EXPANSIONS OF REAL CLOSED FIELDS WITH THE BANACH FIXED POINT PROPERTY

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Abstract. An expansion of a real closed ordered field is said to have the Banach Fixed Point Property when for every locally closed definable set E, if every definable contraction on E has a fixed point, then E is closed. We prove that an expansion of a real closed ordered field has o-minimal open core if and only if it is definably complete and has the Banach Fixed Point Property. As consequences, we obtain that the possession of o-minimal open cores is a first-order property in languages extending the language of the ordered rings and is preserved under elementary equivalence.

Let (X, d) be a metric space and $E \subseteq X$. Recall that a contraction $f\colon E \to E$ on E is a function such that there is $0 < \tau < 1$ such that if $x, y \in E$, then $d(f(x), f(y)) \leq \tau \cdot d(x, y)$; and a fixed point of f is a point $a \in E$ such that $f(a) = a$. In 1922, S. Banach introduced Banach Fixed Point Theorem, which implies that if (X, d) is complete and E is closed then every contraction on E has a fixed point. This theorem later became an important tool in many branches of mathematics, especially in Analysis and Differential Equation (see e.g. [3] and [16]). There are many researches on converses of Banach Fixed Point Theorem (see e.g. [2], [7], [11], and [15]). By [7], we obtain that for every $E \subseteq \mathbb{R}$ if E is **locally closed** (that is, $E = U \cap F$ for some open U and closed F) and every contraction on E has a fixed point, then E is closed. In this article, we consider an analogue of this version of converses of Banach Fixed Point Theorem and its connection to tame expansions of real closed ordered fields.

Throughout, let $\mathfrak{R}$ denote a fixed, but arbitrary, expansion of a real closed ordered field $(R, 0, 1, <, +, \cdot)$. Unless indicated otherwise, "definable in $\mathfrak{R}$" means "definable in $\mathfrak{R}$ possibly with parameters"; in addition, we will omit "in $\mathfrak{R}$" when it causes no confusion. For a definable set E, we say that E has the **Banach Fixed Point Property** if every definable contraction on E has a fixed point. The expansion $\mathfrak{R}$ possesses the **Banach Fixed Point Property** (**BFPP** for short) if every locally closed definable set having the Banach Fixed Point Property is closed. First, we obtain the following:

Proposition A. *If $\mathfrak{R}$ is* **o-minimal** *(that is, every unary definable set is a finite union of points and open intervals), then $\mathfrak{R}$ possesses the BFPP.*

We refer to [6] for backgrounds on o-minimality. In fact, Proposition A is a consequence of more general results we present in this paper.

Besides o-minimality, researchers found many notions of tameness. Here we are interested in the following two notions. We say that the structure $\mathfrak{R}$ is **definably complete** if every

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nonempty unary definable set that is bounded above has a supremum in R. Next, the **open core** of $\mathfrak{R}$ (denoted by $\mathfrak{R}^o$) is the structure on R generated by all definable open sets in $\mathfrak{R}$. We say that $\mathfrak{R}$ has **o-minimal open core** if $\mathfrak{R}^o$ is o-minimal. By [4, Theorem A], we know that if $\mathfrak{R}$ is definably complete and has uniform finiteness, then $\mathfrak{R}$ has o-minimal open core. In this paper, we prove that:

Theorem A. *$\mathfrak{R}$ has o-minimal open core if and only if $\mathfrak{R}$ is definably complete and possesses the BFPP.*

By the Cell Decomposition Theorem, definable sets in o-minimal structures are locally closed. Therefore, Proposition A can be strengthened as follows:

Corollary A. *If $\mathfrak{R}$ is o-minimal, then every definable set possessing the Banach Fixed Point Property is closed.*

We now proceed to more model theoretic points of this paper. Let $\mathfrak{L}$ be a first-order language extending the language of ordered rings. Let T be an $\mathfrak{L}$-theory. We write $T \models \mathbf{DC}$ if every model of T is definably complete; and write $T \models \mathbf{BFPP}$ if every model of T possesses the BFPP. Recall that (1) a structure $\mathfrak{M}$ has **no dense graphs** if the graph of every function definable in $\mathfrak{M}$ is nowhere dense; we write $T \models \mathbf{NDG}$ if every model of T has no dense graphs; and (2) a structure $\mathfrak{M}$ satisfies **uniform finiteness** if for all $A \subseteq M^n \times M^m$ definable in $\mathfrak{M}$, if every fiber $A_x = \{y \in M^m : (x, y) \in A\}$ is finite ($x \in M^n$), then there is $N \in \mathbb{N}$ such that the cardinality of every A_x is at most N; we write $T \models \mathbf{UF}$ if every model of T satisfies uniform finiteness. In [4], A. Dolich, C. Miller and C. Steinhorn gave a nice characterization:

if $T \models \mathbf{NDG}$ and extends the theory of densely ordered groups, then $T \models \mathbf{DC} + \mathbf{UF}$ if and only if every model of T has o-minimal open core.

However, we know that dense pairs of o-minimal structures have o-minimal open core but does not have no dense graphs (see [5]); while there are expansions of the real field by generic predicates that has o-minimal open core and has no dense graphs (see [14]). This gives rise to the following question:

Can we get a characterization that does not depend on **NDG***?*

Here, by Theorem A, we obtain a surprising result:

Theorem B. *Suppose T extends the theory of real closed ordered fields. Then $T \models \mathbf{DC} + \mathbf{BFPP}$ if and only if every model of T has o-minimal open core.*

(Since the proof of this theorem is not difficult once we have Theorem A, we include it here.)

Proof of Theorem B. Recall that the theory of real closed ordered fields can be axiomatized by a theory in the language of ordered rings (see e.g. [12]). We denote this theory by **RCF**. Hence, $T \models \mathbf{RCF}$. Let $\mathfrak{M} \models T$. Then $\mathfrak{M}$ is an expansion of a real closed ordered field. By Theorem A, $\mathfrak{M}$ is definably complete and has the BFPP if and only if $\mathfrak{M}$ has o-minimal open core. This proof is completed. $\square$

Observe that a set E is locally closed if and only if the frontier of E (which is the relative complement E in its closure) is closed. Therefore, both definable completeness and BFPP are first-order schema. In particular, we can conclude that the possession of o-minimal open cores of expansions of real closed ordered fields is also a first-order scheme in $\mathfrak{L}$ and is preserved under elementary equivalence.

Theorem C. *Suppose $\mathfrak{R}$ is an $\mathfrak{L}$-structure and has o-minimal open core. If $\mathfrak{M}$ is an $\mathfrak{L}$-structure and $\mathfrak{M} \equiv \mathfrak{R}$, then $\mathfrak{M}$ has o-minimal open core.*

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Conventions and notations. Throughout this paper, m and n will range over the set $\mathbb{N} = \{0, 1, 2, 3, \dots\}$ of natural numbers. Let $S \subseteq R^n$. We denote by $\operatorname{cl} S = \operatorname{cl}(S)$ the closure, by $\operatorname{fr} S = \operatorname{fr}(S) := \operatorname{cl}(S) \setminus S$ the frontier, and by $\operatorname{int} S = \operatorname{int}(S)$ the interior of S. For $a \in R^n$, let $S - a := \{x - a : x \in S\}$ be the translation of S by a. We denote the Euclidean norm on R^n by $\|\cdot\|$. For $S \subseteq R^n$, we denote the diameter of S by

$$\operatorname{diam} S := \sup\{\|x - y\| : x, y \in S\}.$$

1. Preliminaries

In this section, we recall fundamental properties that will be used in the proof of Theorem A.

1.1. Definable completeness.

Throughout this section, assume $\mathfrak{R}$ is definably complete. First, we recall several properties of definably complete structures.

1.1. *[Miller,* [13, Proposition 1.10]*] Let E be a closed, bounded and definable subset of R^n and $f\colon E \to R^m$ be continuous and definable. Then the image $f(E)$ is also closed, bounded and definable.*

1.2. *[Aschenbrenner, Fischer,* [1, Corollary 1.5]*] Let E be a nonempty, closed, bounded and definable subset of R^n and $f\colon E \to R$ be continuous and definable. Then f achieves a maximum and a minumum on E.*

Let $E \subseteq R^n$ be definable. We say that E is **definably connected** if for all disjoint definable open nonempty sets $U, V \subseteq R^n$ such that $E = (E \cap U) \cup (E \cap V)$, either $E \subseteq U$ or $E \subseteq V$.

1.3. *[Miller,* [13, Proposition 1.6]*] Let $E \subseteq R^n$ be definably connected and $f\colon E \to R^m$ be definable. Then $f(E)$ is also definably connected.*

Since the BFPP is related to converses of Banach Fixed Point Theorem, it is natural to check whether there is a definable version of Banach Fixed Point Theorem in definably complete context.

1.4. *Let E be nonempty, closed and definable. If f is a definable contraction on E, then it has a fixed point.*

Proof. Let $f\colon E \to E$ be a definable contraction. Then there is $0 < \tau < 1$ such that $\|f(x) - f(y)\| \le \tau \cdot \|x - y\|$ for all $x, y \in E$. By replacing E by $E - a$ (for some $a \in E$), we may assume that $0 \in E$. Observe that if $\|x\| \ge \|f(0)\|/(1-\tau)$ then $\|f(x)\| \le \|f(x) - f(0)\| + \|f(0)\| \le \tau \cdot \|x\| + (1-\tau)\|x\| = \|x\|$. Let $E_0 = \{x \in E : \|x\| \le \|f(0)\|/(1-\tau)\}$. By 1.1, the image $f(E_0)$ is bounded. Let $\delta \in R$ such that $\delta \ge \|f(0)\|/(1-\tau)$ and $f(E_0) \subseteq \{x \in R^n : \|x\| \le \delta\}$. Therefore, we may assume further that E is bounded.

To complete this proof, it remains to prove that f has a fixed point. Suppose to the contrary that f has no fixed points. Define $g\colon E \to R^n$ by $g(x) := f(x) - x$ for every $x \in E$. Then $g(x) \ne 0$ for every $x \in E$. Since E is closed, bounded and definable and g is continuous and definable, by 1.2, $\|g\|$ has a minimum on E. Suppose $\|g\|$ achieves a minimum at $x_0 \in E$. Therefore,

$$\begin{aligned}\|g(x_0)\| \le \|g(f(x_0))\| &= \|f(f(x_0)) - f(x_0)\| \\ &\le \tau \cdot \|f(x_0) - x_0\| \\ &\le \tau \cdot \|g(x_0)\| < \|g(x_0)\|,\end{aligned}$$

which is absurd. □

We now know that a definable version of Banach Fixed Point Theorem holds in definably complete expansions of a real closed ordered field. This result will be used in the proof of Theorem A.

1.2. **O-minimality.**

Throughout this part, assume $\mathfrak{R}$ is o-minimal and $E \subseteq R^n$ is definable. We know o-minimal structures possess many good geometric properties. Here are some examples.

1.5 (Definable Choice, van den Dries, [6, Proposition 1.2]). *Let $\{A_x\}_{x\in X}$ be a definable family of nonempty subsets of R^n. Then there is a definable function $f\colon X \to R^n$ such that $f(x) \in A_x$ for all $x \in X$.*

1.6 (Curve Selection, van den Dries, [6, Corollary 1.5]). *Let $a \in \operatorname{fr} E$. Then there is a definable continuous path $\gamma\colon [0,1] \to R^n$ such that $\gamma(0) = a$ and $\gamma(0,1] \subseteq E$.*

Let $L \in R$. A function $f\colon E \to R^m$ is L-Lipschitz if for every $x, y \in E$, $\|f(x) - f(y)\| \le L \cdot \|x - y\|$. We say that a subset C of R^n is an **L-Lipschitz cell** if there exist a linear orthogonal isomorphism $\pi\colon R^n \to R^n$, a definable open set $U \subseteq R^k$, and a definable L-Lipschitz map $g\colon U \to R^{n-k}$ such that $k \le n$, $C = \pi\{(x, g(x)) : x \in U\}$. In [8], A. Fischer proved the Λ^m-regular Stratification Theorem. The following is an immediate consequence of this theorem.

1.7. *E is a finite disjoint union of $2n^{3/2}$-Lipschitz cells.*

Combining the Curve Selection and 1.7, we obtain:

1.8. *Let $a \in \operatorname{fr} E$. Then there is a definable $2n^2$-Lipschitz path $\gamma\colon [0,b] \to R^n$ such that γ is injective, $\gamma(0) = a$, and $\gamma(0,b] \subseteq E$.*

Proof. Observe that the case $n = 1$ is trivial. Therefore, we assume that $n \ge 2$.

Let $a \in \operatorname{fr} E$. By the Curve Selection, we have a definable continuous path $\lambda\colon [0,1] \to R^n$ such that $\lambda(0) = a$ and $\lambda(0,1] \subseteq E$. Let $A = \gamma[0,1]$. Then we have that $\dim A = 1$.

Therefore, by 1.7, there exist a linear orthogonal isomorphism $\pi\colon R^n \to R^n$ and a definable $2n^{3/2}$-Lipschitz map $g\colon (\alpha_1, \alpha_2) \to R^{n-1}$ such that $\pi((t, g(t))) \in A$ for all $t \in (\alpha_1, \alpha_2)$ and $\lim_{t\to\alpha_1} \pi((t, g(t))) = a$. Let $b = \alpha_2 - \alpha_1$. We define $\gamma\colon [0, b] \to R^n$ by

$$\gamma(s) = \begin{cases} \pi((\alpha_1 + s, g(\alpha_1 + s))) & \text{if } s \neq 0; \\ a & \text{if } s = 0. \end{cases}$$

Obviously, we have that γ is a definable injection, $\gamma(0) = a$ and $\gamma(0, b] \subseteq E$. To complete this proof, we show that γ is $2n^2$-Lipschitz. Since π is a linear orthogonal isomorphism, it is 1-Lipschitz. Therefore, it is enough to prove that for all $t_1, t_2 \in [\alpha_1, \alpha_2]$,

$$\|(t_1, g(t_1)) - (t_2, g(t_2))\| \leq 2n^2 \cdot |t_1 - t_2|\,.$$

Let $t_1, t_2 \in [\alpha_1, \alpha_2]$. Since g is $2n^{3/2}$-Lipschitz, we obtain that

$$\begin{aligned} \|(t_1, g(t_1)) - (t_2, g(t_2))\| &= \sqrt{(t_1 - t_2)^2 + \|g(t_1) - g(t_2)\|^2} \\ &\leq \sqrt{(1 + 4n^3)\,|t_1 - t_2|^2} \\ &= 2n^2 \cdot |t_1 - t_2| \end{aligned}$$

as desired. This completes the proof. □

2. Proof of main theorems

For convenience of readers, we recall the statement of Theorem A and Theorem C here.

Theorem A. *$\mathfrak{R}$ has o-minimal open core if and only if $\mathfrak{R}$ is definably complete and possesses the BFPP.*

Theorem C. *Suppose $\mathfrak{R}$ is an $\mathfrak{L}$-structure and has o-minimal open core. If $\mathfrak{M}$ is an $\mathfrak{L}$-structure and $\mathfrak{M} \equiv \mathfrak{R}$, then $\mathfrak{M}$ has o-minimal open core.*

We now begin the proofs.

2.1. **Proof of Theorem A (forward direction).** First assume that $\mathfrak{R}$ has o-minimal open core. Notice that for every unary definable set A, the set $\{x \in R : x \leq a \text{ for some } a \in A\}$ is definable in $\mathfrak{R}^o$. Since every o-minimal structure is definably complete, we have that $\mathfrak{R}$ is definably complete. To prove that $\mathfrak{R}$ possesses the BFPP, let E be a locally closed, definable set. Since E is locally closed, we have that E is also definable in $\mathfrak{R}^o$. Suppose E is not closed. We will show that there is a definable contraction on E that has no fixed points.

Let $a \in \operatorname{fr} E$. Since $\mathfrak{R}^o$ is o-minimal and E is definable in $\mathfrak{R}^o$, by 1.8, we obtain an injective $2n^2$-Lipschitz path $\gamma\colon [0, b] \to R^n$ such that γ is definable in $\mathfrak{R}^o$, $\gamma(0) = a$, and $\gamma(0, b] \subseteq E$.

Define $H\colon E \to E$ by

$$H(x) := \gamma(\min(b, \|x - a\|)/4n^2).$$

We first show that H is a contraction. Let $x, y \in E$. Then

$$\begin{aligned}
\|H(x) - H(y)\| &= \left\|\gamma(\min(b, \|x-a\|)/4n^2) - \gamma(\min(b, \|y-a\|)/4n^2)\right\| \\
&\le \frac{2n^2}{4n^2} \cdot |\min(b, \|x-a\|) - \min(b, \|y-a\|)| \\
&\le \frac{1}{2}\|x-y\|.
\end{aligned}$$

To prove that H has no fixed points, suppose to the contrary that c is a fixed point for H. Since $H(E) \subseteq \gamma(0, b]$ and γ is injective, $c = H(c) = \gamma(\|c-a\|/4n^2)$. Recall that γ is $2n^2$-Lipschitz. Therefore, we have

$$\begin{aligned}
\|c-a\| &= \left\|\gamma(\|c-a\|/4n^2) - \gamma(0)\right\| \\
&\le \frac{2n^2}{4n^2} \cdot \|c-a\| \\
&< \|c-a\|
\end{aligned}$$

which is impossible.

Hence, H is a contraction on E that has no fixed points.

2.2. **Proof of Theorem A (backward direction).** Conversely, assume $\mathfrak{R}$ is definably complete and does not have o-minimal open core. By [4], there is a definable, infinite and discrete subset of R. By the same argument as in [10, Lemma 3] (which was inspired by [9]), we obtain a definable, infinite, closed and discrete set D. We may assume that $1 \in D \subseteq \{x \in R : x \ge 1\}$. Let $D^{-1} = \{1/x : x \in D\}$ and the predecessor $P\colon D^{-1} \to D^{-1}$ be defined by $P(t) = \sup\{s \in D^{-1} : s < t\}$ for all $t \in D^{-1}$. Set

$$E = (D \times [0,1]) \cup \{(st + (1-s)P(t), s) : s \in [0,1], t \in D^{-1}\} \text{ (see Figure 1).}$$

(The construction of the set E is inspired by the fact that every contraction on the graph of $x \mapsto \sin(1/x)\colon (0,1] \to \mathbb{R}$ has a fixed point; see [7, Theorem 1.2])

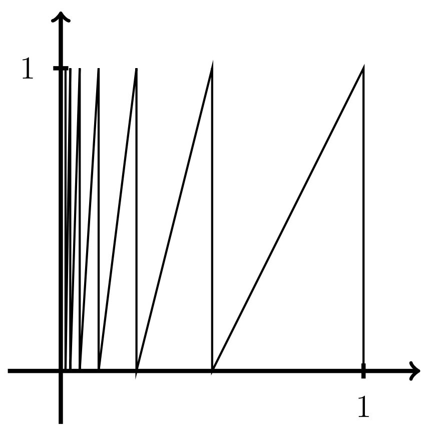

FIGURE 1. Illustration of the set E

It is routine to check that E is locally closed but not closed. To finish this proof, we show that E has the Banach Fixed Point Property.

Let $f\colon E \to E$ be a definable contraction. Then there is $0 < \tau < 1$ such that $\|f(x) - f(y)\| \leq \tau \cdot \|x - y\|$ for all $x, y \in E$. For each $\epsilon \in [0,1]$, let $E_\epsilon = E \cap ([0,\epsilon] \times [0,1])$ and $E^\epsilon = E \cap ([\epsilon, 1] \times [0,1])$. Observe that (1) $\operatorname{diam}(E_\epsilon) \leq \sqrt{1+\epsilon^2}$, (2) E_ϵ is definably connected, and (3) E^ϵ is closed and bounded.

Claim. *There is $\delta > 0$ such that $f(E) \subseteq E^\delta$.*

Suppose we have the above claim for now. Let $\delta > 0$ such that $f(E) \subseteq E^\delta$. Hence, $f\restriction E^\delta$ is a contraction on E^δ. Since E^δ is closed and definable, by 1.4, $f\restriction E^\delta$ has a fixed point; therefore, f also has a fixed point.

To finish this proof, we give:

Proof of Claim. Let $\epsilon = \sqrt{(1-\tau^2)/2\tau^2}$. It is enough to prove that there exist $\delta_1, \delta_2 > 0$ such that $f(E_\epsilon) \subseteq E^{\delta_1}$ and $f(E^\epsilon) \subseteq E^{\delta_2}$. Note that $\epsilon < \sqrt{(1-\tau^2)/\tau^2}$. Therefore, $\operatorname{diam}(E_\epsilon) < 1/\tau$; and so we have $\operatorname{diam}(f(E_\epsilon)) < 1$. Hence, $E_\epsilon \cap ([0,1] \times \{0\}) = \emptyset$ or $E_\epsilon \cap ([0,1] \times \{1\}) = \emptyset$. Since $f(E_\epsilon)$ is definably connected, there exists $\delta_1 > 0$ such that $f(E_\epsilon) \subseteq E^{\delta_1}$. Next, we consider $f(E^\epsilon)$. Since E^ϵ is closed and bounded, $f(E^\epsilon)$ is closed and bounded. Therefore, there is $\delta_2 > 0$ such that $f(E^\epsilon) \subseteq E^{\delta_2}$. $\square$

2.3. **Proof of Theorem C.** Suppose $\mathfrak{R}$ has o-minimal open core. Let $\mathfrak{M}$ be an $\mathfrak{L}$-structure such that $\mathfrak{M} \equiv \mathfrak{R}$. Let T be the theory of $\mathfrak{R}$ in the language $\mathfrak{L}$. Hence $\mathfrak{M} \models T$. Since $\mathfrak{R}$ is an expansion of a real closed field, $T \models \mathbf{RCF}$; so $\mathfrak{M}$ is an expansion of a real closed field. In addition, since $\mathfrak{R}$ has o-minimal open core, by Theorem A, $\mathfrak{R}$ is definably complete and has the BFPP. As discussed in the introduction, both definable completeness and BFPP are first-order schema. Hence $T \models \mathbf{DC} + \mathbf{BFPP}$. Since $\mathfrak{M} \models T$, by Theorem B, $\mathfrak{M}$ has o-minimal open core.

3. Open question

3.1. *We say that $\mathfrak{R}$ possesses the* **strong Banach Fixed Point Property** *("***SBFPP***" for short) if every definable set possessing the BFPP is closed. As mentioned in Corollary A, we know that if $\mathfrak{R}$ is o-minimal, then it has the SBFPP. Therefore, a question arises naturally: does the converse hold? To be precise, we ask:*

If $\mathfrak{R}$ is definably complete and has the SBFPP, is $\mathfrak{R}$ o-minimal?

Now we do not know the answer. We suspect that the answer is 'no'. Obviously, if $\mathfrak{R}$ is definably complete and possesses the SBFPP, then it has o-minimal open core. One of our candidates for counterexamples is the expansion of the real field by the set of all real algebraic numbers.

3.2. *It is natural to ask whether a similar result holds in expansions of ordered abelian groups. Since every abelian group can be considered as a vector space over $\mathbb{Q}$, we can obtain a weak form of contractions in this context. However, the major problem is the backward direction in the proof of Theorem A. The given construction of non-closed, locally closed sets that have the BFPP does not seem to work without multiplication.*

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