset P^r$ of dimension 1 or the image is a point. We can clearly exclude the second case, otherwise we can find $\{\phi_\lambda, \phi_\mu\}$ differing by a constant of proportionality ρ on U, therefore $\phi_\lambda - \rho\phi_\mu$ contains C. Let $C' = \overline{V}$, then C' is an irreducible projective algebraic curve. We claim that C' is birational to C. If not, then, using Lemma 1.21 or just the definition of birationality, in characteristic 0, for generic $x \in C$, we can find a distinct $x' \in U$ such that $\Phi_\Sigma(x) = \Phi_\Sigma(x')$. The choice of basis for Σ determines an isomorphism of Par_Σ with P^r. Using the parametrisation of Par_Σ given by this isomorphism, if ϕ_λ passes through x, the corresponding hyperplane $H_\lambda \subset P^r$ would pass through $\Phi_\Sigma(x)$ and, therefore, ϕ_λ would pass through x' as well. This contradicts simplicity. The lemma may, of course, fail in non-zero characteristic. We refer the reader to the final section for the problems associated to Frobenius. □

We also have the following transfer results;

Lemma 2.31. *Let a simple g_n^r on C be given, as in the previous lemma, defining a birational projective image $C' \subset P^r$. Let V_{ϕ_Σ} and W_{ϕ_Σ} denote the canonical sets associated to this birational map. Then, given* *a g^d_m on C', without fixed points, there exists a corresponding g^d_m on C, without fixed points, and, for any corresponding pair $\{x,x'\}$ in $\{V_{\phi_\Sigma},W_{\phi_\Sigma}\}$, x is counted s times in V_λ iff x' is counted s times in W_λ, where $\{V_\lambda,W_\lambda\}$ are the weighted sets parametrised by the g^d_m, on $\{C,C'\}$ respectively.*

Proof. Let the g^d_m on C' be defined by a linear system Σ' of dimension d, with basis $\{\psi_0,\ldots,\psi_d\}$, after removing all the fixed point contribution. We then obtain a corresponding linear system Σ'' defined by;

$$\{\theta_\lambda=\lambda_0\psi_0(\phi_0,\ldots,\phi_r)+\ldots+\lambda_d\psi_d(\phi_0,\ldots,\phi_r)=0\}$$

Clearly, Σ'' has finite intersection with C, hence it defines a $g^d_{m'}$ after removing all fixed point contributions. Now, let $\{x,x'\}$ be a corresponding pair in $\{V_{\phi_\Sigma},W_{\phi_\Sigma}\}$. Suppose that x' is counted s times in W_λ, then, as the g^d_m on C' has no fixed points, we have that;

$$I^{mobile}_{italian}(x',C',\psi_\lambda)=s$$

Therefore, we can find $\lambda'\in\mathcal{V}_\lambda$ generic in $Par_{\Sigma'}$ and

$$\{x'_1,\ldots,x'_s\}=C'\cap\psi_{\lambda'}\cap\mathcal{V}_{x'}\setminus\{x'\}$$

By properties of infinitesimals, $\{x'_1,\ldots,x'_s\}\subset W_{\phi_\Sigma}$, hence, we can find corresponding $\{x_1,\ldots,x_s\}\subset V_{\phi_\Sigma}\cap\theta_{\lambda'}\cap\mathcal{V}_x\setminus\{x\}$. It follows that $I^{mobile}_{italian}(x,C,\theta_\lambda)\geq s$, and equality follows from the converse argument. Therefore, as the corresponding $g^d_{m'}$ has no fixed points, x is counted s times in V_λ. The converse uses the same argument. It remains to show that $m=m'$. By Lemmas 2.16 and 2.17 and the fact that the $g^d_{m'}$ has no fixed points, a generic V_λ consists of m' points each counted once inside V_{ϕ_Σ}. Therefore, by the above argument, these points are each counted once inside the corresponding W_λ. Hence, $m'\leq m$. We obtain $m\leq m'$ by the reversal of this argument.

□

Lemma 2.32. *Let a simple g^r_n on C be given, defining a birational map $\Phi_\Sigma : C \leftrightsquigarrow C'\subset P^r$. Then, if d is the degree of C', we have that $d'=d+I$ for the $g^r_{d'}(\Sigma)$ defined by Σ, where I is the total fixed point contribution from $Base(\Sigma)$.*

Proof. Let k be the degree of ϕ_Σ and v the degree of C, then we claim that;

$d + I = kv$

By Lemma 2.4, if H_λ is a generic hyperplane, it cuts C' transversely in d distinct points. We may also assume that these points lie inside the canonical set W_{Φ_Σ} as this is defined over the data of Φ_Σ. Let $\{p_1, \ldots, p_d\}$ be the corresponding points of V_{Φ_Σ} and ϕ_λ the corresponding form in Σ. By, for example, Lemma 2.17;

$I_{italian}(p_j, C, \phi_\lambda) = 1$ for $1 \leq j \leq d$

There can be no more intersections of ϕ_λ with C outside $Base(\Sigma)$, otherwise one would obtain a corresponding intersection of H_λ with C' outside W_{Φ_Σ}. Hence, the total multiplicity of intersection $I_{italian}(C, \phi_\lambda)$ between C and ϕ_λ is exactly $I + d$, using Lemma 2.14. By Theorem 2.3 and the fact that ϕ_λ is a form of degree k, we also have that the total multiplicity $I_{italian}(C, \phi_\lambda)$ is kv. As, by definition, $d' = kv$, The result follows.

□

3. The Construction of a Birational Model of a Plane Projective Algebraic Curve without Multiple Points

We first make the following definition.

Definition 3.1. *Let $C \subset P^w$ be a projective algebraic curve, not contained in any hyperplane of P^w. We define a point $p \in C$ to be s-fold on C if, for every hyperplane H passing through p;*

$I_{italian}(p, C, H) \geq s.$

and equality is attained. We define p to be a multiple point if it is s-fold for some $s \geq 2$. We define p to be simple if it is not multiple.

We have the following lemma;

Lemma 3.2. *Let $C \subset P^2$ be a projective algebraic curve and p a point on C. Let $F(X, Y) = 0$ be an affine representation of C such that the point p corresponds to $(0, 0)$. Then p is s-fold on C iff we can write F in the form;*

$F(X, Y) = \Sigma_{(i+j) \geq s} a_{ij} X^i Y^j$

In particular, p is non-singular iff it is not multiple.

Proof. Suppose that p is s-fold on C, then, for a generic line l defined by $aX - bY = 0$, we have that;

$I_{italian}(p, C, l) \geq s$

It follows, by the results of [6], that we also have;

$I_{italian}(p, l, C) = I_{italian}(p, C, l) \geq s$ $(*)$

Now, suppose that $F(X, Y)$ has the expansion;

$F(X, Y) = \Sigma_{i+j \leq deg(F)} a_{ij} X^i Y^j$.

and that $a_{ij} \neq 0$ for some (i, j) with $i + j < s$ (**). We can parametrise the branch at $(0, 0)$ of l algebraically by;

$(x(t), y(t)) = (bt, at)$

We make the substitution of this parametrisation into $F(X, Y)$ and obtain;

$F(x(t), y(t)) = \Sigma a_{ij} x(t)^i y(t)^j = \Sigma a_{ij} a^j b^i t^{i+j}$

By $(**)$ and generic choice of $\{a, b\}$, this expansion has order $s_1 < s$. In characteristic 0, it then follows, by the method of [6], see also Theorem 6.1 of this paper, that, for the pencil Σ_1 defined by $\{F_t \equiv F(X, Y) + t = 0\}$;

$I^{\Sigma_1}_{italian}(p, l, F) = s_1$.

In particular, as $p \notin Base(\Sigma_1)$, we have, by Lemma 2.10, that;

$I_{italian}(p, l, F) = s_1$

and therefore, by $(*)$, that;

$I_{italian}(p, C, l) = s_1$

contradicting the fact that p was s-fold on C. It follows that $(**)$ doesn't hold, as required. See, however, the final section for the problem in non-zero characteristic. For the converse direction, suppose that $F(X,Y)$ has the expansion;

$$F(X,Y) = \Sigma_{i+j\geq s} a_{ij} X^i Y^j$$

Then, by a direct calculation and using the argument above, one has that, for *any* line;

$$I_{italian}(p,l,F) \geq s.$$

Hence, by $(*)$ again, p must be s-fold on C.

Using this result, it follows immediately, by considering the vector $(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y})$ evaluated at $(0,0)$, that p is non-singular on C iff p is not multiple.

□

Our main result in this section will be the following;

Theorem 3.3. *Let $C \subset P^2$ be a projective algebraic curve, then there exists $C_1 \subset P^w$ such that C and C_1 are birational and C_1 has no multiple points.*

The proof will proceed using the basic theory of g_n^r that we developed in the previous section. We require the following definition;

Definition 3.4. *Let a g_n^r without fixed points be given on C. We will call the g_n^r transverse if the following property holds;*

There does not exist a subordinate $g_{n'}^{r-1} \subset g_n^r$ such that $n' \leq n-2$.

We then have;

Lemma 3.5. *Let a g_n^r on C be given without fixed points. Then, if the g_n^r is transverse, it must be simple.*

Proof. Suppose that the g_n^r, defined by Σ, is not simple, then it must be composite. Therefore, for generic $x \in C$, there exists an $x'(x) \notin$

$Base(\Sigma)$ such that every weighted set W_λ containing x also contains $x'(x)$. As x is generic, $x \notin Base(\Sigma)$, hence the subsystem $\Sigma_x \subset \Sigma$, consisting of forms in Σ passing through x, has dimension $r-1$. Define a $g^{r-1}_{n'}$ from Σ_x by removing the fixed point contribution of $g^{r-1}_{n''}(\Sigma_x)$. We claim that $g^{r-1}_{n'} \subseteq g^r_n$ $(*)$. We clearly have that $\Sigma_x \subset \Sigma$. Suppose that p appears in a W_λ defined by $g^{r-1}_{n'}$ with multiplicity s. Then, by definition, $I^{mobile}_{italian}(p, C, \phi_\lambda) = s$, calculated with respect to Σ_x. It follows easily from the above lemmas that $I^{mobile}_{italian}(p, C, \phi_\lambda) = s$, calculated with respect to Σ, as well. Hence, p also appears in the corresponding V_λ with multiplicity s. This gives the claim $(*)$. We show that $n' \leq n-2$. Let $\{p_1, \ldots, p_r\}$ be the base points of Σ. Then Σ_x has base points containing the set $\{p_1, \ldots, p_r, x, x'(x)\}$. Clearly, the fixed point contribution of Σ_x at $\{p_1, \ldots, p_r\}$ can only increase the fixed point contribution of Σ and, moreover, the fixed point contribution of Σ_x at $\{x, x'\}$ must be at least 2. Hence, using the Hyperspatial Bezout Theorem and the definition of $\{g^r_n, g^{r-1}_{n'}\}$, we have that $n' \leq n-2$. This contradicts the fact that the original g^r_n was transverse. Hence, the g^r_n must be simple. $\square$

Following from this, we have the following;

Lemma 3.6. *Let a g^r_n be given on C without fixed points such that g^r_n is transverse. Then the g^r_n defines a birational map $\Phi : C \leftrightsquigarrow C_1 \subset P^r$ and, moreover, C_1 has no multiple points.*

Proof. The first part of the lemma follows from the previous Lemma 3.5 and Lemma 2.30. See the final section, though, for the problem in non-zero characteristic. It remains to prove that C_1 has no multiple points. Consider the $g^r_{n'}$ without fixed points on C_1, defined by the linear system Σ of hyperplane sections. By Lemma 2.31, there exists a corresponding $g^r_{n'}$ on C without fixed points. By its construction in Lemma 2.31, it equals the original g^r_n. Now suppose that there exists a multiple point p of C_1. We consider the subsystem Σ_p of hyperplane sections passing through p. This defines a $g^{r-1}_n \subset g^r_n$ and, as p is multiple, after removing the fixed point contribution, we obtain a $g^{r-1}_{n'} \subset g^r_n$ with $n' \leq n-2$. Using Lemma 2.31 again, we obtain a corresponding $g^{r-1}_{n'}$ on C. We claim that $g^{r-1}_{n'} \subset g^r_n$ on C. This follows easily from the fact that $\Sigma_p \subset \Sigma$, the $g^{r-1}_{n'}$ has no fixed points and the argument of the previous lemma. This contradicts the fact that the original g^r_n on C was transverse. Hence, C_1 has no multiple points.

$\square$

We now give the proof of Theorem 3.3. We find a transverse g_n^r on C using a combinatorial argument;

Proof. (Theorem 3.3)

Suppose that C has order $n \geq 2$ (the case when C has order $n = 1$ is obvious.) Consider the independent system Σ consisting of all plane algebraic curves defined by homogeneous forms of order $n-1$. Clearly, no form in Σ may contain C, and the system has no fixed points on C, hence the system defines a $g_{n_1}^{r_1}$, without fixed points, of dimension $r_1 = \frac{(n-1)(n+2)}{2}$ and order $n_1 = n(n-1)$. If this $g_{n_1}^{r_1}$ is transverse, the proof is complete. Hence, we may suppose that there exists a $g_{n_2}^{r_2} \subset g_{n_1}^{r_1}$, without fixed points, such that $r_2 = r_1 - 1$ and $n_2 \leq n_1 - 2$. Again, if this $g_{n_2}^{r_2}$ is transverse, the proof is complete. Hence, we may suppose that there exists a sequence;

$$g_{n_i}^{r_i} \subset g_{n_{i-1}}^{r_{i-1}} \subset \ldots \subset g_{n_1}^{r_1}$$

with $r_i = r_1 - (i-1)$ and $n_i \leq n_1 - 2(i-1)$. $(*)$

We need to show that this sequence terminates. By Lemma 2.24, we always have that $r_i \leq n_i$. Combining this with $(*)$, we have that;

$$r_1 - (i-1) \leq n_1 - 2(i-1)$$

Therefore;

$$i \leq n_1 - r_1 + 1 = n(n-1) - \frac{(n-1)(n+2)}{2} + 1 = \frac{(n-1)(n-2)}{2} + 1 \ (**)$$

Now suppose that equality is attained in $(**)$, then we would have that;

$$r_i = r_1 - (i-1) = \frac{(n-1)(n+2)}{2} - \frac{(n-1)(n-2)}{2} = 2(n-1) \geq 2$$

$$n_i \leq n_1 - 2(i-1) = n(n-1) - (n-1)(n-2) = 2(n-1) \ (***)$$

This implies that $n_i \leq r_i$ and hence $n_i = r_i$. Therefore, we must have that there exists a $g_m^m \subset g_{n_i}^{r_i}$ for each i with $m \geq 2$ and the sequence terminates in fewer than $\frac{(n-1)(n-2)}{2} + 1$ steps. The final $g_{n_i}^{r_i}$ in the sequence then defines a birational map from C to $C_1 \subset P^w$ without multiple points $(w \geq 2)$, as required.

□

Remarks 3.7. *The terminology of transverse g_n^r is partly motivated by the following fact. Suppose that a transverse g_n^r on C is defined by a linear system Σ, possibly after removing some fixed point contribution, then, if $x \in C \setminus Base(\Sigma)$ is non-singular, there exists an algebraic form ϕ_λ from Σ such that $\phi_\lambda(x) = 0$ but ϕ_λ is not algebraically tangent to C at x. The proof of this follows straightforwardly from the definition of transversality and the fact that, if ϕ_λ is algebraically tangent to C at x, then $I_{italian}(x, C, \phi_\lambda) \geq 2$ (see the proof of Lemma 4.2). In modern terminology, one calls this property separating tangent vectors. See, for example Proposition 7.3 of* [2]. *The full motivation, however, comes from Theorem 6.10.*

4. The Method of Conic Projections

The purpose of this section is to explore the Italian technique of projecting a curve onto a plane. The method of conic projections is extremely old and can be found in [1], where projective notions are explicitly incorporated in the discussion of perspective. Severi himself also wrote an article on the subject of [1] in [11]. We assume that we are given a projective algebraic curve $C \subset P^w$ for some $w > 2$ and that C is not contained in any hyperplane section (otherwise reduce to this lower dimension).

The construction;

Let $\Omega \subset P^w$ be a plane of dimension $w - k - 1$ and $\omega \subset P^w$ a plane of dimension $2 \leq k < w$ such that $\Omega \cap \omega = \emptyset$. We define the projection of C from Ω to ω as follows;

Let $P \in C$. We may assume that P does not lie on Ω. Let $< \Omega, P >$ be the intersection of all hyperplanes containing Ω and P. It is a plane of dimension $w - k$. Now, by elementary dimension theory, we must have that;

$$dim(< \Omega, P > \cap \omega) \geq k + (w - k) - w = 0$$

We may exclude the case that $< \Omega, P >$ and ω intersect in a line l, as then Ω and l would intersect in a point Q, contradicting the fact that $\Omega \cap \omega = \emptyset$. Hence, $< \Omega, P > \cap \omega$ defines a point $pr(P)$. We may repeat this construction for the cofinitely many points $U \subset C$ which do not lie on Ω. Now, consider the statement;

$$\phi(y) \equiv [y \in \omega \wedge \exists x \exists w(x \in \Omega \wedge w \in U \wedge y \in l_{xw})]$$

By elimination of quantifiers for algebraically closed fields, this clearly defines an algebraic set consisting of $\{pr(w) : w \in U\}$. We call this the projection $pr(U)$ of U. As U is irreducible, it follows that $pr(U)$ is irreducible. Moreover, $pr(U)$ has dimension 1, otherwise $pr(U)$ would consist of a single point Q, in which case U and therefore C would be contained in the plane $< \Omega, Q >$, contradicting the assumption. We define the projection $proj(C)$ of C from Ω to ω to be the closure $\overline{pr(U)}$ of $pr(U)$ in ω. We can define a correspondence $\Gamma \subset U \times pr(U)$ by;

$$\Gamma(w, y) \equiv [y \in pr(U) \wedge w \in U \wedge \exists x(x \in \Omega \wedge y \in l_{xw})]$$

We define the associated correspondence $\Gamma_{pr} \subset C \times proj(C)$ to be the Zariski closure $\overline{\Gamma}$. Note that, in the case when $\Omega \cap C = \emptyset$, the correspondence Γ defines an algebraic *function* (in the sense of model theory) pr from C to $pr(C)$. By the model theoretic description of definable closure in algebraically closed fields of characteristic 0, pr defines a morphism from C to $pr(C)$. See the final section for the problem in non-zero characteristic.

Remarks 4.1. *Note that when Ω is a point P and ω is a hyperplane, the construction is equivalent to forming the cone;*

$$Cone(C) = \bigcup_{x \in C} l_{xP}$$

and taking the intersection $\omega \cap Cone(C)$. This is the reason for the terminology of "conic projections". The case when $w = 3$ is explicitly discussed in [1]*, P represents the eye of the observer wishing to obtain a representation of a curve on a plane.*

We now prove the following general lemma on conic projections;

Lemma 4.2. *Let $w \geq 3$ and suppose that $\{A, B\}$ are independent generic points of C. Then the line l_{AB} does not otherwise meet the curve.*

Proof. Suppose, for contradiction, that we can find independent generic points $\{A, B\}$ on C such that l_{AB} intersects C in a new point P. As A and B are generic, they are non-singular points on the curve C, hence we can define the tangent lines l_A and l_B at these points. We claim the following;

For any hyperplane H_λ containing l_A and passing through A;

$I_{italian}(A, C, H_\lambda) \geq 2$ (*)

This is the converse to a result already proved in Lemma 2.10 above. It can be proved in a similar way, namely fix a birational map Φ_Σ between C and $C_1 \subset P^2$ such that A and its corresponding non-singular point A' on C_1 lie in the fundamental sets V_{Φ_Σ} and W_{Φ_Σ} (use the fact that A is generic). By the chain rule, the corresponding ϕ_λ passes through A' and is tangent to the curve C_1. Hence, by results of [6], we have that;

$I_{italian}(A', C_1, \phi_\lambda) \geq 2$

Using the technique of Section 2, it follows easily that $(*)$ holds as required.

Now choose a generic hyperplane H in P^w. Let pr be the projection of C from P to H. Let $D = pr(A) = pr(B)$. We have that $pr(C)$ is defined over P and, moreover, that D is generic in $pr(C)$. This follows as, otherwise, $dim(D/P) = 0$, therefore, as $\{A, B\} \subset l_{PD}$, $dim(A, B/P) = 0$, which implies $dim(A, B) = 1$, contradicting the fact that $\{A, B\}$ were independent generic. Hence, D defines a non-singular point on the curve $pr(C)$. Now, let l_1 and l_2 be the projections of the lines l_A and l_B. (We will deal with the degenerate case when one of these projections is a point below (†)). Clearly l_1 and l_2 pass through D. We claim that they have the property $(*)$. Let H_λ be a hyperplane of H passing through l_1. Then the hyperplane $< H_\lambda, P >$ of P^w passes through l_A. Let $\lambda' \in \mathcal{V}_\lambda$ be generic in Par_H, then, we may assume that $D \notin pr(C) \cap H_{\lambda'}$. Therefore, the corresponding hyperplane $< H_{\lambda'}, P >$ does not pass through A. Now, using the property $(*)$ of l_A, Lemma 2.10 and the fact that $< H_{\lambda'}, P >$ is an infinitesimal variation of $< H_\lambda, P >$, we can find distinct $\{A_1, A_2\} \subset \mathcal{V}_A \cap C \cap < H_{\lambda'}, P >$. It follows that $\{pr(A_1), pr(A_2)\} \subset \mathcal{V}_D \cap pr(C) \cap H_{\lambda'}$. We claim that $pr(A_1)$ and $pr(A_2)$ are distinct. Suppose not, then $pr(A_1) = pr(A_2) = D' \in \mathcal{V}_D \cap pr(C)$. Consider the following finite cover $F \subset pr(C) \times P^w$;

$F(y, x) \equiv y \in pr(C) \wedge x \in C \cap l_{yP}$

As we have shown, (DA) is generic for this cover, hence, by properties of Zariski structures, $mult_{(DA)}(F/pr(C)) = 1$. However, this contradicts the fact that we can find $D' \in \mathcal{V}_D \cap pr(C)$ and distinct $\{A_1, A_2\} \subset \mathcal{V}_A$ such that $F(D'A_1)$ and $F(D'A_2)$. Therefore, we have shown that $I_{italian}(D, pr(C), H_\lambda) \geq 2$. Hence, l_1 and l_2 have the property $(*)$. We now claim that $l_1 = l_2$ (**). Suppose not, then, if l_D is *the* tangent line to $pr(C)$ at D, it must be distinct from say l_1. Choose a hyperplane H_λ of ω passing passing through l_1 but not through l_D. We have that $I_{italian}(D, pr(C), H_\lambda) \geq 2$, but, using part of the proof of Lemma 2.10, H_λ must then pass through l_D. Hence, $(**)$ is shown.

Now, by $(**)$, we have that the tangent lines l_A and l_B both lie on the plane spanned by l_B and l_{AB}. Consider the statement;

$x \in NonSing(C)$ and the tangent line l_x lies on the plane spanned by l_B and l_{xB}.

It defines an algebraic subset of $NonSing(C)$ over B, and, moreover, as it holds for A, which was assumed to be generic in C and independent from B, it defines an open subset V of C. (***)

Now choose a generic hyperplane H and let $pr_B(C)$ be the projection of C from B to H. By the proof (below) that the degenerate case (†) cannot occur, and the argument above, we can find an open $W \subset V$, such that, for each $x \in W$, l_x is projected to $l_{pr_B(x)}$. We claim that;

For $y \in pr_B(W)$, the l_y intersect in a point Q. $(****)$

Let y_1 and y_2 be in $pr_B(W)$, then we can find corresponding x_1 and x_2 in W such that $< l_{By_1}, l_{y_1} >=< l_{Bx_1}, l_{x_1} >$ and $< l_{By_2}, l_{y_2} >=< l_{Bx_2}, l_{x_2} >$. As x_1 and x_2 lie in V, by $(*)$, we have that;

$< l_{Bx_1}, l_{x_1} > \cap < l_{Bx_2}, l_{x_2} >= l_B$

Note that we can exclude the case that $< l_{Bx}, l_x >=< l_{Bx}, l_B >$ for $x \in V$, as in this case C would clearly be contained in the plane $< l_{Bx}, l_B >$, use the fact that $pr_B(W)$ would have the property that all its tangents intersected in a line l and the proof below that the degenerate case (†) cannot occur. Hence, it follows that;

$l_{y_1} \cap l_{y_2} = pr_B(l_B) = Q$

as required. Now, to obtain the final contradiction, we need to show that $(****)$ cannot occur. This also covers the degenerate case $(\dagger)$. We show;

If C is a projective algebraic curve in P^w, with the property that there exists an open $W \subset Nonsing(C)$ such that each l_x for $x \in W$ intersects in a point Q, then C is a line l. $(\dagger)$

Choose a generic hyperplane H and consider the projection $pr_Q(C)$ of C from Q to H. If this projection is a point, then C is a line l. Hence, we may assume that $pr_Q(C)$ is a projective algebraic curve in H. Suppose that x is generic in $pr_Q(C)$, with $pr_Q(y) = x$ and y generic in C. Then x and y are nonsingular and, moreover, if $H_\lambda \subset H$ is *any* hyperplane passing through x, the corresponding hyperplane $< H_\lambda, Q >$ passes through y. By the assumption, $< H_\lambda, Q >$ contains the tangent line l_y. Hence, by the proof given above,

$$min\{I_{italian}(y, C, < H_\lambda, Q >), I_{italian}(x, C, H_\lambda)\} \geq 2.$$

Now, as x is nonsingular, we can find a hyperplane H_λ passing through x, not containing l_x. By the usual argument, given above, we obtain a contradiction. Therefore, $(\dagger)$ is shown.

As we assumed that our original C was not contained in a P^2, the proof of the lemma is shown. The lemma also holds in non-zero characteristic even though, surprisingly, the result $(\dagger)$ turns out to be false. We will deal with these problems in the final section.

□

Remarks 4.3. *The proof of the above lemma is attributed to Castelnuovo in* [10]*, but I have been unable to find a convenient reference.*

Definition 4.4. *We will define a correspondence $\Gamma \subset C_1 \times C_2$, where C_1 and C_2 are projective algebraic curves, to be generally biunivocal, if, for generic $x \in C_1$ there exists a unique generic $y \in C_2$ such that $\Gamma(x, y)$ and $\Gamma(x', y)$ implies that $x = x'$ (*). We will say that Γ is biunivocal at x if $(*)$ holds.*

We then have as an immediate consequence of the above that;

Lemma 4.5. *Let $C \subset P^w$, $(w \geq 3)$, be a projective algebraic curve, not contained in any hyperplane section. Fix some hyperplane H. Then,* *if P is a generic point of the curve, the projection pr_P from P to H is generally biunivocal on C.*

We now note the following;

Lemma 4.6. *Let $C \subset P^w$ be as in the above lemma. Fix a plane ω of dimension $w-k \geq 2$. Then, if $\{P_1, \ldots, P_k\}$ are independent generic points of C, the projection from $\Omega =< P_1, \ldots, P_k >$ to ω is generally biunivocal.*

Proof. Choose a sequence of planes $\omega = \omega_0 \subset \omega_1 \subset \ldots \subset \omega_{k-1}$ such that $dim(\omega_i) = dim(\omega) + i$ for $i \leq k-1$, with field of definition equal to that of ω. The projection pr_{P_1} from P_1 to ω_{k-1} is generally biunivocal, and $\{pr_{P_1}(P_2), \ldots, pr_{P_1}(P_k)\}$ define independent generic points on $pr_{P_1}(C)$. Repeating the argument k times, we obtain that the composition $pr_{P_k} \circ \ldots \circ pr_{P_1}$ is biunivocal as a projection to ω. We claim that $pr_{<P_1,\ldots,P_k>} = pr_{P_k} \circ \ldots \circ pr_{P_1}$ (*). This can be checked for a generic point x of C. Suppose, inductively, that $pr_{<P_1,\ldots,P_i>}(x) = pr_{P_i} \circ \ldots \circ pr_{P_1}(x) = z$. Let $u = pr_{P_{i+1}}(z)$, then there exists $z' =< P_1, \ldots, P_{i+1} > \cap \omega_{k-i}$ such that $l_{zz'} \cap \omega_{k-i-1} = u$. We have that $< P_1, \ldots, P_i, x > \cap \omega_{k-i} = z$, hence $< P_1, \ldots, P_i, P_{i+1}, x > \cap \omega_{k-i} = l_{zz'}$, therefore $< P_1, \ldots, P_{i+1}, x > \cap \omega_{k-i-1} = u$. This shows that $pr_{<P_1,\ldots,P_{i+1}>}(x) = u$ and, therefore, by induction, that $pr_{<P_1,\ldots,P_k>}(x) = pr_{P_k} \circ \ldots \circ pr_{P_1}(x)$. Hence $(*)$ is shown. It follows immediately that the projection from $\Omega =< P_1, \ldots, P_k >$ to ω is biunivocal on C as required. $\square$

As a consequence, we note the following, which is of independent interest;

Lemma 4.7. *Let $C \subset P^w$ be as in the previous lemma. Let $\{P_1, \ldots, P_k\}$ be independent generic points on C for $k \leq w-1$. Then the hyperplane $< P_1, \ldots, P_k >$ does not otherwise encounter C.*

Proof. By the previous lemma, the projection $pr_{<P_1,\ldots,P_{k-1}>}$ is generally biunivocal. Suppose there existed another intersection Q of $< P_1, \ldots, P_k >$ with C. Then $pr_{<P_1,\ldots,P_{k-1}>}(Q) = pr_{<P_1,\ldots,P_{k-1}>}(P_k)$, contradicting the definition of generally biunivocal. $\square$

Using Lemma 4.6, we obtain an alternative proof of Theorem 1.33;

Theorem 4.8. *Let $C \subset P^w$ be a projective algebraic curve, then C is birational to a plane projective curve.*

Proof. By Lemma 4.6, we can find a generally biunivocal correspondence between C and $C_1 \subset P^2$. In characteristic 0, we can therefore find $U \subset C$ and $V \subset C_1$ such that $U \cong V$. This defines a birational map $\Phi_\Sigma : C \leftrightsquigarrow C_1$. $\square$

We now use the techniques of this section to prove some further results which will be required later. We have first;

Lemma 4.9. *Let $C \subset P^w$ be a projective algebraic curve, not contained in any hyperplane section, and $x \in C$. Then there exists a plane Ω of dimension $w-3$ and a plane ω of dimension 2 such that the projection pr from Ω to ω is generally biunivocal and, moreover, biunivocal at x.*

Proof. Let H be a hyperplane containing x, then, by the assumption on C, it intersects C in a finite number r of points. It follows that we can find P generic in H such that l_{Px} does not otherwise encounter the curve C. Now choose a further hyperplane H' not containing P. Let Q be generic on C with Q independent from P. Suppose that l_{PQ} intersects C in a new point R, $(*)$. Then $\{Q, R\}$ must form a generic independent pair. Otherwise, as $P \in l_{QR}$, we would have that $dim(P/Q) = 0$ and therefore $dim(P) = 0$ which is a contradiction. Now, we can imitate the proof of Lemma 4.2 for the independent pair $\{Q, R\}$ and the projection pr_P to obtain a contradiction. It follows that $(*)$ cannot occur, hence the projection pr_P is generally biunivocal and, moreover, by construction, biunivocal at x. Now, repeat this argument $w-2$ times, to find a sequence of points $\{P_1, \ldots, P_{w-2}\}$ and planes $\{\omega = \omega_1 \subset \omega_2 \subset \ldots \subset \omega_{w-1} = P^w\}$ such that the projection pr_{P_j} from ω_{j+1} to ω_j is generally biunivocal on C and, moreover, biunivocal at x, for $1 \leq j \leq w-2$. It follows that the projection $pr_{<P_1,\ldots,P_{w-2}>}$ from $< P_1, \ldots, P_{w-2} >$ to ω has the required property. $\square$

Definition 4.10. *Let $C \subset P^w$ be a projective algebraic curve. Let pr be a projection from Ω to ω such that pr is biunivocal at x. We call $< \Omega, x >$ the axis of the projection at x.*

We now refine Lemma 4.9 to ensure that the axis of projection is not in "special position" with respect to x.

Definition 4.11. *Special Position*

Let $C \subset P^w$ be a projective algebraic curve and x an s-fold point on C, (see Definition 3.1). Let Ω be a plane passing through x. We *say that Ω is in special position if for every hyperplane H containing Ω;*

$I_{italian}(x, C, H) \geq s + 1.$

Lemma 4.12. *Let hypotheses be as in Lemma 4.9. Let x be an s-fold point of C. Then we can obtain the conclusion of Lemma 4.9, with the extra requirement that the axis of projection $< \Omega, x >$ is* not *in special position.*

Proof. As x is s-fold on C, we can find a hyperplane H passing through x such that $I_{italian}(x, C, H) = s$. Now imitate the proof of Lemma 4.9 by choosing a point P_1, generic on H such that $l_{P_1 x}$ does not otherwise encounter the curve C. Repeating the argument for the projected hyperplane $pr_{P_1}(H)$ in P^{w-1}, we obtain a series $\{P_1, \ldots, P_{w-2}\}$ as in the conclusion of Lemma 4.9 and, by construction, the axis of projection $< P_1, \ldots, P_{w-2}, x > \subset H$. Hence, $< \Omega, x >$ is not in special position. $\square$

We can now prove;

Lemma 4.13. *Let $C \subset P^w$ be a projective algebraic curve. Suppose that $x \in C$ and pr is a projection from Ω to $\omega = P^2$ satisfying the conclusion of Lemma 4.12. Then x is s-fold on C iff $pr(x)$ is s-fold on $pr(C)$.*

Proof. Let $V = \{x \in C : pr\ is\ biunivocal\ at\ x\}$ and $W = pr(V)$. V and W are open subsets of C and $pr(C)$ respectively and are in bijective correspondence. Let $\{H_\lambda : \lambda \in Par_H\}$ be the independent system Σ of lines in P^2. We then obtain a corresponding independent system Σ' on P^w defined by $\{< \Omega, H_\lambda >: \lambda \in Par_H\}$. Clearly, Σ has no base points on $pr(C)$. We claim that Σ' has no base points on C. Suppose that there existed a base point $y \in C$ for Σ', then clearly $pr(y)$ would be a base point for Σ on $pr(C)$, which is a contradiction. We now claim that for $y \in V$ and corresponding $pr(y) \in W$, that;

$I_{italian}(y, C, < \Omega, H_\lambda >) = I_{italian}(pr(y), pr(C), H_\lambda)\ (*)$

By the fact that Σ' has no base points, and results of Section 2, we have that;

$I_{italian}(y, C, < \Omega, H_\lambda >) = I^{\Sigma'}_{italian}(y, C, < \Omega, H_\lambda >)$

Therefore, the result follows immediately from the definition of $I^{\Sigma'}_{italian}$ and the fact that V and W are in bijective correspondence.

We now claim that $deg(C) = deg(pr(C))$. Suppose that $deg(pr(C)) = d$, then, by the proof of Lemma 2.4, we can find a generic plane H_λ intersecting $pr(C)$ transversely at d distinct points inside W. By $(*)$, the corresponding hyperplane $< \Omega, H_\lambda >$ intersects C transversely at d distinct points inside V. Hence, by general properties of infinitesimals, $deg(C) = d$ as well.

Now, let Σ'' be the independent system defined by the lines in P^2 passing through $pr(x)$. By Bezout, it defines a $g^r_d(\Sigma'')$ on $pr(C)$. Suppose that $pr(x)$ is s_1-fold on $pr(C)$, then, after removing the fixed point contribution at $pr(x)$, we obtain a $g^r_{d-s_1}$ on $pr(C)$ without fixed points. Let Σ''' be the independent system defined by $< \Omega, \Sigma'' >$. It consists exactly of the hyperplanes passing through the axis of projection $< \Omega, x >$. Again, by hyperspatial Bezout, it defines a $g^r_d(\Sigma''')$ on C. As pr was assumed to be biunivocal at x, this $g^r_d(\Sigma''')$ has a unique fixed point at x. Moreover, by the assumption on pr that the axis of projection is not in special position, if x is s_2-fold on C, then its fixed point contribution at x is s_2. Hence, after removing this contribution, we obtain a $g^r_{d-s_2}$ on C without fixed points. Now, using Lemma 2.17, for generic λ, the weighted set W_λ for $g^r_{d-s_2}$ consists of $d - s_2$ points, each counted once, lying inside V. Using $(*)$, the corresponding V_λ for $g^r_{d-s_1}$ consists of $d - s_2$ points, each counted once, lying inside W. By Lemma 2.17 again, we must have that $d - s_1 = d - s_2$, hence $s_1 = s_2$ as required. □

As an easy consequence of the above lemma, we have;

Lemma 4.14. *Let $C \subset P^w$ be a projective algebraic curve and suppose that $x \in C$, then x is non-singular iff x is not multiple.*

Proof. Choose a projection pr as in Lemma 4.13. Then, x is not multiple iff $pr(x)$ is not multiple. By Lemma 3.2, $pr(x)$ is not multiple iff $pr(x)$ is non-singular. In characteristic 0, using the fact that pr is generally biunivocal and an elementary model theoretic argument, we can find an inverse morphism $\phi : W' \rightarrow V'$, where $\{W', V'\}$ are open subsets of $\{W, V\}$ given in the previous lemma. It follows that $V' \cong W'$. As $pr(x)$ is non singular, we can extend the morphism ϕ to include $pr(x)$ in its domain, by biunivocity of pr at x, we must have that $\phi(pr(x)) = x$. Hence, we may assume that x lies in V' and $pr(x)$

lies in W'. Therefore, $pr(x)$ is non-singular iff x is non-singular. Hence, the result is shown.

□

Combining this result with Theorem 3.3, we then have the following;

Theorem 4.15. *Let $C \subset P^w$ be a projective algebraic curve, then C is birational to a non-singular projective algebraic curve $C_1 \subset P^{w'}$.*

We finish this section with the following application of the method of Conic Projections, the result will not be required later in the paper. Some part of the proof will require methods developed in Sections 5 and 6, we refer the reader to Definition 6.3 for the terminology "node". I have not seen a reasonable algebraic proof of this result.

Theorem 4.16. *Let $C \subset P^w$ be a projective algebraic curve, then C is birational to a plane projective algebraic with at most nodes as singularities.*

Proof. We may, by the previous theorem, assume that C is non-singular. We first reduce to the case $w = 3$. That is, we claim that C is birational to a non-singular projective algebraic curve $C' \subset P^3$. Let V_3 be the variety of chords on C. That is;

$$V_3 = \overline{\{\bigcup l_{ab} : (a, b) \in C^2 \setminus \Delta\}}$$

We may assume that C is not contained in any plane of dimension 2 (†) and that $w \geq 4$. We then claim that V_3 has dimension 3. Let $a \in C$ and define;

$$Cone_a = \overline{\{\bigcup l_{ab} : b \in (C \setminus a)\}}$$

As C is irreducible, so is $Cone_a$. (use the fact that for any component W of $Cone_a$, $\{b \in (C \setminus a) : l_{ab} \subset W\}$ is a closed subset of $C \setminus a$). Suppose that $Cone_a$ has dimension 1. Then there exists b such that;

$$Cone_a = l_{ab}$$

It follows immediately that C is contained in l_{ab}, contradicting the hypothesis (†). Hence, $Cone_a$ has dimension 2. Now, suppose that V_3 has dimension 2. Then there exist finitely many points $\{a_1, \ldots, a_m\}$ such that;

$V_3 = Cone_{a_1} \cup \ldots \cup Cone_{a_m}$

In particular, we can find distinct $\{a, b\} \subset C$ such that $Cone_a = Cone_b$. Choose $a' \in C$ distinct from $\{a, b\}$. Then we may assume that $l_{aa'}$ has infinite intersection with $Cone_b \setminus \delta(Cone_b)$ and therefore $C \subset H_{aa'b}$, contradicting the hypothesis (†). Hence, V_3 has dimension 3 as required. Now define the tangent variety on C;

$Tang(C) = \{\bigcup l_a : a \in C, l_a \textit{ is the tangent line to } C \textit{ at } a\}$

We claim that $Tang(C)$ is a closed subvariety of V_3 of dimension 2. In order to see that $Tang(C) \subset V_3$, it is sufficient to prove that $l_a \subset Cone_a$. Consider the following covers $F, F^* \subset (C \setminus \{a\}) \times P^w$;

$F(x, y) \equiv x \in (C \setminus \{a\}) \wedge y \in l_{ax}$

$F^*(x, \lambda) \equiv x \in (C \setminus \{a\}) \wedge H_\lambda \textit{ contains } l_{ax}$

Let $\bar{F}$ and $\bar{F}^*$ be the closures of these covers inside $C \times P^w$. We have the incidence relation $I \subset P^w \times P^{w^*}$ given by;

$I(x, \lambda) \equiv x \in H_\lambda$

By definition, for $x \in (C \setminus \{a\})$, we have that;

$I(\bar{F}(x), \bar{F}^*(x))$;

Hence, this relation holds at $\{a\}$ as well $(*)$. Now, by the proof of Theorem 6.7, the fibre $\bar{F}^*(a)$ consists exactly of the hyperplanes H_λ containing l_a. Hence, by $(*)$, we must have that the fibre $\bar{F}(a)$ defines l_a. In order to complete the proof, observe that the projection $pr : C \times P^w \to P^w$ defines a morphism from $\bar{F}$ to $Cone_a$. Let;

$\Gamma_{pr}(x, y, z) \subset \bar{F} \times Cone_a \subset C \times P^w \times P^w$

be the graph of this projection. Then, for $x \in (C \setminus \{a\})$, the fibre $\Gamma_{pr}(x) = l_{ax} \times l_{ax}$. Hence, the fibre $\Gamma_{pr}(a) = l_a \times l_a$. This proves that $l_a \subset Cone_a$ as required. We clearly have that $Tang(C)$ has dimension 2 if C is not contained in a line l, contradicting the hypothesis (†). The fact that $Tang(C)$ is a closed subvariety of V_3 follows immediately from the fact that $Tang(C)$ is a closed subvariety of P^w.

Now choose a plane Ω of dimension $w-4$ such that $\Omega \cap V_3 = \emptyset$. Let pr be the projection from Ω to ω where ω is a plane of dimension 3 and $\Omega \cap \omega = \emptyset$. Let $pr(C) \subset \omega$ be the projection of C. We claim that $pr(C)$ is birational to C and non-singular. First, observe that pr is defined everywhere on C as $C \subset V_3$ and $\Omega \cap V_3 = \emptyset$. Secondly, we have that pr is biunivocal everywhere on C. For suppose that $pr(x) = pr(x')$, then $< \Omega, x > = < \Omega, x' > = < \Omega, pr(x) >$, hence $l_{xx'}$ intersects Ω, contradicting the fact that $\Omega \cap V_3 = \emptyset$. Finally, we have that for every $x \in C$, the axis of projection $< \Omega, x >$ is *not* in special position, see Definition 4.11. This follows from the fact that, as x is non-singular, $< \Omega, x >$ would only be in special position if it contained the tangent line l_x. As $Tang(C) \subset V_3$, this would contradict the fact that $\Omega \cap V_3 = \emptyset$. Now, by Lemmas 4.13 and 4.14, we must have that $pr(x)$ is non-singular for every $x \in C$, that is $pr(C)$ is non-singular. We may, therefore, using the argument of Lemma 4.14 invert the morphism pr to obtain an isomorphism between C and $pr(C)$. In particular, C and $pr(C)$ are birational.

We now consider the curve $C' = pr(C) \subset P^3$. In order to prove the theorem, it will be sufficient to show that C' is birational to a plane projective curve with at most nodes as singularities. We now define the following 5 varieties, we use the notation l to denote a line and P to define a 2-dimensional plane;

(i). $Tangent(C') = \{\bigcup l_a : a \in C, l_a \textit{ the tangent line at } C\}$

(ii). $Trisecant(C') = \overline{\{\bigcup l_{abc} : (a,b,c) \in C^3 \textit{ distinct}, l_{abc} \supset \{a,b,c\}\}}$

By a bitangent plane P_{ab}, we mean a hyperplane passing through the tangent lines l_a and l_b for distinct $\{a,b\}$ in C. By an osculatory plane P_a, we refer the reader to Definition 6.3.

We define $Supp(P_{ab}) = \{a,b\}$ and $Supp(P_a) = C' \cap P_a$. We then consider;

(iii). $Bitangent\,Chord(C') = \overline{\{\bigcup l_{ab} : (a,b) \in (C^2 \setminus \Delta), l_{ab} \supset Supp(P_{ab})\}}$

(iv). $Osculatory\,Chord(C') = \overline{\{\bigcup_{a \in C} l_{ab} : (a,b) \in (Supp(P_a)^2 \setminus \Delta)\}}$.

By Remark 6.6, there exists a finite set $W \subset C'$, consisting of points which are the origins of non-ordinary branches. We let;

(v). $Singular\ Cone(C') = \bigcup_{y \in W} Cone_y$

As we have already seen, $Tangent(C')$ defines a 2-dimensional algebraic variety, unless C' is contained in a line, which we may assume is not the case. By Lemma 4.2, there are no trisecant lines passing through an independent generic pair (a,b) of C'^2, unless C' is contained in a plane, which again we may exclude. The statement $D(x,y) \subset C'^2 \setminus \Delta$, given by;

$$D(x,y) \equiv \{(x,y) \in C'^2 : (x \neq y) \wedge \nexists w(w \neq x \wedge w \neq y \wedge w \in C' \cap l_{xy})\}$$

is clearly constructible and has the property that $\overline{D} = C'^2$. This clearly implies that $dim(Trisecant(C')) \leq 2$. We now claim that, if (a,b) is an independent generic pair in C'^2, there is *no* bitangent plane $P_{a,b}$. Let $\Sigma = \{H_\lambda : \lambda \in P^{3*}\}$ be the system of hyperplanes in P^3. Σ has no fixed points on C', hence the linear condition that H_λ passes through l_a has codimension 2 and defines a 1-dimension family $\Sigma_1 \subset \Sigma$. As we may assume that C' is not contained in any hyperplane section,Σ_1 has finite intersection with C'. As b is independent generic from a, it cannot be a base point for the new system Σ_1. Hence, using the results of Theorems 6.2 and 6.5, the condition that $H_\lambda \in \Sigma_1$ passes through l_b is also a codimension 2 linear condition. As Σ_1 is 1-dimensional, this implies the claim. Now consider the statement $D'(x,y) \subset C'^2 \setminus \Delta$, given by;

$$D'(x,y) \equiv \{(x,y) \in C'^2 : (x \neq y) \wedge \nexists \lambda(H_\lambda \supset l_a \cup l_b)\}$$

Again, this is constructible and has the property that $\overline{D'} = C'^2$. This clearly implies that $dim(Bitangent\ Chord(C')) \leq 2$. For any given $a \in C$, there exists a unique osculatory plane P_a passing through a. Hence, there exist finitely many lines of the form l_{ab} for $b \in (C^2 \setminus \Delta) \cap Supp(P_a)$. This clearly implies that $dim(Osculatory\ Chord(C')) \leq 2$. Finally, by the above consideration, we have that $dim(Singular\ Cone(C')) = 2$

We now choose a point P in P^3 such that P lies outside the above defined varieties. This is clearly possible as they all have dimension at most 2. Let ω be a plane of dimension 2 not containing P and let $pr_P(C')$ be the projection of C' from P to ω. We claim that $pr_P(C')$ is birational to C' with at most nodes as singularities. First, suppose that $y \in pr_P(C')$ is singular, then, by Lemma 4.14, it must be multiple. By Lemma 4.13 and the fact that C' is non-singular, this can only occur if either $|pr_P^{-1}(y)| \geq 2$ or there exists a unique x such that $pr_P(x) = y$ and the axis of projection $< P, x >$ is in special position. We may exclude the second possibility on the grounds that P is disjoint from $Tang(C')$. We may therefore assume that $|pr_P^{-1}(y)| \geq 2$. Now, we may exclude the possibility that $|pr_P^{-1}(y)| \geq 3$, on the grounds that P is disjoint from $Trisecant(C')$. Hence, we may assume that $|pr_P^{-1}(y)| = 2$. Now, as C' is non-singular, by Definition 5.2, there exist exactly 2 branches centred at y on $pr_P(C')$. Moreover, we may assume that both elements of $pr_P^{-1}(y) = \{x_1, x_2\}$ are the origins of ordinary branches, otherwise P would lie inside $Singular\ Cone(C')$. If P is situated on the osculatory plane P_{x_1} of say x_1, then $(x_1x_2) \subset Supp(P_{x_1})$ and, hence, as $P \in l_{x_1x_2}$, we would have that P lies in $Osculatory\ Chord(C')$, which is not the case. Hence, we may apply Theorem 6.4, to obtain that both branches have character $(1,1)$. Finally, let l_a and l_b be the tangent lines to the 2 branches $\{\gamma_y^1, \gamma_y^2\}$ at y. We claim that l_a and l_b are distinct $(*)$. By definition, we have that;

$$I_{italian}(y, \gamma_y^1, pr_P(C'), l_a) = 2$$

Hence, using Definition 5.15 and Lemma 5.17, we can find an infinitesimal variation l_a' of l_a intersecting γ_y^1 in distinct points $\{y', y''\}$. By Definition 5.15, we can find distinct points $\{x_1', x_1''\}$ in $C' \cap \mathcal{V}_{x_1}$ such that $pr_P(x_1') = y'$ and $pr_P(x_1'') = y''$. We clearly have that that $< l_a', P >$ is an infinitesimal variation of $< l_a, P >$ which intersects C' in $\{x_1', x_1''\}$. Therefore;

$$I_{italian}(x_1, C', < l_a, P >) \geq 2$$

and hence, by previous arguments, $< l_a, P >$ contains the tangent line l_{x_1}. As the axis of projection l_{Px_1} was not in special position, we must have that $pr_P(l_{x_1}) = l_a$. Similarly, $pr_P(l_{x_2}) = l_b$. Now, if $l_a = l_b$, we would have that $< l_a, P >$ is a bitangent plane to C', $l_{x_1x_2}$ is a bitangent chord and P would belong to the variety $Bitangent\ Chord(C')$. As this is not the case, the claim $(*)$ is proved. This completes the theorem.

$\square$

5. A Theory of Branches for Algebraic Curves

We now develop a theory of branches for algebraic curves, using the techniques of the previous sections.

Let $C \subset P^w$ be a projective algebraic curve. Then, by Theorem 4.15, we can find $C^{ns} \subset P^{w_1}$ which is non-singular and birational to C. Let

$\Phi^{ns} : C^{ns} \leftrightsquigarrow C$ be the birational map between C^{ns} and C. As C^{ns} is non-singular, Φ^{ns} extends to a totally defined morphism from C^{ns} to C. As usual, we let $\Gamma_{\Phi^{ns}}$ denote the graph of the correspondence between C^{ns} and C. It has the property that, given any $x \in C^{ns}$, there exists a unique $y \in C$ such that $\Gamma_{\Phi^{ns}}(x, y)$. We claim the following;

Lemma 5.1. *Let $C_1 \subset P^{w_1}$ and $C_2 \subset P^{w_2}$ be* any *two non-singular birational models of C, with corresponding morphisms Φ_1^{ns} and Φ_2^{ns}. Then there exists a unique* isomorphism *$\Phi : C_1 \leftrightarrow C_2$ with the property that $\Phi_2^{ns} \circ \Phi = \Phi_1^{ns}$ and $\Phi_1^{ns} \circ \Phi^{-1} = \Phi_2^{ns}$.*

Proof. As Φ_2^{ns} is a birational map, we can invert it to give a birational map $\Phi_2^{ns-1} : C \leftrightsquigarrow C_2$. Then the composition $\Phi_2^{ns-1} \circ \Phi_1^{ns} : C_1 \leftrightsquigarrow C_2$ is birational as well. As C_1 is non-singular, we can extend this birational map to a totally defined morphism $\Phi : C_1 \rightarrow C_2$. By the same argument, we can find a totally defined morphism $\Psi : C_2 \rightarrow C_1$ with the property that Ψ inverts Φ as a birational map. We claim that Ψ inverts Φ as a morphism. We have that $\Psi \circ \Phi : C_1 \rightarrow C_1$ is a morphism with the property that there exists an open $U \subset C_1$ such that $\Psi \circ \Phi | U = Id_U$. Then $Graph(\Psi \circ \Phi) \subset C_1 \times C_1$ is closed irreducible and intersects the diagonal Δ in an open dense subset. Therefore, $Graph(\Psi \circ \Phi) = \Delta$, hence $\Psi \circ \Phi = Id_{C_1}$. Similarly, one shows that $\Phi \circ \Psi = Id_{C_2}$. Therefore, Φ is an isomorphism. By construction, Φ_1^{ns} and $\Phi_2^{ns} \circ \Phi$ agree as birational maps and are totally defined, hence, by a similar argument, they agree as morphisms. Similarly, Φ_2^{ns} and $\Phi_1^{ns} \circ \Phi^{-1}$ agree as morphisms. For the uniqueness statement, use the fact that any 2 isomorphisms Φ_1 and Φ_2, satisfying the properties of Φ, would agree on an open subset U of C_1, hence must be identical. □

Now fix a point O of C. Let $\Gamma_{\Phi^{ns}} \subset C^{ns} \times C$ be the graph of the correspondence defined above. We denote by $\{O_1, \ldots, O_t\}$ the fibre $\Gamma_{\Phi^{ns}}(x, O)$. Note that, by the previous lemma, if we are given 2 non-singular models with correspondences $\Gamma_{\Phi_1^{ns}}$ and $\Gamma_{\Phi_2^{ns}}$, then we have a correspondence $O_1 \sim O_1', O_2 \sim O_2', \ldots, O_t \sim O_t'$ between the fibres $\Gamma_{\Phi_1^{ns}}(x, O)$ and $\Gamma_{\Phi_2^{ns}}(x, O)$, given by $O_j \sim O_j'$ iff $\Phi(O_j) = O_j'$, where Φ is the isomorphism given by the previous lemma. Moreover, as Φ is an isomorphism, we also have a correspondence of infinitesimal neighborhoods $\mathcal{V}_{O_1} \sim \mathcal{V}_{O_1'}, \ldots, \mathcal{V}_{O_t} \sim \mathcal{V}_{O_t'}$, given by $\mathcal{V}_{O_j} \sim \mathcal{V}_{O_j'}$ iff $\Phi : \mathcal{V}_{O_j} \cong \mathcal{V}_{O_j'}$, here we mean that Φ is a bijection of sets. We now make the following definition;

Definition 5.2. *Let $C \subset P^w$ be a projective algebraic curve. Suppose that $O \in C$. For $1 \leq j \leq t$, we define the branches γ_O^j at O to be the equivalence classes $[\mathcal{V}_{O_j}]$ of the infinitesimal neighborhoods of O_j in the fibre $\Gamma_{\Phi^{ns}}(x, O)$ of* any *non-singular model C^{ns} of C. We define γ_O to be the union of the branches at O.*

Remarks 5.3. *Note that the definition does not depend on the choice of a non-singular model C^{ns}, however, for computational purposes, it* is *convenient to think of a branch γ as the infinitesimal neighborhood of some O_j in a fixed non-singular model C^{ns}.*

We now have the following lemmas concerning branches;

Lemma 5.4. *Let $C \subset P^w$ and $O \in C$ be an s-fold point. Then O is the origin of at most s-branches. In particular, a non-singular point is the origin of a single branch.*

Proof. Suppose that O is the origin of t branches. Then there exists a non-singular model C^{ns} and a birational map $\Phi_\Sigma : C^{ns} \rightarrow C$ such that $\Phi_\Sigma^{-1}(O) = \{O_1, \ldots, O_t\}$. By a slight extension to Remarks 1.32, and the fact that $\{O_1, \ldots, O_t\}$ are non-singular, we may assume that $Base(\Sigma)$ is disjoint from this set (*). Let Σ define the system of hyperplanes in P^w. It defines a $g_d^r(\Sigma)$ without fixed points on C, where $d = deg(C)$. By Lemma 2.32, there is a corresponding $g_{d+I}^r(\Sigma)$ on C^{ns}, where I is the total fixed point contribution at $Base(\Sigma)$. Now let $\Sigma' \subset \Sigma$ define the system of hyperplanes passing through O. It defines a $g_d^{r-1}(\Sigma')$ on C with corresponding $g_{d+I}^{r-1}(\Sigma')$ on C^{ns}. As O is s-fold on C, we can find $g_{d-s}^{r-1} \subset g_d^{r-1}(\Sigma')$ on C without fixed points. By Lemma 2.31, we can find a corresponding $g_{d-s}^{r-1} \subset g_{d+I}^{r-1}(\Sigma')$ on C^{ns}, without fixed points (**). Now each ϕ_λ in Σ' must pass through $\{O_1, \ldots, O_t\}$. Moreover, the fixed point contribution from Σ' at $Base(\Sigma)$ must be at least I. Hence, by the assumption (*), the total fixed point contribution from $Base(\Sigma')$ must be at least $t+I$. Hence, by $(**)$, $d-s \leq (d+I)-(t+I)$, therefore $t \leq s$. The lemma is proved.

□

We now make the following definition;

Definition 5.5. *Let $C_1 \subset P^{w_1}, C_2 \subset P^{w_2}, C_3 \subset P^{w_3}$ be birational projective algebraic curve with correspondences $\Gamma_{\Phi_1} \subset C_1 \times C_2$ and $\Gamma_{\Phi_2} \subset C_2 \times C_3$. We define the composition $\Gamma_{\Phi_2} \circ \Gamma_{\Phi_1} \subset C_1 \times C_3$ to be;*

$\Gamma_{\Phi_2} \circ \Gamma_{\Phi_1}(xz) \equiv \exists y(y \in C_2 \wedge \Gamma_{\Phi_1}(xy) \wedge \Gamma_{\Phi_2}(yz))$

Lemma 5.6. *Let hypotheses be as in the previous lemma, then if $\Phi_1 : C_1 \rightarrow C_2$ and $\Phi_2 : C_2 \rightarrow C_3$ are birational maps;*

$\Gamma_{\Phi_2} \circ \Gamma_{\Phi_1} = \Gamma_{\Phi_2 \circ \Phi_1}$

Proof. The proof is a straightforward consequence of the fact that both correspondences obviously agree on a Zariski open subset. □

Lemma 5.7. *Birational Invariance of Branches*

Let $C_1 \subset P^{w_1}$ and $C_2 \subset P^{w_2}$ be birational projective algebraic curves with correspondence $\Gamma_{[\Phi]} \subset C_1 \times C_2$. Fix $O \in C_2$ and let $\{P_1, \ldots, P_s\} = \Gamma_{[\Phi]}(x, O)$. Then, $[\Phi]$ induces an injective map;

$[\Phi]^* : \gamma_O \rightarrow \bigcup_{1 \leq k \leq s} \gamma_{P_k}$

and, moreover;

$[\Phi]^* : \bigcup_{O \in C_2} \gamma_O \rightarrow \bigcup_{O \in C_1} \gamma_O$

is a bijection, with inverse given by $[\Phi^{-1}]^*$.

Proof. We first claim that there exists a non-singular model C^{ns} of C_1 and C_2 with morphisms $\Phi_1 : C^{ns} \rightarrow C_1$ and $\Phi_2 : C^{ns} \rightarrow C_2$ such that $\Phi_2 = \Phi \circ \Phi_1$ and $\Phi_1 = \Phi^{-1} \circ \Phi_2$ as birational maps (*). Choose a non-singular model C^{ns} of C_1 with morphism $\Phi_1 : C^{ns} \rightarrow C_1$. Then $\Phi \circ \Phi_1$ defines a birational map from C^{ns} to C_2. As C^{ns} is non-singular, it extends(uniquely) to a birational morphism $\Phi_2 : C^{ns} \rightarrow C_2$. Clearly, $\{C^{ns}, \Phi_1, \Phi_2\}$ have the property $(*)$. In order to define the map $[\Phi]_*$, let $O \in C_2$ and suppose that γ_O^j is a branch corresponding to the equivalence class $[\mathcal{V}_{O_j}]$, where O_j lies in the fibre $\Phi_2^{-1}(O)$. We claim that $\Phi_1(O_j)$ lies in the fibre $\Gamma_{[\Phi]}(x, O)$ $(**)$. As $\Phi \circ \Phi_1 = \Phi_2$ as birational maps, it follows, by Lemma 1.21, that $\Gamma_{\Phi \circ \Phi_1} = \Gamma_{\Phi_2}$. However, by the previous lemma, $\Gamma_{\Phi \circ \Phi_1} = \Gamma_\Phi \circ \Gamma_{\Phi_1}$, which gives $(**)$. Then, if $\Phi_1(O_j) = P_k$, and $\Phi_1^{-1}(P_k) = \{P'_{k1}, \ldots, P'_{kr}, \ldots, P'_{kl}\}$, for $1 \leq r \leq l$, we must have that $[\mathcal{V}_{O_j}] = [\mathcal{V}_{P'_{kr}}]$, for some r. Hence, we can set $[\Phi]^*(\gamma_O^j) = \gamma_{P_k}^r$. One needs to check that this definition does not depend either on the choice of the non-singular model C^{ns} or the choice of the birational map Φ representing the correspondence $\Gamma_{[\Phi]}$. For the first claim, after fixing a choice of Φ, note that by the requirement $(*)$, Φ_1 is uniquely determined by the choice of $\{C^{ns}, \Phi_2\}$. By Lemma 5.1, for any 2 choices $\{C_1^{ns}, \Phi_2^1\}$ and $\{C_2^{ns}, \Phi_2^2\}$, there exists a connecting isomorphism $\Theta : (C_1^{ns}, \Phi_2^1) \leftrightarrow (C_2^{ns}, \Phi_2^2)$, which must therefore be a connecting isomorphism $\Theta : (C_1^{ns}, \Phi_1^1) \leftrightarrow (C_2^{ns}, \Phi_1^2)$. The result then follows immediately by definition of the branch as an equivalence class of infinitesimal neighborhoods. For the second claim, after fixing a choice of $\{C^{ns}, \Phi_1, \Phi_2\}$ and replacing Φ by an equivalent Φ', one still obtains $(*)$ for the original $\{C^{ns}, \Phi_1, \Phi_2\}$ and the result follows trivially. The rest of the lemma now follows by proving that $[\Phi^{-1}]^*$ inverts $[\Phi]^*$. This is trivial to check using the particular choice $\{C^{ns}, \Phi_1, \Phi_2, \Phi\}$. $\square$

Remarks 5.8. *Note that points are* not *necessarily preserved by birational maps, but, by the above, branches* are *always preserved.*

We now refine the Italian definition of intersection multiplicity, in order to take into account the above construction of a branch. Suppose that $C \subset P^w$ is a projective algebraic curve. We let Par_F be the projective parameter space for *all* hypersurfaces of a given degree e. Now, fix a particular form F_λ of degree e, such that F_λ has finite intersection with C. The condition that a hypersurface of degree e contains C is clearly linear on Par_F, (use Theorem 2.3 to show that the condition is equivalent to the plane condition P on Par_F of passing through $de+1$ distinct points on the curve C.) Now fix a maximal linear system Σ containing F_λ, such that Σ has empty intersection with P. It is trivial to check that the corresponding $g_n^r(\Sigma)$ on C has no fixed points. Now fix a non-singular model C^{ns} of C, with corresponding birational morphism $\Phi_{\Sigma'}$. By the transfer result Lemma 2.31, we obtain a corresponding g_n^r without fixed points on C^{ns}. Let $Card(O, V_\lambda, g_n^r)$ denote the number of times $O \in C^{ns}$ is counted for this g_n^r in the weighted set V_λ. We then define;

Definition 5.9. $I_{italian}(p, \gamma_p^j, C, F_\lambda) = Card(p_j, V_\lambda, g_n^r)$ $(*)$

where the branch γ_p^j *corresponds to* $[\mathcal{V}_{p_j}]$ *in the fibre* $\Gamma_{[\Phi_{\Sigma'}]}(x, p)$.

Remarks 5.10. *One may, without loss of generality, assume that* $Base(\Sigma')$ *is disjoint from the fibre* $\Gamma_{[\Phi_{\Sigma'}]}(x, p)$. *In which case, the definition becomes the more familiar;*

$I_{italian}(p, \gamma_p^j, C, F_\lambda) = Card(C^{ns} \cap \overline{F}_{\lambda'} \cap \mathcal{V}_{p_j})$, $\lambda' \in \mathcal{V}_\lambda$ *generic in* Par_Σ.

Here, we have used the notation $\{\overline{F}_\lambda : \lambda \in Par_\Sigma\}$ to denote the family of "lifted" forms on C^{ns} corresponding to the family $\{F_\lambda : \lambda \in Par_\Sigma\}$ on C, by way of the birational morphism $\Phi_{\Sigma'}$.

Lemma 5.11. *Given Σ containing F_λ. The definition $(*)$ does not depend on the choice of the non-singular model C^{ns} and birational morphism $\Phi_{\Sigma'}$.*

Proof. We divide the proof into the following 2 cases;

Case 1. (C^{ns}, Φ) is fixed and we have 2 presentations Φ_{Σ_1} and Φ_{Σ_2} of the birational morphism Φ, as given by Lemma 1.20, such that $Base(\Sigma_1)$ possibly includes some of the points in $\Gamma_{[\Phi]}(x,p)$, while $Base(\Sigma_2)$ is disjoint from this set.

Let $V = V_{\Phi_{\Sigma_1}} \cap V_{\Phi_{\Sigma_2}}$ with corresponding $W \subset C$. Denote the weighted sets of the *2* given g_n^r on C^{ns},in the definition $(*)$, corresponding to the presentations Φ_{Σ_1} and Φ_{Σ_2}, by $\{V_\lambda^1\}$ and $\{V_\lambda^2\}$. By the proof of Lemma 2.31, if $x \in V$, then x is counted s-times in V_λ^1 iff x is counted s-times in V_λ^2 $(\dagger)$. Now, we claim that $Card(p_j, V_\lambda^1, g_n^r) = Card(p_j, V_\lambda^2, g_n^r)$. This follows by $(\dagger)$ and an easy application of, say Lemma 2.16, to witness both these cardinalities in the canonical set V.

Case 2. We have 2 non-singular models (C_1^{ns}, Φ_1) and (C_2^{ns}, Φ_2), with presentations Φ_{Σ_1} and Φ_{Σ_2}, such that $Base(\Sigma_1)$ is disjoint from $\Gamma_{[\Phi_1]}(x,p)$ and $Base(\Sigma_2)$ is disjoint from $\Gamma_{[\Phi_2]}(x,p)$.

Using Lemma 5.1, we can find an isomorphism $\Phi : C_1 \rightarrow C_2$, such that $\Phi_2 \circ \Phi = \Phi_1$. Let $\{G_\lambda : \lambda \in Par_\Sigma\}$ denote the lifted forms on C_1^{ns} corresponding to the morphism Φ_{Σ_1} and $\{H_\lambda : \lambda \in Par_\Sigma\}$ denote the lifted forms on C_2^{ns} corresponding to the morphism Φ_{Σ_2}. We need to check that;

$$Card(C_1^{ns} \cap G_{\lambda'} \cap \mathcal{V}_{p_j}) = Card(C_2^{ns} \cap H_{\lambda'} \cap \mathcal{V}_{q_j}),\ \lambda' \in \mathcal{V}_\lambda \cap Par_\Sigma \text{ generic.}$$

where $\Phi(p_j) = q_j$ and $\{p_j, q_j\}$ in $\{C_1^{ns}, C_2^{ns}\}$ respectively, correspond to the branch γ_p^j.

Suppose that $Card(C_2^{ns} \cap H_{\lambda'} \cap \mathcal{V}_{q_j}) = n$. We may, without loss of generality, assume that $Base(\Phi_{\Sigma_3})$ is disjoint from $\Gamma_{[\Phi_1]}(x,p)$ in a particular presentation Φ_{Σ_3} of the morphism Φ $(\dagger)$. Hence, $Base(\Phi_{\Sigma_2} \circ \Phi_{\Sigma_3})$ is disjoint from $\Gamma_{[\Phi_1]}(x,p)$ as well $(\dagger\dagger)$. Let $\{\Phi_{\Sigma_3}^* H_\lambda : \lambda \in Par_\Sigma\}$

denote the lifted forms on C_1^{ns} corresponding to this presentation. By the fact that Φ is an isomorphism and (†), we have $Card(C_1^{ns}\cap\Phi^*H_{\lambda'}\cap\mathcal{V}_{p_j})=n$. Now let $V=V_{\Phi_{\Sigma_1}}\cap V_{\Phi_{\Sigma_2}\circ\Phi_{\Sigma_3}}$. By (††) and Lemma 2.13, we may witness $Card(C_1^{ns}\cap\Phi^*H_{\lambda'}\cap\mathcal{V}_{p_j})=n$ inside the canonical set V. Now, using the fact that $\Phi_2\circ\Phi=\Phi_1$ as birational maps, and Lemma 2.31, it follows that $Card(C_1^{ns}\cap G_{\lambda'}\cap\mathcal{V}_{p_j})=n$.

□

Lemma 5.12. *The definition* $(*)$ *does not depend on the choice of a maximal independent system* Σ *containing* F_λ*, having finite intersection with* C*.*

Proof. Let Σ be a maximal independent system containing F_λ. We claim first that Σ has no base points on C, (†). For suppose that $w\in Base(\Sigma)\cap C$. Let F_μ be *any* form of degree e having finite intersection with C. Then $<F_\mu,\Sigma>$ defines a system of higher dimension, hence must intersect H in a point. That is, we can find parameters $\{\alpha,\beta\}$ and a fixed F_λ in Σ such that $\alpha F_\mu+\beta F_\lambda$ contains C. It follows immediately that w must also be a base point for F_μ. Clearly, we can find a form of degree e, having finite intersection with C, which doesn't contain w. This gives a contradiction and proves (†). Now, given a choice of Σ containing F_λ, by the proof of the previous lemma, we may assume that;

$$I_{italian}(p,\gamma_p^j,C,F_\lambda)=Card(C^{ns}\cap\overline{F_{\lambda'}}\cap\mathcal{V}_{p_j}),\ \lambda'\in\mathcal{V}_\lambda \text{ generic in } Par_\Sigma.$$

As p is not a base point for Σ on C, it follows immediately that p_j is not a base point for Σ on C^{ns}. Hence, by Lemma 2.12, we have that;

$$I_{italian}(p,\gamma_p^j,C,F_\lambda)=I_{italian}(p_j,C^{ns},\overline{F_\lambda})$$

Clearly, this equality does not depend on the particular choice of Σ containing F_λ but only on the presentation of the morphism $\Phi:C^{ns}\to C$. The result follows.

□

Following from the definition of the Italian intersection multiplicity at a branch, we obtain a more refined version of Bezout's theorem;

Theorem 5.13. *Hyperspatial Bezout, Branched Version*

Let $C\subset P^w$ *be a projective algebraic curve of degree* d *and* F_λ *a hypersurface of degree* e*, intersecting* C *in finitely many points. Let* p

be a point of intersection with branches given by $\{\gamma_p^1, \ldots, \gamma_p^n\}$*. Then;*

$I_{italian}(p, C, F_\lambda) = \sum_{1 \leq j \leq n} I_{italian}(p, \gamma_p^j, C, F_\lambda)$

In particular;

$\sum_{p \in C \cap F_\lambda} \sum_{1 \leq j \leq n_p} I_{italian}(p, \gamma_p^j, C, F_\lambda) = de$

Proof. Fix a non-singular model (C^{ns}, Φ) of C. Let $\Phi_{\Sigma'}$ be a particular presentation of the morphism Φ such that $Base(\Phi_{\Sigma'})$ is disjoint from the fibre $\Gamma_\Phi(x, p) = \{p_1, \ldots, p_n\}$ $(*)$. Let Σ be a linear system containing F_λ such that Σ has finite intersection with C and;

$I_{italian}(p, C, F_\lambda) = I^{\Sigma}_{italian}(p, C, F_\lambda)$

Let $V_{\Phi_{\Sigma'}} \subset C^{ns}$ and $W_{\Phi_{\Sigma'}} \subset C$ be the canonical sets associated to the morphism $\Phi_{\Sigma'}$. By Lemma 2.13, we can witness $m = I_{italian}(p, C, F_\lambda)$ by transverse intersections $\{x_1, \ldots, x_m\} = C \cap F_{\lambda'} \cap \mathcal{V}_p$ inside the canonical set $W_{\Phi_{\Sigma'}}$, for λ' generic in Par_Σ. Again, by Lemma 2.13 and $(*)$, we can find λ' such that for each $p_j \in \Gamma_{[\Phi]}(x, p)$, $m_j = I_{italian}(p, \gamma_p^j, C, F_\lambda)$ is witnessed by transverse intersections $\{y_j^1, \ldots, y_j^{m_j}\} = C^{ns} \cap \overline{F}_{\lambda'} \cap \mathcal{V}_{p_j}$ inside the canonical set $V_{\Phi_{\Sigma'}}$. We need to show that $m_1 + \ldots + m_j + \ldots + m_n = m$. By properties of infinitesimals and the fact that $\{p_1, \ldots, p_n\}$ are distinct, the sets $\{y_1^1, \ldots, y_1^{m_1}\}, \ldots, \{y_n^1, \ldots, y_n^{m_n}\}$ are disjoint. If y_i^j belongs to one of these sets, then $y_i^j \in \mathcal{V}_{p_i}$, hence $\Phi_{\Sigma'}(y_i^j) \in \mathcal{V}_p$. Moreover, $\Phi_{\Sigma'}(y_i^j) \in C \cap F_{\lambda'}$. It follows that y_i^j corresponds to a unique x_k in $\{x_1, \ldots, x_m\}$. As each y_i^j lies in $V_{\Phi_{\Sigma'}}$, this clearly gives an injection from $\{y_1^1, \ldots, y_1^{m_1}, \ldots, y_n^1, \ldots, y_n^{m_n}\}$ to $\{x_1, \ldots, x_m\}$. Hence, $m_1 + \ldots + m_n \leq m$. To prove equality, suppose that x_k lies in $\{x_1, \ldots, x_m\}$. Then there exists a unique y_k with $\Phi_{\Sigma'}(y_k) = x_k$. By a similar argument to the above, y_k must appear in one of the sets $\{y_1^1, \ldots, y_1^{m_1}\}, \ldots, \{y_n^1, \ldots, y_n^{m_n}\}$. This gives the result.

□

Remarks 5.14. *One may use this version of Bezout's theorem in order to develop a more refined theory of* g_n^r.

We also simplify the branch terminology for later applications;

Definition 5.15. *Let $C \subset P^w$ be a projective algebraic curve and $C^{ns} \subset P^{w_1}$ some non-singular birational model with birational morphism $\Phi^{ns} : C^{ns} \rightarrow C$. Let γ_p^j correspond to the infinitesimal neighborhood $\mathcal{V}_{p_j}$ of p_j in $\Gamma_{\Phi^{ns}}(x,p)$. Then we will also define the branch γ_p^j by the formula;*

$$\gamma_p^j := \{x \in C : \exists y(\Gamma_{\Phi^{ns}}(y,x) \wedge y \in \mathcal{V}_{p_j})\}$$

Remarks 5.16. *Note that the definition uses the language $\mathcal{L}_{spec}$, and, in particular, is not algebraic.*

We have the following;

Lemma 5.17. *Definition 5.15 does not depend on the choice of non-singular model C^{ns} and morphism Φ^{ns}. Lemma 5.7 may be reformulated replacing the old definition of a branch with Definition 5.15. Finally, we have, with hypotheses as for the old definition of intersection multiplicity at a branch, that;*

$$I_{italian}(p, \gamma_p^j, C, F_\lambda) = Card(C \cap F_{\lambda'} \cap \gamma_p^j),\ \lambda' \in \mathcal{V}_\lambda \text{ generic in } Par_\Sigma \ (*)$$

Proof. The first part follows immediately from Lemma 5.1 and the fact that all the data of the lemma may be taken inside a standard model. The second part is similar, follow through the proof of Lemma 5.7. The final part may be checked by following carefully through the proofs of Lemmas 5.11 and 5.12. □

Remarks 5.18. *Note that we could not have simplified the above presentation by taking $(*)$ as our original definition of intersection multiplicity at a branch. The main reason being that the arguments on Zariski multiplicities require us to count intersections inside $C \cap \mathcal{V}_p$, rather than the smaller $\mathcal{L}_{spec}$ definable $C \cap \gamma_p^j$.*

We now reformulate the preliminary definitions of Section 2 in terms of branches.

Let $C \subset P^w$ be a projective algebraic curve and let Σ be a linear system, having finite intersection with C. Let $g_{n'}^r(\Sigma)$ be defined by this linear system Σ and let $g_n^r \subset g_{n'}^r$ be obtained by removing its fixed point contribution. Fix a non-singular model (C^{ns}, Φ) of C, with corresponding presentation $\Phi_{\Sigma'}$. By the transfer result Lemma 2.31, we obtain a corresponding g_n^r without fixed points on C^{ns}. Let $Card(O, V_\lambda, g_n^r)$ denote the number of times $O \in C^{ns}$ is counted for this g_n^r in the weighted set V_λ. We then make the following definition;

Definition 5.19. $I_{italian}^{\Sigma,mobile}(p, \gamma_p^j, C, \phi_\lambda) = Card(p_j, V_\lambda, g_n^r)$ $(*)$

where the branch γ_p^j corresponds to $[\mathcal{V}_{p_j}]$ in the fibre $\Gamma_{[\Phi_{\Sigma'}]}(x, p)$.

As before, one needs the following lemma;

Lemma 5.20. *The definition $(*)$ does not depend on the choice of non-singular model C^{ns} and birational morphism $\Phi_{\Sigma'}$.*

Proof. The proof is the same as Lemma 5.11, we leave the details to the reader. $\square$

We now make the following definition;

Definition 5.21. *Let hypotheses be as in Definition 5.19, then we define;*

$$I_{italian}^{\Sigma}(p, \gamma_p^j, C, \phi_\lambda) = I_{italian}^{\Sigma,mobile}(p, \gamma_p^j, C, \phi_\lambda) + 1 \text{ if } p \in Base(\Sigma)$$

$$I_{italian}^{\Sigma}(p, \gamma_p^j, C, \phi_\lambda) = I_{italian}^{\Sigma,mobile}(p, \gamma_p^j, C, \phi_\lambda) \quad \text{ if } p \notin Base(\Sigma)$$

Lemma 5.22. *Let notation be as in the previous definition. As in Remarks 5.10, if (C^{ns}, Φ) is a non-singular model of C, with presentation $\Phi_{\Sigma'}$ such that $Base(\Sigma')$ is disjoint from the fibre $\Gamma_{[\Phi_{\Sigma'}]}(x, p)$, then we have that;*

$$I_{italian}^{\Sigma}(p, \gamma_p^j, C, \phi_\lambda) = Card(C^{ns} \cap \overline{\phi_{\lambda'}} \cap \mathcal{V}_{p_j}),\ \lambda' \in \mathcal{V}_\lambda \text{ generic in } Par_\Sigma.$$

where again we have used the notation $\{\overline{\phi_\lambda} : \lambda \in Par_\Sigma\}$ to denote the family of "lifted" forms, as in Remarks 5.10.

Proof. We divide the proof into the following cases;

Case 1. $p \notin Base(\Sigma)$.

Then, by Definition 5.21, we have that;

$I^{\Sigma}_{italian}(p, C, \gamma^j_p, \phi_\lambda) = I^{\Sigma,mobile}_{italian}(p, C, \gamma^j_p, \phi_\lambda)$ (1).

By the assumption on Σ', we have that $p_j \notin Base(\Sigma)$ for the "lifted" family of forms on C^{ns}, corresponding to Σ, $(\dagger)$. By the transfer result, Lemma 2.31, the g^r_n on C^{ns}, used to define $I^{\Sigma,mobile}_{italian}(p, C, \gamma^j_p, \phi_\lambda)$, is obtained from the "lifted" family of form on C^{ns} after removing all fixed point contributions. Therefore, as by $(\dagger)$, this lifted family has no fixed point contribution at p_j, we must have that;

$I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I^{\Sigma,mobile}_{italian}(p, C, \gamma^j_p, \phi_\lambda)$ (2)

Combining (1),(2) and using Lemma 2.10, we have that;

$I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I^{\Sigma}_{italian}(p, C, \gamma^j_p, \phi_\lambda)$

The result for this case now follows immediately from the definition of $I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$.

Case 2. $p \in Base(\Sigma)$.

Then, by Definition 5.21, we have that;

$I^{\Sigma}_{italian}(p, C, \gamma^j_p, \phi_\lambda) = I^{\Sigma,mobile}_{italian}(p, C, \gamma^j_p, \phi_\lambda) + 1$ (1).

In this case, we have that $p_j \in Base(\Sigma)$ for the "lifted" family of forms on C^{ns}, corresponding to Σ, $(\dagger)$. Let I_{p_j} be the fixed point contribution for this family, as defined in Lemma 2.14. Then, by a similar argument to the above, we have that;

$I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I_{p_j} + I^{\Sigma,mobile}_{italian}(p, C, \gamma^j_p, \phi_\lambda)$ (2)

Using Lemma 2.15, we have that;

$I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I_{p_j} + I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) - 1$ (3)

Combining $(1), (2), (3)$ gives that;

$I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I^{\Sigma}_{italian}(p, C, \gamma^j_p, \phi_\lambda)$

Again, the result for this case follows immediately from the definition of $I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$.

$\square$

As an easy consequence of the previous lemma, we have that;

Lemma 5.23. *Let $C \subset P^w$ be a projective algebraic curve. Let Σ be a linear system, having finite intersection with C. Then, if γ_p^j is a branch centred at p and ϕ_λ belongs to Σ;*

$I^{\Sigma}_{italian}(p, C, \gamma_p^j, \phi_\lambda) = Card(C \cap \gamma_p^j \cap \phi_{\lambda'})$ *for* $\lambda' \in \mathcal{V}_\lambda$ *generic in* Par_Σ.

$I^{\Sigma,mobile}_{italian}(p, C, \gamma_p^j, \phi_\lambda) = Card(C \cap (\gamma_p^j \setminus p) \cap \phi_{\lambda'})$ *for* $\lambda' \in \mathcal{V}_\lambda$ *generic in* Par_Σ.

where γ_p^j was given in Definition 5.15.

Proof. The first part of the lemma follows immediately from Lemma 5.22 and the Definition 5.15 of a branch. The second part follows from Definition 5.21 and the first part. $\square$

We then reformulate the remaining results of Section 2 in terms of branches. The notation of Lemma 5.23 will be use for the remainder of this section.

Lemma 5.24. *Non-Existence of Coincident Mobile Points along a Branch*

Let $C \subset P^w$ be a projective algebraic curve. Let Σ be a linear system, having finite intersection with C, such that $p \in C \setminus Base(\Sigma)$. Then, if γ_p^j is a branch centred at p and ϕ_λ belongs to Σ;

$$I_{italian}(p, C, \gamma_p^j, \phi_\lambda) = I^{\Sigma}_{italian}(p, C, \gamma_p^j, \phi_\lambda) = I^{\Sigma,mobile}_{italian}(p, C, \gamma_p^j, \phi_\lambda)$$

Proof. Let (C^{ns}, Φ) be a non-singular model of C, with presentation $\Phi_{\Sigma'}$, such that $Base(\Sigma')$ is disjoint from $\Gamma_{[\Phi]}(x, p)$. Let $\{\overline{\phi_\lambda}\}$ be the "lifted" family of algebraic forms on C^{ns} defined by Σ. By Lemma 5.22, we have that;

$I^{\Sigma}_{italian}(p, C, \gamma_p^j, \phi_\lambda) = I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$ (1)

By Remarks 5.10, we have that;

$I_{italian}(p, C, \gamma_p^j, \phi_\lambda) = I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$ (2)

Using the fact that $p \notin Base(\Sigma)$ and the hypotheses on $\Phi_{\Sigma'}$, it follows that $p_j \notin Base(\Sigma)$, for the lifted system defined by Σ. Hence, we may apply Lemma 2.10, to obtain that;

$I^{\Sigma}_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$ (3)

The result follows by combining (1), (2) and (3) and using Definition 5.21.

□

Lemma 5.25. *Branch Multiplicity at non-base points witnessed by transverse intersections along the branch*

Let $p \in C \setminus Base(\Sigma)$ and let γ_p^j be a branch centred at p. Then, if $m = I_{italian}(p, C, \gamma_p^j, \phi_\lambda)$, we can find $\lambda' \in \mathcal{V}_\lambda$, generic in Par_Σ, and distinct $\{p_1, \ldots, p_m\} = C \cap \phi_{\lambda'} \cap (\gamma_p^j \setminus p)$ such that the intersection of C with $\phi_{\lambda'}$ at each p_i is transverse for $1 \leq i \leq m$.

Proof. By Lemmas 5.23 and 5.24, for $\lambda' \in \mathcal{V}_\lambda$, generic in Par_Σ, the intersection $C \cap \phi_{\lambda'} \cap \gamma_p^j$ consists of m distinct points $\{p_1, \ldots, p_m\}$. The condition on Par_Σ that ϕ_λ passes through p defines a proper closed subset, hence we may assume these points are all distinct from p. Finally, the transversality result follows from, say Lemma 2.17, using the fact that $\{p_1, \ldots, p_m\}$ cannot lie inside $Base(\Sigma)$.

□

Again, we have analogous results to Lemmas 5.24 and 5.25 for points in $Base(\Sigma)$;

We first require the following;

Lemma 5.26. *Let $p \in C \cap Base(\Sigma)$ and γ_p^j a branch centred at p. Then there exists an open subset $U_{\gamma_p^j} \subset Par_\Sigma$ and an integer $I_{\gamma_p^j} \geq 1$ such that;*

$I_{italian}(p, C, \gamma_p^j, \phi_\lambda) = I_{\gamma_p^j}$ *for* $\lambda \in U_{\gamma_p^j}$

and

$$I_{italian}(p, C, \gamma_p^j, \phi_\lambda) \geq I_{\gamma_p^j} \text{ for } \lambda \in Par_\Sigma$$

Proof. Let (C^{ns}, Φ) be a non-singular model with presentation $\Phi_{\Sigma'}$ such that $Base(\Sigma')$ is disjoint from $\Gamma_{[\Phi]}(x, p)$. Then, by the proof of Lemma 5.12, we have that, for $\lambda \in Par_\Sigma$;

$$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) = I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$$

By properties of Zariski structures;

$$W_k = \{\lambda \in Par_\Sigma : I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) \geq k\}$$

are definable and Zariski closed in Par_Σ. The result then follows by taking $I_{\gamma_{p_j}} = min_{\lambda \in Par_\Sigma} I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda})$ and using the fact that Par_Σ is irreducible.

□

We can now formulate analogous results to Lemmas 5.24 and 5.25 for base points;

Lemma 5.27. *Let $p \in C \cap Base(\Sigma) \cap \phi_\lambda$ and γ_p^j a branch centred at p. Then;*

$$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) = I_{\gamma_p^j} + I_{italian}^{\Sigma}(p, \gamma_p^j, C, \phi_\lambda) - 1$$

and

$$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) = I_{\gamma_p^j} + I_{italian}^{\Sigma, mobile}(p, \gamma_p^j, C, \phi_\lambda)$$

Proof. In order to prove the first part of the lemma, suppose that $m = I_{italian}^{\Sigma}(p, \gamma_p^j, C, \phi_\lambda)$. By Lemma 5.23, choosing $\lambda' \in \mathcal{V}_\lambda$ generic in Par_Σ, we can find $\{p_1, \ldots, p_{m-1}\} = C \cap \phi_{\lambda'} \cap (\mathcal{V}_p \setminus p)$, distinct from p, witnessing this multiplicity. Using the fact that $\{p_1, \ldots, p_{m-1}\}$ lie outside $Base(\Sigma)$, we may apply Lemma 2.17 to obtain that the intersections at these points are transverse. By the previous Lemma 5.26, we have that $I_{italian}(p, \gamma_p^j, C, \phi_{\lambda'}) = I_{\gamma_p^j}$. Now choose a nonsingular model (C^{ns}, Φ), with presentation $\Phi_{\Sigma'}$, such that $Base(\Sigma')$ is disjoint from $\Gamma_{[\Phi]}(x, p)$. By definition 5.15 of a branch, we can find $\{p_1', \ldots, p_m'\} = C^{ns} \cap \overline{\phi_{\lambda'}} \cap (\mathcal{V}_p \setminus p)$. By properties of specialisations, $Base(\Sigma')$ is also disjoint from this set. We then have that the intersections between C^{ns} and $\overline{\phi_{\lambda'}}$ are also transverse at these points and that

$I_{italian}(p_j, C^{ns}, \overline{\phi_{\lambda'}}) = I_{\gamma_p^j}$. It then follows by summability of specialisation, see [6], that;

$$I_{italian}(p_j, C^{ns}, \overline{\phi_\lambda}) = I_{\gamma_p^j} + (m-1).$$

Again, using the presentation of (C^{ns}, Φ), we obtain that;

$$I_{italian}(p, \gamma_p^j, C, \phi_\lambda) = I_{\gamma_p^j} + (m-1).$$

Hence, the result follows. The second part of the lemma follows immediately from the Definition 5.21 of $I^{\Sigma}_{italian}(p, C, \gamma_p^j, \phi_\lambda)$ at a base point. □

Lemma 5.28. *Let $p \in C \cap Base(\Sigma)$ and let γ_p^j be a branch centred at p. Then, if $m = I^{\Sigma}_{italian}(p, C, \gamma_p^j, \phi_\lambda)$, we can find $\lambda' \in \mathcal{V}_\lambda$, generic in Par_Σ, and distinct $\{p_1, \ldots, p_{m-1}\} = C \cap \phi_{\lambda'} \cap (\gamma_p^j \setminus p)$ such that the intersection of C with $\phi_{\lambda'}$ at each p_i is transverse for $1 \leq i \leq m-1$. If $m = I^{\Sigma,mobile}_{italian}(p, C, \gamma_p^j, \phi_\lambda)$, then the same results for distinct $\{p_1, \ldots, p_m\}$ with the same properties.*

Proof. The first part of the lemma is a straightforward consequence of Lemma 5.23, properties of infinitesimals, (to show that $\{p_1, \ldots, p_{m-1}\}$ lie outside $Base(\Sigma)$) and Lemma 2.17 (to obtain transversality). The second part of the lemma also follows from Lemma 5.23 and Lemma 2.17 (to obtain transversality). □

6. Cayley's Classification of Singularities

The purpose of this section is to develop a theory of singularities for algebraic curves based on the work of Cayley. In order to make this theory rigorous, one first needs to find a method of parametrising the branches of an algebraic curve. This is the content of the following theorem;

Theorem 6.1. *Analytic Representation of a Branch*

Let $C \subset P^w$ be a projective algebraic curve. Suppose that C is defined by equations $\{F_1(x_1, \ldots, x_w), \ldots F_m(x_1, \ldots, x_w)\}$ in affine coordinates $x_i = \frac{X_i}{X_0}$. Let $p \in C$ correspond to the point $\bar{0}$ in this coordinate system. Then there exist algebraic power series $\{x_1(t), \ldots, x_w(t)\}$ such that $x_1(t) = \ldots = x_w(t) = 0$, $F_j(x_1(t), \ldots, x_w(t)) = 0$ for $1 \leq j \leq m$ and with the property that, for any algebraic function $F_\lambda(x_1, \ldots, x_w)$;

$F_\lambda(x_1(t),\ldots,x_w(t))\equiv 0$ iff F_λ vanishes on C.

Otherwise, F_λ has finite intersection with C and

$ord_tF_\lambda(x_1(t),\ldots,x_w(t))=I_{italian}(p,\gamma_p^j,C,F_\lambda)$ $(*)$

We refer to the power series as parametrising the branch γ_p^j.

Proof. We first prove the theorem in the case when $C\subset P^w$ is a non-singular projective algebraic curve. By Lemma 4.13, we can find a plane projective algebraic curve $C_1\subset P^2$ such that $\{C,C_1\}$ are birational and there exists a corresponding point $p_1\in C_1$ such that p_1 is non-singular. Let Φ_Σ and $\Psi_{\Sigma'}$ be presentations such that $\Psi_{\Sigma'}=\Phi_\Sigma^{-1}$ as a birational map. Without loss of generality, we may assume that $V_{\Phi_\Sigma}=W_{\Psi_{\Sigma'}}\subset C$ and $V_{\Psi_{\Sigma'}}=W_{\Phi_\Sigma}\subset C_1$. Moreover, we may assume that $\{p,p_1\}$ lie in $\{V_{\Phi_\Sigma},V_{\Psi_{\Sigma'}}\}$ and correspond to the origins of the affine coordinate systems $(x_1,\ldots,x_w)$ and (y_1,y_2). Let $\Sigma'=\{\psi_0,\psi_1,\ldots,\psi_w\}$ and let $(y_1(t),y_2(t))$ be an analytic representation of $p_1\in C_1$, given by the inverse function theorem. We obtain an analytic representation of $p\in C$ by the formula;

$$(x_1(t),\ldots,x_w(t))=(\tfrac{\psi_1}{\psi_0}(y_1(t),y_2(t)),\ldots,\tfrac{\psi_w}{\psi_0}(y_1(t),y_2(t)))$$

First, note that as $p\notin Zero(\psi_0)$, we may assume that $\psi_0(0,0)\neq 0$. Hence, we can invert the power series $\psi_0(y_1(t),y_2(t))$. This clearly proves that $x_j(t)$ is a formal power series in $L[[t]]$. That $x_j(0)=0$ for $1\leq j\leq w$ follows from the corresponding property for $(y_1(t),y_2(t))$ and the fact that p is situated at the origin of the coordinate system $(x_1,\ldots,x_w)$. Finally, we need to check that $x_j(t)$ define algebraic power series. This follows obviously from the fact that ψ_j and ψ_0 define algebraic functions. Now, suppose that $\{F_1,\ldots,F_m\}$ are defining equations for C. Let $\{F_1',\ldots,F_m'\}$ be the corresponding equations written in homogeneous form for the variables $\{X_0,\ldots,X_w\}$, where $x_j=\frac{X_j}{X_0}$. Let $G(Y_0,Y_1,Y_2)$ be the defining equation for C_1. We can homogenise the power series representation of $p_1\in C_1$ by $(Y_0(t):Y_1(t):Y_2(t))=(1:y_1(t):y_2(t))$. Then we must have that $G(1:y_1(t):y_2(t))=0$. Now $F_k'(\psi_0,\ldots,\psi_w)$ vanishes identically on C_1, hence, by the projective Nullstellensatz, there exists a homogeneous $H_k(Y_0,Y_1,Y_2)$ such that;

$$F_k'(\psi_0,\ldots,\psi_w)=H_kG$$

It follows that;

$$F'_k(\psi_0(1:y_1(t):y_2(t)),\ldots,\psi_w(1:y_1(t):y_2(t)))\equiv 0$$

therefore;

$$F'_k(1:\tfrac{\psi_1(y_1(t),y_2(t))}{\psi_0(y_1(t),y_2(t))}:\ldots:\tfrac{\psi_w(y_1(t),y_2(t))}{\psi_0(y_1(t),y_2(t))})\equiv 0$$

which gives;

$$F_k(x_1(t),\ldots,x_w(t))\equiv 0$$

as required. The property that an algebraic function F_λ vanishes on $(x_1(t),\ldots,x_w(t))$ iff it vanishes on C can be proved in a similar way to the above argument, invoking Theorem 5.1 of the paper [6]. Alternatively, it can be proved directly, using the fact that, as $(x_1(t),\ldots,x_w(t))$ define algebraic power series, $(y_1-x_1(t),\ldots,y_w-x_w(t))$ defines the equation of a curve C' on some etale cover $i:(A^w_{et},(\bar{0})^{lift})\rightarrow(A^w,(\bar{0}))$ such that $i(C')\subset C$. If F_λ vanishes on C', then it must vanish on an open subset U of C, hence as F_λ is closed, must vanish on all of C as required. Finally, we need to check the property $(*)$. Suppose that F_λ is an algebraic function with $m=ord_tF_\lambda(x_1(t),\ldots,x_w(t))$, passing through p. Choose Σ_1 containing F_λ such that Σ_1 has finite intersection with C and $p\notin Base(\Sigma_1)$. It follows, using Lemma 2.12, that $I_{italian}(p,C,F_\lambda)=I^{\Sigma_1}_{italian}(p,C,F_\lambda)$, $(\dagger)$. Using Lemma 2.31, we can transfer the system Σ_1 to a system on C_1. Let G_λ be the corresponding algebraic curve to the algebraic form F_λ. We must have that $p_1\notin Base(\Sigma_1)$, otherwise, as p_1 belongs to the canonical set $V_{\Psi_{\Sigma'}}$, we would have that p belongs to $Base(\Sigma_1)$ as well. Hence, using Lemma 2.12 again, we must have that $I_{italian}(p_1,C_1,G_\lambda)=I^{\Sigma_1}_{italian}(p_1,C_1,G_\lambda)$, $(\dagger\dagger)$. By direct calculation, we have that;

$$G_\lambda(y_1(t),y_2(t))=\psi_0^r(y_1(t),y_2(t))F_\lambda(x_1(t),\ldots,x_w(t))$$

for some $r\leq 0$. As $ord_t\psi_0(y_1(t),y_2(t))=0$, we have that;

$$ord_tG_\lambda(y_1(t),y_2(t))=ord_tF_\lambda(x_1(t),\ldots,x_w(t))=m$$

Now, by Theorem 5.1 of the paper [6] and $(\dagger\dagger)$, it follows that;

$I_{italian}(p_1, C_1, G_\lambda) = I^{\Sigma_1}_{italian}(p_1, C_1, G_\lambda) = m$

Now, using Lemma 2.31 and the fact that $\{p, p_1\}$ lie in the canonical sets V_{Φ_Σ} and W_{Φ_Σ}, we must have that $I^{\Sigma_1}_{italian}(p, C, F_\lambda) = m$ as well. Hence, by (†), it follows that $I_{italian}(p, C, F_\lambda) = m$. As C is a non-singular model of itself, this proves the claim $(*)$ in this special case.

We now assume that $C \subset P^w$ is *any* projective algebraic curve. Suppose that $C^{ns} \subset P^{w'}$ is a non-singular model of C with birational morphism $\Phi_{\Sigma'} : C^{ns} \rightarrow C$ such that the branch γ^j_p corresponds to $\mathcal{V}_{p_j}$ in the fibre $\Gamma_{[\Phi]}(x, p)$, disjoint from $Base(\Sigma')$. As before, we may assume that $\{p, p_j\}$ correspond to the origins of the coordinate systems $(x_1, \ldots, x_w)$ and $(y_1, \ldots, y_{w'})$. Let $\Sigma' = \{\phi_0, \ldots, \phi_w\}$. By the previous argument, we can find an analytic representation $(y_1(t), \ldots, y_{w'}(t))$ of p_j in C^{ns}, with the properties given in the statement of the theorem. As before, we obtain an analytic representation of the corresponding $p \in C$, by the formula;

$$(x_1(t), \ldots, x_w(t)) = (\tfrac{\phi_1}{\phi_0}(y_1(t), \ldots, y_{w'}(t)), \ldots, \tfrac{\phi_w}{\phi_0}(y_1(t), \ldots, y_{w'}(t)))$$

One checks that this has the required properties up to the property $(*)$ by a direct imitation of the proof above, with the minor modification that the projective Nullstellensatz for C^{ns} gives that, if $\{G_1, \ldots, G_k\}$ are defining equations for C^{ns}, then, if F vanishes on C^{ns}, there must exist homogeneous $\{H_1, \ldots, H_k\}$ such that $F = H_1G_1 + \ldots + H_kG_k$. Alternatively, one can refer the parametrisation to a non-singular point of a plane projective curve, in which case the argument up to $(*)$ is identical.

We now verify the property $(*)$. Suppose that F_λ is an algebraic function with $ord_tF_\lambda(x_1(t), \ldots, x_w(t)) = m$. Let $\overline{F}_\lambda$ be the corresponding function on C^{ns}, obtained from the presentation $\Phi_{\Sigma'}$. By Lemmas 5.11 and 5.12;

$I_{italian}(p, \gamma^j_p, C, F_\lambda) = I_{italian}(p_j, C^{ns}, \overline{F_\lambda}) \quad (**)$

We claim that $ord_t\overline{F}_\lambda(y_1(t), \ldots, y_{w'}(t)) = m$. This follows by repeating the argument given above. By the properties of $(y_1(t), \ldots, y_{w'}(t))$ and the result verified in the case of a non-singular curve, we obtain immediately that $I_{italian}(p_j, C^{ns}, \overline{F_\lambda}) = m$. Combined with $(**)$, this gives the required result.

□

Using the analytic representation, we obtain the following classification of singularities due to Cayley;

Theorem 6.2. *Cayley's Classification of Singularities*

Let $C \subset P^w$ be a projective algebraic curve which is not contained in any hyperplane section. Let γ_p^j be a branch of the algebraic curve centred at p. Then we can assign a sequence of non-negative integers $(\alpha_0, \alpha_1, \ldots, \alpha_{w-1})$, called the character *of the branch, which has the following property;*

Let Σ be the system of hyperplanes passing through p. Then there exists a filtration of Σ into subsystems of hyperplanes;

$\Sigma_{w-1} \subset \Sigma_{w-2} \subset \ldots \subset \Sigma_1 \subset \Sigma_0 = \Sigma$

with $dim(\Sigma_i) = (w-1)-i$, such that, for any hyperplane H_λ passing through p, we have that;

$H_\lambda \in \Sigma_i$ *iff* $I_{italian}(p, \gamma_p^j, C, H_\lambda) \geq \alpha_0 + \alpha_1 + \ldots + \alpha_i$. $(\dagger)$

Moreover, these are the only multiplicities which occur.

Proof. Without loss of generality, we may assume that p is situated at the origin of the coordinate system $(x_1, \ldots, x_w)$. By the previous theorem, we can find an analytic parameterisation of the branch γ_p^j of the form;

$x_k(t) = a_{k,1}t + a_{k,2}t^2 + \ldots + a_{k,n}t^n + \ldots$, for $1 \leq k \leq w$

Let $\sum_{k=1}^{w} \lambda_k x_k = 0$ be the equation of a hyperplane H_λ passing through p. Then, we can write $H_\lambda(x_1(t), \ldots, x_k(t))$ in the form;

$(\sum_{k=1}^{w} \lambda_k a_{k,1})t + \ldots + (\sum_{k=1}^{w} \lambda_k a_{k,n})t^n + O(t^n)$ (*)

Let $\bar{a}_n = (a_{1,n}, \ldots, a_{k,n}, \ldots, a_{w,n})$. We claim that we can find a sequence $(\bar{a}_{m_1}, \ldots, \bar{a}_{m_w})$, for $m_1 < \ldots < m_i < \ldots < m_w$, such that;

(i). $\{\bar{a}_{m_1},\ldots,\bar{a}_{m_w}\}$ is linearly independent

(ii). $V_{m_i}=<\bar{a}_1,\ldots,\bar{a}_{m_i}>=<\bar{a}_{m_1},\ldots,\bar{a}_{m_i}>$, for $1\leq i\leq w$

(iii). If there exist $\{n_1,\ldots,n_i\}$ for $n_1<\ldots<n_i$ with $n_1\leq m_1,\ldots,n_i\leq m_i$ such that

$V_{m_i}=<\bar{a}_{n_1},\ldots,\bar{a}_{n_i}>$,

then $n_1=m_1,\ldots,n_i=m_i$.

(iv). $V_k=V_{m_i}$ for $m_i\leq k<m_{i+1}$.

The first three properties may be proved by induction on i.For $i=1$, choose the first non-zero vector $\bar{a}_{m_1}$. For the inductive step, assume we have found $\{\bar{a}_{m_1},\ldots,\bar{a}_{m_i}\}$ with the required properties $(i)-(iii)$. We claim that there exists $\bar{a}_{m_{i+1}}$, with $m_{i+1}>m_i$, such that $\bar{a}_{m_{i+1}}\notin V_{m_i}$, $(**)$. Suppose not. As $i<w$, the condition;

$\bigwedge_{s=1}^{i}(\sum_{k=1}^{w}\lambda_k a_{k,m_s}=0)$

defines a non-empty plane P in the dimension $w-1$ parameter space Par_H of hyperplanes passing through p. Choosing $\lambda\in P$ and using $(*)$, it follows that $ord_t H_\lambda(x_1(t),\ldots,x_w(t))\geq n$ for all $n>0$. Therefore, $H_\lambda(x_1(t),\ldots,x_w(t))\equiv 0$ and, by Theorem 6.1, H_λ must contain C. This contradicts the assumption that C is *not* contained in any hyperplane section. Using $(**)$, choose m_{i+1} minimal such that $\bar{a}_{m_{i+1}}\notin V_{m_i}$. Properties (i) and (ii) are trivial to verify. For (iii), assume that $\{\bar{a}_{n_1},\ldots,\bar{a}_{n_{i+1}}\}$ are as given in the hypotheses. Then, the sequence must form a linearly independent set. Hence, we must have that $V_{m_i}=<\bar{a}_{n_1},\ldots,\bar{a}_{n_i}>$. By the induction hypothesis, $n_1=m_1,\ldots,n_i=m_i$. Then, $\bar{a}_{n_{i+1}}\notin V_{m_i}$, Now, by minimality of m_{i+1}, we also have that $\bar{a}_{m_{i+1}}=\bar{a}_{n_{i+1}}$ as required.

Property (iv) follows easily from properties $(i)-(iii)$. We clearly have that $V_{m_i}\subseteq V_k\subseteq V_{m_{i+1}}$, for $m_i\leq k<m_{i+1}$. If $V_k\neq V_{m_i}$, then, by $(i),(ii)$, $V_k=V_{m_{i+1}}$. This clearly contradicts (iii).

Now, define;

$\Sigma_i=\{\lambda\in Par_{H,p}:\bigwedge_{s=1}^{i}(\sum_{k=1}^{w}\lambda_k a_{k,m_s}=0)\}$, for, $1\leq i\leq w-1$, Then, we obtain a filtration;

$\Sigma_{w-1} \subset \ldots \subset \Sigma_i \subset \ldots \subset \Sigma_0 = \Sigma$

with $dim(\Sigma_i) = (w-1)-i$ as in the statement of the theorem. Define;

$\alpha_0 = m_1$ and $\alpha_i = m_{i+1} - m_i$ for $1 \leq i \leq w-1$.

We need to verify the property (†). Suppose that $H_\lambda \in \Sigma_i$, for $i \geq 1$. Then H_λ contains the plane V_{m_i} spanned by $\{\bar{a}_{m_1}, \ldots, \bar{a}_{m_i}\}$. Hence, by (iv), it contains the plane V_k for $k < m_{i+1}$. Then, by $(*)$, $ord_t(H_\lambda(x_1(t), \ldots, x_w(t))) \geq m_{i+1}$ and ,by Theorem 6.1,

$I_{italian}(p, \gamma_p^j, C, H_\lambda) \geq m_{i+1} = \alpha_0 + \ldots + \alpha_i$.

Conversely, suppose that

$ord_t H_\lambda(x_1(t), \ldots, x_w(t)) \geq \alpha_0 + \ldots \alpha_i = m_{i+1}$, for some $i \geq 1$.

Then H_λ contains the plane V_k for $k < m_{i+1}$. In particular, it contains the plane V_{m_i}. Hence $H_\lambda \in \Sigma_i$. The remaining case amounts to showing that $I_{italian}(p, \gamma_p^j, C, H_\lambda) \geq \alpha_0$ for any hyperplane H_λ passing through p. This follows immediately from $(*)$, Theorem 6.1 and the fact that $\bar{a}_{m_1}$ was the first non-zero vector in the sequence $\{\bar{a}_n : n < \omega\}$. The remark made after the property (†) follows immediately from the property (iv) of the sequence $\{\bar{a}_{m_1}, \ldots, \bar{a}_{m_w}\}$.

□

Definition 6.3. *In accordance with the Italian terminology, we refer to* α_0 *as the order of the branch,* α_j *as the* j*'th range of the branch, for* $1 \leq j \leq w-2$*, and* α_{w-1} *as the final range or class of the branch. We define* $< \bar{a}_{m_1}, \ldots, \bar{a}_{m_k} >$ *to be the* k*'th osculatory plane at* p *for* $1 \leq k \leq w-1$*. We also define the* $w-1$*'th osculatory plane to be the osculatory plane. We define the tuple* $(\alpha_0, \ldots, \alpha_{w-1})$ *to be the character of the branch. Cayley referred to branches of order* 1 *as linear and superlinear otherwise. He referred to branches having a character of the form* $(1, 1 \ldots, 1)$ *as ordinary. The Italian geometers refer to a simple point, which is the origin of an ordinary branch, as an ordinary simple point. Note that for a simple (equivalently non-singular) point, the* 1*'st osculatory plane is the same as the tangent line. We define a node of a plane curve to be the origin of at most* 2 *ordinary branches with distinct tangent lines, this definition was used in Theorem 4.16.*

We also used, in Theorem 4.16, the fact that a generic point of an algebraic curve is an ordinary simple point, (this is not true when the field has non-zero characteristic, see the final section) a rigorous proof of this result requires duality arguments, we postpone this proof for another occasion.

We have the following important results on the projection of a branch, see Section 4 for the relevant definitions.

Theorem 6.4. *Let $C \subset P^w$ be a projective algebraic curve, as defined in the previous theorems of this section, and γ_O a branch centred at O with character $(\alpha_0, \ldots, \alpha_{w-1})$. Let P be chosen generically in P^w, then the projection $pr_P(\gamma_O)$ has character;*

$$(\alpha_0, \ldots, \alpha_{w-2})$$

If P is situated generically on the osculatory plane, then $pr_P(\gamma_O)$ has character;

$$(\alpha_0, \ldots, \alpha_{w-2} + \alpha_{w-1})$$

More generally, if P is situated on the k'th osculatory plane for $1 \leq k \leq w-2$, and not on an osculatory plane of lower order, then the projection $pr_P(\gamma_O)$ has character;

$$(\alpha_0, \ldots, \alpha_{k-2}, \alpha_{k-1} + \alpha_k, \alpha_{k+1}, \ldots, \alpha_{w-1}).$$

Proof. First note that, by Lemma 4.9, if P is situated generically on the k'th-osculatory plane for any $k \geq 1$, and H' is any hyperplane not containing P, the projection pr_P is generally biunivocal. Hence, by Lemma 5.7, the projection $pr_P(\gamma_O)$ is well defined. We first claim that for any hyperplane H_λ in P^{w-1};

$$I_{italian}(O, \gamma_O, C, pr_P^{-1}(H_\lambda)) = I_{italian}(pr_P(O), pr_P(\gamma_O), pr_P(C), H_\lambda) \ (*)$$

This follows by using the proof of Lemma 4.13 and the fact that the multiplicity is calculated at a branch, to replace the use of biunivocity. Now, if P is situated in generic position in P^w, then $\{pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{w-1}})\}$ forms a linearly independent sequence passing through $pr_P(O) \in P^{w-1}$ and, for any hyperplane $H_\lambda \subset P^{w-1}$, we have that, for $i \leq w-2$;

$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_i}) >\subset H_\lambda$ iff $< \bar{a}_{m_1}, \ldots, \bar{a}_{m_i} >\subset< P, H_\lambda >$

It follows by $(*)$ and Theorem 6.2 that;

$I_{italian}(pr_P(O), pr_P(\gamma_O), pr_P(C), H_\lambda) \geq \alpha_0 + \ldots \alpha_i$

iff

$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_i}) >\subset H_\lambda \quad (i \leq w-2)$

Hence, by Theorem 6.2 again, we have that the branch $pr_P(\gamma_O)$ has character $(\alpha_0, \ldots, \alpha_{w-2})$.

If P is situated generically on the k'th oscillatory plane, but not on an oscillatory plane of lower order, then, the sequence $\{pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{k-1}}), pr_P(\bar{a}_{m_{k+1}}), \ldots, pr_P(\bar{a}_{m_{w-1}})\}$ forms a linearly independent set, and, for $1 \leq i \leq k-2$;

$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_i}) >\subset H_\lambda$ iff $< \bar{a}_{m_1}, \ldots, \bar{a}_{m_i} >\subset< P, H_\lambda >$

whereas;

$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{k-1}}) >\subset H_\lambda$ iff $< \bar{a}_{m_1}, \ldots, \bar{a}_{m_k} >\subset< P, H_\lambda >$

and, for $1 \leq j \leq (w-2-k)$;

$< pr_P(\bar{a}_{m_1}), \ldots, pr_P(\bar{a}_{m_{k-1}}), pr_P(\bar{a}_{m_{k+1}}), \ldots, pr_P(\bar{a}_{m_{k+j}}) >\subset H_\lambda$

iff

$< \bar{a}_{m_1}, \ldots, \bar{a}_{m_{k+j}} >\subset< P, H_\lambda >$

The rest of the theorem then follows immediately by the same argument as given above.

$\square$

We also have the following important consequence of Theorem 6.1.

Theorem 6.5. *Linearity of Multiplicity at a Branch*

Let $C \subset P^w$ be a projective algebraic curve and γ_p^j a branch of C, centred at p. Let Σ be an independent system having finite intersection *with C. Then, for any $k \geq 1$, the condition;*

$$\{\lambda \in Par_{\Sigma} : I_{italian}(p, \gamma_p^j, C, F_{\lambda}) \geq k\} \ (*)$$

is linear and definable.

Proof. That $(*)$ is definable follows from the definition of multiplicity at a branch and elementary facts about Zariski structures. Moreover, $(*)$ is definable over the parameters of C and the point p. In order to prove linearity, suppose that F_{λ} and F_{μ} belong to Σ and satisfy $(*)$. Let $(x_1(t), \ldots, x_w(t))$ be a parameterisation of the branch γ_p^j as given by Theorem 6.1. Then;

$$k \leq min\{ord_t F_{\lambda}(x_1(t), \ldots, x_w(t)), ord_t F_{\mu}(x_1(t), \ldots, x_w(t))\}$$

Then, for any constant c;

$$(F_{\lambda}+cF_{\mu})(x_1(t), \ldots, x_w(t)) = F_{\lambda}(x_1(t), \ldots, x_w(t))+cF_{\mu}(x_1(t), \ldots, x_w(t))$$

Hence,

$$ord_t(F_{\lambda} + cF_{\mu})(x_1(t), \ldots, x_w(t)) \geq k$$

as well. This shows that the pencil of curves generated by $\{F_{\lambda}, F_{\mu}\}$ satisfies $(*)$. Let W be the closed projective subvariety of Par_{Σ} defined by the condition $(*)$. Then, W has the property that for any $\{a, b\} \subset W$, $l_{ab} \subset W$. It follows easily that W defines a plane H in Par_{Σ} as required. (Use the fact that for any tuple $\{a_1, \ldots, a_n\}$ in W, the plane $H_{a_1,\ldots,a_n} \subset W$ and a dimension argument)

□

Remarks 6.6. *Note that, given Σ of dimension n as in the statement of the theorem, we can, using the above theorem, find a sequence $\{\beta_0, \ldots, \beta_{n-1}\}$ and a filtration;*

$$\Sigma_{n-1} \subset \ldots \subset \Sigma_i \subset \ldots \subset \Sigma_0 \subset \Sigma$$

with $dim(\Sigma_i) = (n-1) - i$ such that;

$$I_{italian}(p, \gamma_p^j, C, F_{\lambda}) \geq \beta_0 + \ldots + \beta_i \text{ iff } F_{\lambda} \in \Sigma_i$$

and these are the only multiplicities which can occur. Note also that, as an easy consequence of the theorem, given any tuple $(\beta_0, \ldots, \beta_{n-1})$

and Σ as above;

$\{x \in C : x$ *has character* $(\beta_0, \ldots, \beta_{n-1})$ *with respect to* $\Sigma\}$

is constructible and defined over the field of definition of C and Σ. In particular, it follows from the previous Definition 6.3, in characteristic 0, that there exist only finitely many points on C which are the origins of non-ordinary branches.

We have the following important characterisation of multiplicity at a branch;

Theorem 6.7. *Multiplicity at a Branch as a Specialised Condition*

Let C and Σ be as in the statement of Theorem 6.5 and the following remark. Fix independent generic points (over L) $\{p_{0j}, \ldots, p_{ij}\}$ in γ_p^j. Then the system Σ_i may be obtained by specialising the condition;

$\{\lambda \in Par_\Sigma : F_\lambda = 0$ *passes through* $\{p_{0j}, \ldots, p_{ij}\}\}$.

That is, if $F_\lambda \in \Sigma_i$, there exists $\lambda' \in \mathcal{V}_\lambda$ such that $F_{\lambda'}$ passes through $\{p_{0j}, \ldots, p_{ij}\}$, while, if $F_{\lambda'}$ passes through $\{p_{0j}, \ldots, p_{ij}\}$, then its specialisation F_λ belongs to Σ_i.

Proof. We will first assume that the curve C is non-singular, (hence, by Lemma 5.4, there exists a single branch at p). Consider the cover $F_i \subset (C \setminus p) \times Par_\Sigma$ given by;

$F_{i+1}(x, \lambda) \equiv (x \in C \cap F_\lambda) \wedge (F_\lambda \in \Sigma_i)$ $(i \geq 0)$

For generic $q \in C$, the fibre $F_0(q, y)$ has dimension $n - 1 - (i + 1)$. Hence, we obtain an open subset $U \subset C$ such that $F_{i+1} \subset (U \setminus p) \times Par_\Sigma$ is regular with fibre dimension $n - 1 - (i + 1)$. Let $\bar{F}_{i+1}$ be the closure of F_{i+1} in $U \times Par_\Sigma$. We claim that Σ_{i+1} is defined by the fibre $\bar{F}_{i+1}(p, y)$. First observe that, as p has codimension 1 in U, $dim(\bar{F}_{i+1}(p, y)) \leq n - 1 - (i + 1)$. As p is non-singular, each component of the fibre $\bar{F}_{i+1}(p, y)$ has dimension at least $n - 1 - (i + 1)$. Suppose that $\bar{F}_{i+1}(p, \lambda)$ holds, then, by regularity of p for the cover $\bar{F}_{i+1}$, given $p' \in U \cap \mathcal{V}_p$ generic, we can find $\lambda' \in \mathcal{V}_\lambda$ such that $F_{i+1}(p', \lambda')$, that is $F_{\lambda'}$ passes through p' and $F_{\lambda'}$ belongs to Σ_i. By definition of Σ_i, we have that $I_{italian}(p, C, F_{\lambda'}) \geq \beta_0 + \ldots + \beta_i$. As p' is distinct from p, by summability of specialisation, see the paper [6], we must have that $I_{italian}(p, C, F_\lambda) \geq \beta_0 + \ldots + \beta_i + 1$. Therefore, in fact, by the above theorem, $I_{italian}(p, C, F_\lambda) \geq \beta_0 + \ldots + \beta_{i+1}$ and F_λ belongs to Σ_{i+1}. By dimension considerations, it follows that $\bar{F}_{i+1}(p, y)$ defines the system Σ_{i+1} as required.

We now prove one direction of the theorem by induction on $i \geq 0$. Suppose that $\{p_{0j}, \ldots, p_{i+1,j}\}$ are given independent generic points in $C \cap \mathcal{V}_p$. By the above, if F_λ belongs to Σ_{i+1}, then there exists $\lambda' \in \mathcal{V}_\lambda$ such that $F_{\lambda'}$ belongs to Σ_i and passes through $p_{i+1,j}$. Moreover, as all the covers F_i are defined over the field of definition L of C, we may take λ' to lie in the field $L_1 = L(p_{i+1,j})^{alg}$. Hence, $F_{\lambda'}$ does not pass through any of the other independent generic points $\{p_{0j}, \ldots, p_{ij}\}$. Let $L_2 = L(p_{0j}, \ldots, p_{ij})^{alg}$. As $dim(p_{kj}/L) = 1$, for $1 \leq k \leq i$, we may, without loss of generality, assume that L_2 is linearly disjoint from L_1 over L. Hence, by the amalgamation property for the universal specialisation, see the paper [7], we have that $\{p_{0j}, \ldots, p_{ij}\}$ still belong to $\mathcal{V}_p \cap C$ when taking $P(L_1)$ as the standard model. Now, we consider the subsystem $\Sigma' \subset \Sigma$ defined by;

$$\Sigma' = \{F_\lambda : F_\lambda \textit{ passes through } p_{i+1,j}\}$$

As $p_{i+1,j}$ was chosen to be generic, it cannot be a base point for any of the subsystems;

$$\Sigma_{n-1} \subset \ldots \subset \Sigma_i \subset \ldots \subset \Sigma_0 \subset \Sigma$$

Hence, we obtain a corresponding filtration;

$$\Sigma_{n-2} \cap \Sigma' \subset \ldots \subset \Sigma_i \cap \Sigma' \subset \ldots \subset \Sigma_0 \cap \Sigma' \subset \Sigma'$$

with the properties in Remarks 6.6. We now apply the induction hypothesis to $F_{\lambda'} \in \Sigma_i \cap \Sigma'$. We can find $\lambda'' \in \mathcal{V}_{\lambda'}$ such that $F_{\lambda''}$ passes through $\{p_{0j}, \ldots, p_{ij}\}$ and $F_{\lambda''}$ belongs to Σ'. Hence $F_{\lambda''}$ passes through $\{p_{0j}, \ldots, p_{i+1,j}\}$. Finally, note that $\lambda'' \in \mathcal{V}_\lambda$, if one considers $P(L)$ rather than $P(L_1)$ as the standard model. Hence, one direction of the theorem is proved.

The converse direction may also be proved by induction on $i \geq 0$. Suppose that $F_{\lambda''}$ passes through independent generic points $\{p_{0j}, \ldots, p_{i+1,j}\}$ in γ_p^j. As before, we may consider $p_{i+1,j}$ as belonging to the standard model $P(L_1)$ and $\{p_{0j},\ldots,p_{ij}\}$ as belonging to $\mathcal{V}_p\cap C$, relative to $P(L_1)$. We again consider the subsystem $\Sigma'\subset\Sigma$ as defined above. Let λ' be the specialisation of λ'' relative to $P(L_1)$. Then, $F_{\lambda'}$ belongs to Σ', as Σ' is defined over $p_{i+1,j}$. Moreover, by the inductive hypothesis applied to Σ', $F_{\lambda'}$ also belongs to Σ_i. We now apply the argument at the beginning of this proof, with $P(L)$ as the standard model, to obtain that F_λ belongs to Σ_{i+1}, where λ is the specialisation of λ', relative to $P(L)$. Clearly λ'' specialises to λ, hence the converse direction is proved.

We still need to consider the case for arbitrary $C\subset P^w$. Let $C^{ns}\subset P^{w'}$ be a non-singular model of C and suppose that the presentation $\Phi_{\Sigma'}$ of (C^{ns},Φ^{ns}) has $Base(\Sigma')$ disjoint from the fibre $\Gamma_{[\Phi^{ns}]}(y,p)$. Let γ_p^j correspond to the infinitesimal neighborhood $\mathcal{V}_{p_j}$ of p_j in $\Gamma_{[\Phi^{ns}]}(y,p)$ and let $\{\overline{F}_\lambda\}$ be the system Σ of lifted forms on C^{ns} corresponding to the space of forms $\{F_\lambda\}$ in Σ. By Lemma 5.12, it follows that, for $\lambda\in Par_\Sigma$, $I_{italian}(p,\gamma_p^j,C,F_\lambda)=I_{italian}(p_j,C^{ns},\overline{F_\lambda})$, hence the character $(\beta_0,\ldots,\beta_n)$ of the branch γ_p^j with respect to the system Σ is the same as the character of the branch γ_{p_j} with respect to the lifted system Σ. Moreover, we obtain the same filtration of Par_Σ, as given in Remark 6.6, for both systems with respect to the branches $\{\gamma_p^j,\gamma_{p_j}\}$ (†). Now, suppose that we are given independent generic points $\{p_{0j},\ldots,p_{ij}\}$ in γ_p^j. Then, by definition of γ_p^j, we can find corresponding independent generic points $\{p'_{0j},\ldots,p'_{ij}\}$ in γ_{p_j}. Now suppose that $F_\lambda\in\Sigma_i$, then the corresponding $\overline{F_\lambda}$ belongs to Σ_i. By the proof of the above theorem for non-singular curves, we can find $\lambda'\in\mathcal{V}_\lambda$ such that $\overline{F_{\lambda'}}$ passes through $\{p'_{0j},\ldots,p'_{ij}\}$. Then, by definition, the corresponding F_λ passes through $\{p_{0j},\ldots,p_{ij}\}$. Conversely, suppose that we can find $\lambda'\in\mathcal{V}_\lambda$ such that $F_{\lambda'}$ passes through $\{p_{0j},\ldots,p_{ij}\}$. Then the corresponding $\overline{F_{\lambda'}}$ passes through $\{p'_{0j},\ldots,p'_{ij}\}$. By the proof for non-singular curves, the specialisation $\overline{F_\lambda}$ belongs to Σ_i. Hence, by the observation (†) above, the corresponding F_λ belongs to Σ_i as well. The theorem is then proved. $\square$

Remarks 6.8. *One can give a slightly more geometric interpretation of the preceding theorem as follows;*

Consider the cover $F\subset C^{i+1}\times Par_\Sigma$ *given by;*

$F(p_1,\ldots,p_{i+1},\lambda)\equiv\{p_1,\ldots,p_{i+1}\}\subset C\cap F_\lambda=0$

Generically, the cover F over C^{i+1} has fibre dimension $n-(i+1)$. For a tuple $(p,\ldots,p)\in\Delta^{i+1}$, the dimension of the fibre $F(p,\ldots,p)$ is $n-1$. By the above, $\Sigma_i\subset F(p,\ldots,p)$, which has dimension $n-(i+1)$, is regular for the cover, in the sense of the above theorem.

The theorem may be construed as a generalisation of an intuitive notion of tangency. $i+1$ independent generic points on the branch γ_p^j determine a projective plane H_i of dimension i. As these $i+1$ points converge independently to p, the plane H_i converges to the i'th osculatory plane at p. As with the proofs we have given of many of the original arguments in [10]*, the method using infinitesimals in fact reverses this type of thinking in favour of a more visual approach. In this case, we have shown that, by moving the i'th osculatory plane* away *from p, we can cut the branch γ_p^j in $i+1$ independent generic points.*

The theorem also provides an effective method of computing osculatory planes at a branch γ_p^j for $p\in C$.

We now require the following lemma;

Lemma 6.9. *Let $F_{\lambda'}$ have finite intersection with C, where the parameter λ' is taken inside the non-standard model $P(K)$. Then there exists a maximally independent set $\{p_{0j},\ldots,p_{ij}\}$ of generic intersections (over L) inside γ_p^j.*

Proof. Let W be the finite set of intersections inside γ_p^j of $F_{\lambda'}$ with C. As C is strongly minimal, if $dim(W/L)=i+1$, then there exists a basis $\{p_{0j},\ldots,p_{ij}\}$ of W over L. In particular, we have that $\{p_{0j},\ldots,p_{ij}\}$ are generically independent points of C and are maximally independent in W. □

We finish this paper with the following theorem;

Theorem 6.10. *Intersections along a Branch*

Let hypotheses be as in the previous theorem. Let i be maximal such that $F_{\lambda'}$ belongs to Σ_i and suppose that $F_{\lambda'}$ intersects γ_p^j in the maximally independent set of generic points $\{p_{i+1,j},\ldots,p_{i+r,j}\}$ over L, where $r\leq(n-1)-i$. Then, if the branch γ_p^j has character $(\beta_0,\ldots,\beta_n)$ with respect to the system Σ, $F_{\lambda'}$ intersects $(\gamma_p^j\setminus p)$ in at least $\beta_{i+1}+\ldots+\beta_{i+r}$ points, counted with multiplicity.

Proof. We assume first that C is non-singular. Let F_λ be the specialisation of $F_{\lambda'}$ relative to the standard model $P(L)$. Then by Theorem 6.7, F_λ belongs to Σ_{i+r} (replace the system Σ in Theorem 6.7 by the system Σ_i.) Hence, $I_{italian}(p, C, F_\lambda) \geq \beta_0 + \ldots + \beta_{i+r}$, whereas $I_{italian}(p, C, F_{\lambda'}) = \beta_0 + \ldots + \beta_i$. It follows immediately, by summability of specialisation, see the paper [6], that the total multiplicity of intersections of $F_{\lambda'}$ with C inside the branch $(\gamma_p^j \setminus p)$ is at least;

$$(\beta_0 + \ldots + \beta_{i+r}) - (\beta_0 + \ldots + \beta_i) = \beta_{i+1} + \ldots + \beta_{i+r}$$

as required. If C is singular, let (C^{ns}, Φ^{ns}) be a non-singular model, with a presentation $\Phi_{\Sigma'}$ such that $Base(\Sigma')$ is disjoint from $\Gamma_{[\Phi]}(x, p)$. Then, given the maximally independent set of generic points $\{p_{i+1,j}, \ldots, p_{i+r,j}\}$, in γ_p^j, for the intersection of C with $F_{\lambda'}$, we obtain a maximally independent set for the intersection $\overline{F}_{\lambda'} \cap C \cap \mathcal{V}_p$. By the above, and the fact that the character of the branch γ_p^j with respect to $\{F_\lambda\}$ equals the character of the branch γ_{p_j} with respect to $\overline{F}_\lambda$, we obtain that $\overline{F}_{\lambda'}$ intersects the branch $(\gamma_{p_j} \setminus p_j)$ in at least $\beta_{i+1} + \ldots + \beta_{i+r}$ points with multiplicity. Hence, using for example Lemma 5.12, and the fact that $Base(\Sigma')$ is disjoint from γ_{p_j}, we obtain that $F_{\lambda'}$ intersects $(\gamma_p^j \setminus p)$ in at least $\beta_{i+1} + \ldots + \beta_{i+r}$ points with multiplicity as well. □

Remarks 6.11. *Note that, in the statement of the theorem, one cannot obtain that $F_{\lambda'}$ intersects the branch $(\gamma_p^j \setminus p)$ in exactly $\beta_{i+1} + \ldots + \beta_{i+r}$ points, with multiplicity. For example, consider the algebraic curve C given in affine coordinates by $x^3 - y^2 = 0$. At $(0,0)$, this has a cusp singularity with character $(2,1)$. The tangent line or 1'st osculatory plane, is given by $y = 0$. The line $y - \epsilon = 0$, where ϵ is an infinitesimal, cuts the branch of C at $(0,0)$ in exactly $3 = 2 + 1$ points. However, the total transcendence degree (over L) of these points is clearly 1. Neither can one obtain that $F_{\lambda'}$ intersects the branch $(\gamma_p^j \setminus p)$ transversely. For example, consider the algebraic curve C given in affine coordinates by $y - x^2 = 0$. Let Σ be the 2-dimensional system consisting of (projective) lines. As $(0,0)$ is an ordinary simple point of C, it has character $(1,1)$, which is also the character of $(0,0)$ with respect to the system Σ. Again, the tangent line or 1'st osculatory plane, is given by $y = 0$. The line $y = (2\epsilon)x - \epsilon^2$ cuts the branch of C at $(0,0)$ in exactly one point (ϵ, ϵ^2), with multiplicity 2, and specialises to $y = 0$.*

Remarks 6.12. *The above theorem is critical in calculations relating to duality. We save this point of view for another occasion.*

7. Some Remarks on Frobenius

When the field has non-zero characteristic, many of the above arguments are complicated by the Frobenius morphism. However, we take the point of view that this is an exception rather than a general rule, hence the results are true if we exclude unusual cases. We will consider each of the previous sections separately and point out where to make these modifications. We briefly remind the reader that given, algebraic curves C_1 and C_2, by a generally biunivocal map, denoted for this section only using the repetition of notation, $\phi : C_1 \leftrightsquigarrow C_2$, we mean a morphism ϕ, defined on an open subset $U \subset C_1$, such that ϕ defines a bijective correspondence between U and an open subset V of C_2. In characteristic 0, a generally biunivocal map is birational, in the sense of Definition 1.19. However, this is not true when the field has non-zero characteristic, Frobenius being a counterexample.

Section 1. The results of this section hold in arbitrary characteristic.

Section 2. We encounter the first problem in Theorem 2.3, the proof of which depends on Lemma 2.10. Unfortunately, Lemma 2.10 is *not* true in arbitrary characteristic. However, as we will explain below, Lemma 2.10 is true for a linear system Σ which defines a birational morphism Φ_Σ on C. As this was assumed in Theorem 2.3, its proof *does* hold in non-zero characteristic.

Lemma 2.10 does *not* hold in arbitrary characteristic. Let C be the algebraic curve defined by $y = 0$ in affine coordinates (x, y). Consider the linear system Σ of dimension 1 defined by $\phi_t(x, y) := (y = x^2 + t)$ in characteristic 2. Then ϕ_t is tangent to C at $(t^{1/2}, 0)$ for all t. In particular, $(0, 0)$ is a coincident mobile point for the linear system. The reason for the failure of the lemma is that the function $F(x, y) = y - x^2$ defines a Zariski unramified morphism on $y = 0$ at $(0, 0)$, which is not etale, it is just the Frobenius map in characteristic 2. One can avoid such cases by insisting that the linear system Σ under consideration defines a separable morphism on C $(*)$. With this extra requirement and a result from [8] (Theorem 6.11) that any locally Zariski unramified separable morphism between curves is locally etale, the proof of Lemma 2.10 holds.

The remaining results of the section are unaffected, with the restriction $(*)$ on Σ in non-zero characteristic. In particular, Lemma 2.30 holds with this restriction. The proof of the Lemma gives the existence of a generally biunivocal morphism ϕ. By seperability, this induces an isomorphism of the function fields of the respective curves. By an elementary algebraic and model theoretic argument, see for example [2], (Theorem 4.4 p25), one can then invert the morphism ϕ in the sense of Definition 1.19. Therefore, ϕ will define a birational map.

Section 3. The proof of Lemma 3.2 requires results from Section 2 which may not hold in certain exceptional cases. However, the Lemma is *still* true in arbitrary characteristic. One should replace the use of Lemma 2.30 by invoking general results for plane curves in [6]. Lemma 3.6 and Theorem 3.3 also holds, if we replace birational with biunivocal. In order to obtain the full statement of Theorem 3.3 in arbitrary characteristic, one can use the following argument;

We obtain from the argument of Theorem 3.3, in arbitrary characteristic, a generally biunivocal morphism ϕ from $C \subset P^2$ to $C_1 \subset P^w$. This induces an inclusion of function fields $\phi^* : L(C_1) \rightarrow L(C)$. We may factor this extension as $L(C_1) \subset L(D) \subset L(C)$, with D an algebraic curve, $L(D) \subset L(C)$ a purely inseperable extension and $L(C_1) \subset L(D)$ a seperable extension. We, therefore, obtain rational maps $\phi_1 : C \rightsquigarrow D$ and $\phi_2 : D \rightsquigarrow C_1$, such that $\phi_2 \circ \phi_1$ is equivalent to ϕ as a biunivocal map between C and C_1. Now, using the method of [8] (Remarks 6.5), we may find an algebraic curve $C' \subset P^2$ (apply some power of Frobenius to the coefficients defining C) and a morphism $Frob^n : C \rightarrow C'$, together with a birational map $\phi_3 : D \leftrightsquigarrow C'$ such that $Frob^n$ and $\phi_3 \circ \phi_1$ are equivalent as biunivocal maps between C and C'. We now obtain a seperable rational map $\phi_4 = \phi_2 \circ \phi_3^{-1} : C' \rightsquigarrow C_1$, such that $\phi_4 \circ Frob^n$ and ϕ are equivalent as biunivocal morphisms. Now let $U \subset NonSing(C)$ be an open set on which ϕ and $\phi_4 \circ Frob^n$ are defined and agree as morphisms. By an elementary application of the chain rule and the fact that the differential of the Frobenius morphism is identically zero, one obtains that, for any $x \in U$, $(D\phi)_x$ contains the tangent line l_x of C. By the methods in the introduction of Section 1, this is in fact a closed condition on $(D\phi)$, hence, in fact $(D\phi)_x$ contains the tangent line l_x of C, for $x \in NonSing(C)$, at any point where ϕ is defined. We can summarise this more generally in the following lemma;

Lemma 7.1. *Let $\phi : C \rightsquigarrow P^w$ be an inseperable rational map, then, for* any *nonsingular point x of C at which ϕ is defined, $(D\phi)_x$ contains the tangent line l_x of C.*

By Remark 3.7, this property is excluded for a transverse g_n^r as used in Theorem 3.3.

Section 4. The projection construction defined at the beginning of the section may fail to define a separable morphism in non-zero characteristic. However, using Lemma 7.1 and methods from Section 1, one can easily show that this only occurs for projective curves C with the property that, for every $x \in NonSing(C)$, the tangent line l_x passes through a given point P. In this case, the projection of C from P onto any hyperplane will be inseparable. In [2] such curves are called *strange*. Non-singular strange curves were completely classified by Samuel in [9];

Theorem 7.2. *The only strange non-singular projective algebraic curves are the line and the conic in characteristic* 2.

However, there are examples of other *singular* strange projective algebraic curves in P^w, for $w \geq 2$, not contained in any hyperplane section. For example, the curves $Fr_w \subset P^w$ obtained by iterating Frobenius, given parametrically by;

$(t, t^p, t^{p^2}, \ldots, t^{p^{w-1}})$ in characteristic p.

For these examples, Lemma 4.2 fails. In order to see this, pick independent points $\{T, S\}$ on Fr_w given by $(t, t^p, \ldots, t^{p^{w-1}})$ and $(s, s^p, \ldots, s^{p^{w-1}})$. Then, the equation of the chord l_{TS} is given parametrically by;

$(t + \lambda s, t^p + \lambda s^p, \ldots, t^{p^{w-1}} + \lambda s^{p^{w-1}})$

If $t + \lambda s = v$, and V is given by $(v, v^p, \ldots, v^{p^{w-1}})$, then we have that the chord l_{TS} meets V, distinct from $\{T, S\}$, iff we can solve $\lambda^{p-1} = 1, \ldots, \lambda^{p^{w-1}-1} = 1$ for $\lambda \neq 1$. This is clearly possible if $p \geq 3$. In this case, we would have that the chord l_{TS} intersects Fr_w in at least p points.

Lemma 4.2 holds in arbitrary characteristic, if we exclude singular strange projective curves, however the proof should be modified as it involves Lemma 2.10 applied to a projection. If C is a non-singular strange curve, using the classification given above, the theorem has no content as we assumed that C was not contained in any hyperplane section.

Lemma 7.3. *Lemma 4.2 in arbitrary characteristic, excluding singular strange projective curves*

Let $C \subset P^w$, for $w \geq 3$, not contained in any hyperplane section and such that C is not *a singular strange projective curve. Suppose that $\{A, B\}$ are independent generic points of C, then the line l_{AB} does not otherwise meet the curve C.*

Proof. We use the same notation as in Lemma 4.2. Let pr_P be the projection defined in this lemma. Suppose that pr_P is inseperable, then, by the above remarks C is a strange projective algebraic curve, such that all its tangent lines l_x, for $x \in NonSing(C)$, pass through P. Hence, we can assume that pr_P is seperable. We now show that the degenerate case (†) cannot occur. Suppose that $pr_P(l_A)$ is a point. We have that $dim_P(A) = 1$, hence we can find an open $W \subset NonSing(C)$, defined over P, such that, for $x \in W$, l_x passes through P. In particular, as $dim_P(B) = 1$, l_B passes through P, hence we must have that $l_A = l_B = AB$. As A and B were independent, it follows easily that C must be a line l, which is a contradiction. We can now follow through the proof of Lemma 4.2 to obtain that;

There exists an open $W \subset NonSing(C)$, defined over the field of definition of C, such that, for $y \in pr_B(W)$, the l_y intersect in a point Q. $(****)$

It follows that, for $x \in W$, the l_x intersect l_B. In particular, l_A intersects l_B. If $l_A = l_B$, we obtain that C is a line, hence we may assume that $l_A \cap l_B = Q$. As B was generic, we can find an open subset $W' \subset NonSing(C)$, defined over AB, such that, for $x \in W'$, the l_x intersect l_A and l_B. Then, either, for such $x \in W'$, the l_x all pass through Q or the l_x all lie in the plane P_{AB} defined by l_A and l_B. In the first case, we have that C is a strange curve, contradicting the hypotheses. In the second case, we use the fact that the plane $Pl = P_{AB}$ must be defined over the field of definition of C and then, by the fact that the generic chord l_{AB} lies in Pl, that C must be contained in Pl as well, contradicting the hypotheses. □

Lemma 4.5 is true if we exclude singular strange projective curves. In order to obtain the corresponding result for a singular strange projective curve C, pick a generic point point $P \in P^w$. Let $x \in C$ be generic and independent from P. We claim that l_{Px} does not otherwise meet the curve $(*)$. If not, we can find $y \in C$, distict from x, such that $P \in l_{xy}$. Hence, $dim(P/xy) \leq 1$ and $dim(P/x) \geq 3$. Now calculate

$dim(Pxy)$ in two different ways;

(i). $dim(Pxy) = dim(P/xy) + dim(xy) \leq 1 + 2 = 3$

(ii).$dim(Pxy) = dim(y/Px) + dim(Px) \geq 0 + 4 = 4$

This clearly gives a contradiction. It follows, using $(*)$, by an elementary model theoretic argument, that the projection pr_P onto any hyperplane H will be generally biunivocal on C. Lemma 4.6 may also be easily modified to include the case of singular strange curves. Theorem 4.8 holds in arbitrary characteristic by the modifications of Lemma 4.5 and Lemma 4.6 and by ensuring that the projections pr_P always define seperable morphisms. In the case of strange curves, we can always ensure this by picking the centre of projection P to be disjoint from the *bad* point Q, defined as the intersection of the tangent lines. Lemma 4.9 is still true in arbitrary characteristic but the proof needs to be modified in order to take into account singular strange projective curves, (we implicitly used Lemma 4.2 in the proof). Using the same notation as in the lemma, given $x \in C$, using the same argument, we can find $P \in P^w$ generic, such that l_{xP} does not otherwise meet the curve. Now using the modification of Lemma 4.5, the projection from P will be generally biunivocal and, by construction, biunivocal at x. We can then obtain the lemma by repeating this argument. Lemma 4.12 holds in arbitrary characteristic provided the projection pr is seperable. As we have already remarked, this can always be arranged in non-zero characteristic. Lemma 4.14 holds in arbitrary characteristic, using the previous modified lemmas, and the fact that a seperable biunivocal map, between C and $pr(C)$, may be inverted to give a birational map including the nonsingular points of $pr(C)$. It follows that Theorem 4.15 holds in arbitrary characteristic as well, by the modifications from Section 3. Finally, Theorem 4.16 holds by checking the result for certain further unusual curves, depending on generalisations of results in later sections, (see (†) below) . We should note that, without these generalisations, Theorem 4.16 still holds if one accepts the weaker definition of a node as the origin of 2 linear branches (see Definition 6.3).

Section 5. The results of this section hold in arbitrary characteristic up to Lemma 5.24. We only make the remark that it is always possible to choice a *maximal* linear system such that it defines a separable morphism on a projective curve C. The proof of Lemma 5.24 has the same complications as Lemma 2.10. Again, we can avoid these complications and recover the remaining results of the section in arbitrary characteristic, by the assumption on the linear system Σ that it defines a seperable morphism on C.

Section 6. The results up to Definition 6.3 hold, by appropriate choices of linear systems Σ. In Definition 6.3, the claim that a generic point of an algebraic curve is an ordinary simple point does not hold in arbitrary characteristic, (†). An example is given by the plane quartic curve $F(x,y,z) = x^3y + y^3z + z^3x = 0$ over a field of characteristic 3. Every point of this curve is an inflection point, that is a point with character $(1,2)$. In this case, the natural duality map;

$$DF : C \rightarrow C^*$$

$$[x : y : z] \mapsto [F_x : F_y : F_z] = [z^3 : x^3 : y^3]$$

is purely inseparable. In order to prove Theorem 4.16 in arbitrary characteristic, one needs to classify such exceptional curves. This can be done, using work of Plucker on the transformation of branches by duality, we save this point of view for another occasion.

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Mathematics Department, The University of Camerino, Camerino, Italy

E-mail address: `tristam.depiro@unicam.it`