LOCALLY DEFINABLE $C^\infty G$ MANIFOLD STRUCTURES OF LOCALLY DEFINABLE $C^r G$ MANIFOLDS

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Abstract. Let G be a finite abelian group and $1 \leq r < \infty$. We prove that every locally definable $C^r G$ manifold admits a unique locally definable $C^\infty G$ manifold structure up to locally definable $C^\infty G$ diffeomorphism.

1. Introduction[1]

Let G be a finite group and $1 \leq r < \infty$. Let $\mathcal{M}$ be an o-minimal exponential expansion $(\mathbb{R}, +, \cdot, >, e^x, \dots)$ of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers admits the C^∞ cell decomposition and has piecewise controlled derivatives.

In this paper we consider existence of locally definable $C^\infty G$ manifold structures of a locally definable $C^r G$ manifold and uniqueness of locally definable $C^\infty G$ manifold structure up to locally definable $C^\infty G$ diffeomorphism. If G is a finite abelian group and $1 \leq s < r < \infty$, then unique existence of locally definable $C^r G$ manifold structure of a locally definable $C^s G$ manifold is studied in [11].

Let $0 \leq r \leq \infty$. A *locally definable C^r manifold* is a C^r manifold admitting a countable system of charts whose gluing maps are of class definable C^r. If this system is finite, then it is called a *definable C^r manifold*. Definable $C^r G$ manifolds are studied in [5], [6], [7], [8], [9] . A locally definable C^r manifold is *affine* if it can be imbedded into some $\mathbb{R}^n$ in a locally definable C^r way. We can define locally definable $C^r G$ manifolds and affine locally definable $C^r G$ manifolds in a similar way of equivariant definable cases. Locally definable $C^r G$ manifolds are generalizations of definable $C^r G$ manifolds and they are studied in [11] when r is a positive integer.

In this paper everything is considered in $\mathcal{M}$, any map is continuous and every manifold does not have boundary unless otherwise stated.

Theorem 1.1. *Let G be a finite group and $1 \leq r < \infty$. Then every affine locally definable $C^r G$ manifold is locally definably $C^r G$ diffeomorphic to some locally definable $C^\infty G$ manifold.*

Theorem 1.2. *Let G be a finite group. Then for any two affine locally definable C^∞ G manifolds, they are $C^1 G$ diffeomorphic if and only if they are locally definably $C^\infty G$ diffeomorphic.*

If $\mathcal{M}$ is polynomially bounded, then Theorem 1.2 is not always true. Even in the non-equivariant Nash category, there exist two affine Nash manifolds such that they are

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not Nash diffeomorphic but C^∞ diffeomorphic [14], and that for any two affine Nash manifolds, they are locally Nash diffeomorphic if and only if they are Nash diffeomorphic.

Existence of $C^\omega G$ manifold structures of proper $C^\infty G$ manifolds and uniqueness of them are studied in [3] and [4], respectively, when G is a C^ω Lie group. Moreover if G is a compact C^ω Lie group, then for any two $C^\omega G$ manifolds, they are $C^\infty G$ diffeomorphic if and only if they are $C^\omega G$ diffeomorphic [13]. Theorem 1.1 and 1.2 are locally definable C^∞ versions of [2] and [3], respectively, when G is a finite group.

The above theorems are locally definable C^∞ versions of results of [10].

In the non-equivariant setting, we have the following.

Theorem 1.3. *If $1 \leq r \leq \infty$, then every n-dimensional locally definable C^r manifold X is locally definably C^r imbeddable into $\mathbb{R}^{2n+1}$.*

The above theorem is the locally definable version of Whitney's imbedding theorem (e.g. 2.14 [2]). The definable C^r version of Theorem 1.1 is known in [8] when r is a non-negative integer.

If $\mathcal{M} = \mathcal{R}$ and $r = \infty$, then Theorem 1.3 is not true. The assumption that $\mathcal{M}$ is exponential is necessary.

As a corollary of Theorem 1.3, we have the following.

Theorem 1.4. *Let G be a finite abelian group and $1 \leq r \leq \infty$. Then every locally definable C^rG manifold is affine.*

By Theorem 1.1-1.4, we have the following theorem.

Theorem 1.5. *Let G be a finite abelian group and $1 \leq r < \infty$. Then every locally definable C^rG manifold admits a unique locally definable $C^\infty G$ manifold structure up to locally definable $C^\infty G$ diffeomorphism.*

2. Locally definable C^rG manifolds

Let $f : U \to \mathbb{R}$ be a definable C^∞ function on a definable open subset $U \subset \mathbb{R}^n$. We say that f has *controlled derivatives* if there exist a definable continuous function $u : U \to \mathbb{R}$, real numbers $C_1, C_2, \dots$ and positive integers $E_1, E_2, \dots$ such that $|D^\alpha f(x)| \leq C_{|\alpha|} u(x)^{E_{|\alpha|}}$ for all $x \in U$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. We say that $\mathcal{M}$ has *piecewise controlled derivatives* if for every definable C^∞ function $f : U \to \mathbb{R}$ defined in a definable open subset U of $\mathbb{R}^n$, there exist definable open sets $U_1, \dots, U_l \subset U$ such that $\dim(U - \cup_{i=1}^l U_i) < n$ and each $f|U_i$ has controlled derivatives.

A subset X of $\mathbb{R}^n$ is called *locally defin-able* if for every $x \in X$ there exists a definable open neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U$ is definable in $\mathbb{R}^n$. Clearly every definable set is locally definable. Remark that any open subset of $\mathbb{R}^n$ is locally definable and that every compact locally definable set is definable. A more general setting of locally definable sets is studied in [1].

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be locally definable sets. We call a map $f : U \to V$ *locally definable* if for any $x \in U$ there exists a definable open neighborhood W_x of x in $\mathbb{R}^n$ such that $f|U \cap W_x$ is definable.

Note that for any locally definable map f between locally definable sets X and Y, if X is compact, then $f(X)$ is a definable set and $f : X \to f(X)$ $(\subset Y)$ is a definable map.

Remark that the maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sin x, f_2(x) = \cos x$, respectively, are analytic but not definable in any o-minimal expansion of $\mathcal{R}$. However they are locally definable in $\mathbf{R}_{an}$. Remark further that the field $\mathbb{Q}$ ($\subset \mathbb{R}$) of rational numbers is not a locally definable subset of $\mathbb{R}$.

Proposition 2.1 ([11])**.** *Let X, Y and Z be locally definable sets and let $f : X \to Y$ and $g : Y \to Z$ be locally definable maps. Then $g \circ f : X \to Z$ is locally definable.*

We can define locally definable groups and affine locally definable groups in a similar way of definable cases. But we do not give their definitions here because we restrict our attention to finite groups.

A *representation map* of G is a group homomorphism from G to some $O(n)$. A *representation* of G means the representation space of a representation map of G. Recall the definition of locally definable $C^r G$ manifolds [11].

Definition 2.2 ([11])**.** Let $1 \le r \le \omega$.
(1) A locally definable C^r submanifold of a representation Ω of G is called a *locally definable $C^r G$ submanifold of* Ω if it is G invariant.
(2) A *locally definable $C^r G$ manifold* is a pair (X, θ) consisting of a locally definable C^r manifold X and a group action θ of G on X such that $\theta : G \times X \to X$ is a locally definable C^r map. For simplicity of notation, we write X instead of (X, θ). Clearly each definable $C^r G$ manifold is a locally definable $C^r G$ manifold.
(3) Let X and Y be locally definable $C^r G$ manifolds. A locally definable C^r map is called a *locally definable $C^r G$ map* if it is a G map. We say that X and Y are *locally definably $C^r G$ diffeomorphic* if there exist locally definable $C^r G$ maps $f : X \to Y$ and $h : Y \to X$ such that $f \circ h = id$ and $h \circ f = id$.
(4) A locally definable $C^r G$ manifold is said to be *affine* if it is locally definably $C^r G$ diffeomorphic to a locally definable $C^r G$ submanifold of some representation of G.

Note that we can define locally definable G manifolds for a locally definable group G, but in this paper we do not use these notions.

Recall existence of definable $C^r G$ tubular neighborhoods.

Theorem 2.3 ([9], [6])**.** *Let r be a non-negative integer, ∞ or ω. Then every definable $C^r G$ submanifold X of a representation Ω of G has a definable $C^r G$ tubular neighborhood (U, θ_X) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta_X : U \to X$ is a definable $C^r G$ map with $\theta_X | X = id_X$.*

Let $G = \{g_1, \dots, g_m\}$ and let f be a $C^r G$ map from a $C^r G$ manifold M to a representation Ω of G. Then the averaging map $A : M \to \Omega$ is

$$A(f)(x) = \frac{1}{m} \sum_{i=1}^{m} g_i^{-1} f(g_i x).$$

By using [7], we have the following lemma.

Proposition 2.4 ([7])**.** *(1) $A(f)$ is equivariant, and $A(f) = f$ if f is equivariant.*
(2) If f is a polynomial map, then so is $A(f)$.
(3) If $0 \le r \le \infty$ and f lies in the set $C^r(M, \Omega)$ of C^r maps from M to Ω, then $A(f) \in C^r(M, \Omega)$.

(4) $A : C^r(M, \Omega) \to C^r(M, \Omega), f \mapsto A(f)$ $(0 \le r \le \infty)$ *is continuous in the* C^r *Whitney topology.*
(5) If M *is a definable* C^rG *manifold,* f *is a definable* C^r *map and* $0 \le r \le \omega$*, then* $A(f)$ *is a definable* C^rG *map.*
(6) If M *is a locally definable* C^rG *manifold,* f *is a locally definable* C^r *map and* $0 \le r \le \omega$*, then* $A(f)$ *is a locally definable* C^rG *map.*

Let K be a subgroup of G. Suppose that S is an affine definable $C^\infty K$ manifold. Then we know that the twisted product $G \times_K S$ with the standard action $G \times (G \times_K S) \to G \times_K S, (g, [g', s]) \mapsto [gg', s]$ is a definable $C^\infty G$ manifold [9].

We need the following proposition to prove Theorem 1.1.

Proposition 2.5. *Let* X *be a locally definable* $C^\infty G$ *manifold. Suppose that* K *is a subgroup of* G *and* N *is an affine definable* $C^\infty K$ *manifold. If* $f : N \to X$ *is a locally definable* $C^\infty K$ *map, then*

$$\mu(f) : G \times_K N \to X, \mu([g, n]) = gf(n)$$

is a locally definable $C^\infty G$ *map.*

Proof. By the property of quotient manifolds, $\mu(f)$ is a $C^\infty G$ map. Thus it suffices to prove that $\mu(f)$ is locally definable. Let π be the orbit map $G \times N \to G \times_K N$. Then π is a definable C^∞ map. Take $x \in G \times_K N$ and $y \in \pi^{-1}(x) \subset G \times N$. By the assumption and the definition of the G action on $G \times N$, $\overline{\mu}(f) : G \times N \to X, \overline{\mu}(f)(g, n) = gf(n)$ is a locally definable $C^\infty G$ map. Hence there exist definable open neighborhoods U of y and V of $\overline{\mu}(f)(y)$, respectively, such that $\overline{\mu}(f)(U) \subset V$ and $\overline{\mu}(f)|U : U \to V$ is a definable C^∞ map. In particular, $\overline{\mu}(f)|U : U \to V$ is definable. Hence $\pi(U)$ is open and definable. Since the graph of $\mu(f)|\pi(U) : \pi(U) \to V \subset X$ is the image of that of $\overline{\mu}(f)|U$ by $\pi \times id_V$, $\mu(f)|\pi(U)$ is definable. □

Definition 2.6. Let X be a definable $C^\infty G$ manifold.
(1) We say that a K invariant definable C^∞ submanifold S of X is a *definable* K *slice* if GS is open in X, S is affine as a definable $C^\infty K$ manifold, and

$$\mu : G \times_K S \to GS \ (\subset X), [g, x] \mapsto gx$$

is a definable $C^\infty G$ diffeomorphism.
(2) A definable $C^\infty K$ slice S is called *linear* if there exist a representation Ω of K and a definable C^rK imbedding $j : \Omega \to X$ such
that $j(\Omega) = S$.
(3) We say that a definable $C^\infty K$ slice (resp. a linear definable $C^\infty K$ slice) S is a *definable* C^∞ *slice* (resp. a *linear definable* C^∞ *slice*) *at* x *in* X if $K = G_x$ and $x \in S$ (resp. $K = G_x$, $x \in S$ and $j(0) = x$).

Recall existence of definable C^∞ slices [9] to prove Theorem 1.2 .

Theorem 2.7 ([9])**.** *Let* X *be an affine definable* $C^\infty G$ *manifold,* $x \in X$*. Then there exists a linear definable* $C^\infty G$ *slice at* x *in* X*.*

3. Proof of Theorem 1.1

The following lemma is obtained by 2.2.8 [2] and Proposition 2.4.

Lemma 3.1. *Let K be a finite group. Suppose that f is a definable $C^\infty K$ map between definable $C^\infty K$ manifolds M and N. Suppose further that V is an open K invariant subset of M and that P is a K invariant definable C^∞ submanifold of N with $f(V) \subset P$. Then there exist an open neighborhood $\mathfrak{N}$ of $f|V$ in the set $Def_K^\infty(V, P)$ of definable $C^\infty K$ maps from V to P such that for any $h \in \mathfrak{N}$, the map $E(h) : M \to N$,*

$$E(h)(x) = \begin{cases} h(x), & x \in V \\ f(x), & x \in M - V \end{cases}$$

is a definable $C^\infty K$ map and $E : \mathfrak{N} \to Def_K^\infty(M, N), h \mapsto E(h)$ is continuous in the C^∞ Whitney topology.

Proposition 3.2. *Let X be a locally definable $C^\infty G$ manifold and Y an affine definable $C^\infty G$ manifold in a representation Ω of G. Then every $C^\infty G$ map $f : X \to Y$ is approximated by a locally definable $C^\infty G$ map $h : X \to Y$ in the C^∞ Whitney topology.*

In the Nash case, if $1 \leq r < \infty$, then locally C^r Nash diffeomorphisms are essentially different from C^r Nash diffeomorphisms because there exist two affine Nash manifolds such that they are C^∞ diffeomorphic but not Nash diffeomorphic [14], and that every C^r Nash diffeomorphism between affine Nash manifolds is approximated by a Nash diffeomorphism [15].

Proposition 3.3 ([12])**.** *Every affine definable $C^\infty G$ manifold is definably $C^\infty G$ diffeomorphic to a definable $C^\infty G$ submanifold closed in some representation Ω of G.*

For the proof of Proposition 3.3, we need the condition that $\mathcal{M}$ is exponential, admits the C^∞ cell decomposition and has piecewise controlled derivatives.

Proof of Proposition 3.2. By Proposition 3.3, replacing Ω if necessary, we may assume that Y a definable $C^\infty G$ submanifold closed in Ω. By a way similar to find a C^∞ partition of unity of C^∞ manifold, we have a locally definable C^∞ partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinates to some locally finite open definable cover $\{X_j\}_{j=1}^\infty$ of X such that $X = \cup_{j=1}^\infty \mathrm{supp}\ \phi_j$ and $\overline{X_j}$ is compact. For any j, take an open neighborhood U_j of $\mathrm{supp}\ \phi_j$ in X such that $\overline{U_j}$ is compact. Applying the polynomial approximation theorem, we have a locally definable C^∞ map $h_j : U_j \to \Omega$ which approximates $f|U_j$. By Theorem 2.3, one can find a definable $C^\infty G$ tubular neighborhood (U, p) of Y in Ω. If our approximation is sufficiently close, then $p \circ \sum_{j=1}^\infty \phi_j h_j$ is a (non-equivariant) C^∞ approximation of f. Since G is a finite group, applying Proposition 2.4, we have the required locally definable $C^\infty G$ map h as a C^∞ Whitney approximation of f. □

Proof of Theorem 1.1. Using Lemma 3.1 and Proposition 3.2, a similar proof of 1.1 [11] proves Theorem 1.1. □

4. Proof of Theorem 1.2

In this section we prove the following theorem.

Theorem 4.1. *Let G be a finite group and let r be a positive integer. Suppose that Y and Z are affine locally definable $C^\infty G$ manifolds and there exists a $C^r G$ diffeomorphism $f : Y \to Z$. Then there exists a locally $C^\infty G$ diffeomorphism $h : Y \to Z$ which is G homotopic to f.*

Theorem 1.2 follows from Theorem 4.1.

Let K be a subgroup of G and let X be an affine definable $C^\infty G$ manifold. By Theorem 2.7, there exists a linear definable $C^\infty K$ slice S, namely there exists a definable $C^\infty K$ diffeomorphism i from some representation Ω of K to S such that GS is open in X, and that $\mu : G \times_K \Omega \to GS\ (\subset X), \mu(i)([g,x]) = gi(x)$ is a definable $C^\infty G$ diffeomorphism.

For simplicity, we use the following notations. Set $B_s := \{x \in \Omega | ||x|| \le s\}, B_s^\circ := \{x \in \Omega | ||x|| < s\}, s > 0, B := B_1,$ and $B^\circ :$
$= B_1^\circ$, and let denote D_s, D_s°, D and D° by $i(B_s), i(B_s^\circ), i(B)$, and $i(B^\circ)$, respectively. Let GD (resp. GD°) denote the closed unit tube (resp. the open unit tube) and let GD_s (resp. GD_s°) stand for the closed tube (resp. the open tube) of radius s.

To prove Theorem 4.1 we prepare two preliminary results.

Lemma 4.2. *Let Ω and Ξ be representations of G and let M (resp. N) be a definable $C^\infty G$ submanifold of Ω (resp. Ξ). Suppose that F is a G invariant definable subset of M and that $\alpha : M \to N$ is a $C^\infty G$ map such that $\alpha|F : F \to N$ is definable. Let $\mathfrak{N}$ be a neighborhood of α in the set $C_G^\infty(M, N)$ of $C^\infty G$ maps from M to N and let V_1 and V_2 be compact G invariant definable subsets of M such that V_1 is properly contained in the interior Int V_2 of V_2. Then there exists $\kappa \in \mathfrak{N}$ such that:*
(a) $\kappa|F \cup V_1 : F \cup V_1 \to N$ is definable.
(b) $\kappa = \alpha$ on $M -$ Int V_2
(c) κ is G homotopic to α relative to $M -$ Int V_2

Proof. Take a non-negative definable C^∞ function $f : M \to \mathbb{R}$ such that $f = 0$ on V_1 and $f = 1$ on $M - \text{Int } V_2$. Notice that if $\mathcal{M}$ is polynomially bounded, then such an f does not necessarily exist. Since G is a finite group and by Proposition 2.4, we may assume that f is G invariant.

We approximate α by a polynomial G map β on V_2 using the polynomial approximation theorem and Proposition 2.4. By Theorem 2.3, one can find a definable $C^\infty G$ tubular neighborhood (U, p) of N in Ξ. If the approximation is sufficiently close, then one can define $\kappa : M \to N, \kappa(x) = p(f(x)\alpha(x) + (1 - f(x))\beta(x))$. Then κ is a $C^\infty G$ map, and κ satisfies Properties (a) and (b). If this approximation is sufficiently close, then $\kappa \in \mathfrak{N}$ because κ and α coincide with outside of a compact set V_2.

The map $H : M \times [0, 1] \to N$ defined by $H(x, t) = p((1 - t)\alpha(x) + t\kappa(x))$ gives a G homotopy relative to $M - \text{Int } V_2$ from α to κ. □

Proposition 4.3. *Let Ω and Ξ be representations of G. Let $M \subset \Omega, N \subset \Xi$ be affine locally definable $C^\infty G$ manifolds and A a closed G invariant locally definable subset of M. Suppose that $f : M \to N$ is a $C^\infty G$ diffeomorphism such that $f|A : A \to N$ is locally definable, and that $x \in M$. Suppose further that $j : \Omega' \to S$ is a linear definable C^∞ slice*

at x in Ω. If $GD_{10} \cap M$ is compact, then there exists a $C^\infty G$ diffeomorphism $h : M \to N$ such that:
(1) $h|A \cup (GD \cap M) : A \cup (GD \cap M) \to N$ is locally definable.
(2) $h = f$ on $M - GD_2^\circ \cap M$.
(3) h is G homotopic to f relative to $M - GD_2^\circ \cap M$.

The condition that $GD_{10} \cap M$ is compact is not essential. By Theorem 2.7, one can find a linear definable C^r slice S at $x \in M$ in Ω. Since S is a linear definable C^∞ slice in Ω, there exists a definable $C^\infty K$ diffeomorphism j from some representation Ω' of G_x onto S such that $j(0) = x$, GS is open in Ω, and that

$$\mu(j) : G \times_{G_x} \Omega' \to GS \ (\subset \Xi),$$
$$\mu(j)([g, x]) = gj(x)$$

is a definable $C^\infty G$ diffeomorphism. Notice that M is locally compact. Thus replacing smaller S, if necessary, $GD_{10} \cap M$ is compact because M is locally compact.

Proof of Proposition 4.3. Since $GD_{10} \cap M$ is compact and A is closed in M, $A \cap GD_{10}$ $(= A \cap (GD_{10} \cap M))$ is a compact G invariant locally definable subset of $GS \cap M$. Thus $A \cap GD_{10}$ is a G invariant definable subset of Ω. Hence

$$E := \mu(j)^{-1}(A \cap GD_{10})$$

is a G invariant definable subset of $G \times_{G_x} \Omega'$. Let $L = j^{-1}(D_{10}^\circ \cap M)$. The map

$$\alpha := f \circ \mu(j)|G \times_{G_x} L : G \times_{G_x} L \to \Xi$$

is a $C^\infty G$ diffeomorphism onto an open G invariant subset $V := f(GD_{10}^\circ \cap M)$ of N. Since $A \cap GD_{10}$ is compact and $f|A$ is locally definable, $f|(A \cap GD_{10}) : A \cap GD_{10} \to f(A \cap GD_{10}) \subset N \subset \Xi$ is definable. The map $\alpha|(E \cap (G \times_{G_x} L)) : E \cap (G \times_{G_x} L) \to \Xi$ is definable because $\mu(j)$ and $f|(A \cap GD_{10}) : A \cap GD_{10} \to \Xi$ are definable. Since V is contained in a G invariant compact set $f(GD_{10} \cap M)$, and since N is a locally definable $C^\infty G$ submanifold of Ξ, there exists a G invariant definable set W of Ξ such that $V \subset W \subset N$ and that W is open in N. Notice that W is an affine definable $C^\infty G$ manifold. Since $G \times_{G_x} L$ is contained in a G invariant compact subset of $G \times_{G_x} j^{-1}(D_{20} \cap M)$, $G \times_{G_x} L$ is an affine definable $C^\infty G$ manifold. Applying Lemma 4.2 to $\alpha : G \times_{G_x} L \to W$, there exists a $C^\infty G$ map $\beta : G \times_{G_x} L \to W$ as a C^∞ Whitney approximation of α such that:
(a) $\beta|(G \times_{G_x} (j^{-1}(A \cap D_{10}^\circ) \cup (B \cap L))) : G \times_{G_x} (j^{-1}(A \cap D_{10}^\circ) \cup (B \cap L)) \to W$ $(\subset N)$ is definable.
(b) $\beta = \alpha$ on $G \times_{G_x} (L - B_2^\circ \cap L)$.
(c) β is G homotopic to α relative to $G \times_{G_x} (L - B_2^\circ \cap L)$.
Then the map $h : M \to N$ defined by

$$h(x) = \begin{cases} \beta \circ \mu(j)^{-1}(x), & x \in GD_5 \cap M \\ f(x), & x \in M - M \cap GD_5^\circ \end{cases}$$

is well-defined, and it is a $C^\infty G$ diffeomorphism if our approximation is sufficiently close. Since $h|(A \cap GD_5)$ and $h|(GD \cap M)$ are definable, and since $h|(A \cap (M - GD_5 \cap M))$ $(= f|(A \cap (M - GD_5 \cap M)))$ is locally definable, $h|A \cup (GD \cap M)$ is locally definable by Proposition 2.1. By the construction of h, h satisfies Properties (2) and (3). $\square$

Proof of Theorem 4.1. Using Proposition 4.3, a similar proof of 4.1 [11] proves Theorem 4.1. □

5. Proof of Theorem 1.3 and 1.4

Proof of Theorem 1.3. By Whitney's imbedding Theorem (e.g. 2.14 [2]), there exists a C^∞ imbedding $f : X \to \mathbb{R}^{2n+1}$. By Proposition 3.2 and since C^∞ imbeddings from X to $\mathbb{R}^{2n+1}$ are open in the set $C^\infty(X, \mathbb{R}^{2n+1})$ of C^∞ maps from X to $\mathbb{R}^{2n+1}$, we have the required a locally definable C^∞ imbedding $h : X \to \mathbb{R}^{2n+1}$. □

Proof of Theorem 1.4. Let $G = \{g_1, \dots, g_m\}$ and X a locally definable $C^\infty G$ manifold of dimension n. By Theorem 1.3, there exists a locally definable C^∞ imbedding $f : X \to \mathbb{R}^{2n+1}$. Let Ω be the representation of G whose underlying space is $\mathbb{R}^{(2n+1)m} = \mathbb{R}^{2n+1} \times \cdots \times \mathbb{R}^{2n+1}$ and its action is defined by the permutation of coordinates $(x_1, \dots, x_m) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(m)})$ induced from $(gg_1, \dots, gg_m) = (g_{\sigma(1)}, \dots, g_{\sigma(m)})$. Then $F : X \to \Omega, F(x) = (f(g_1 x), \dots, f(g_m x))$ is the required locally definable $C^\infty G$ imbedding. □

References

[1] E Baro and M. Otero, *Locally definable homotopy*, preprint.
[2] M.W. Hirsch *Differential topology*, Springer-Verlag (1976).
[3] S. Illman, *Every proper smooth actions of a Lie group is equivalent to a real analytic action: a contribution to Hilbert's fifth problem*, Ann. of Math. Stud. **138** (1995), 189-220.
[4] S. Illman, *Smoothly equivalent real analytic proper G manifolds are subanalytically equivalent*, Math. Ann. **306** (1996), 647-673.
[5] T. Kawakami, *Affineness of definable C^r manifolds and its applications*, Bull. Korean Math. Soc. **40**, (2003), 149-157.
[6] T. Kawakami, *Equivariant definable C^r approximation theorem, definable C^rG triviality of definable C^rG maps and Nash G compactifications*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. No. 55, (2005), 1–14.
[7] T. Kawakami, *Equivariant differential topology in an o-minimal expansion of the field of real numbers*, Topology Appl. **123** (2002), 323-349.
[8] T. Kawakami, *Every definable C^r manifold is affine*, Bull. Korean Math. Soc. **42** (2005), 165–167.
[9] T. Kawakami, *Imbedding of manifolds defined on an o-minimal structures on* $(\mathbb{R}, +, \cdot, <)$, Bull. Korean Math. Soc. **36** (1999), 183-201.
[10] T. Kawakami, *Nash G manifold structures of compact or compactifiable $C^\infty G$ manifolds*, J. Math. Soc. Japan **48** (1996), 321-331.
[11] T. Kawakami, *Locally definable C^sG maifold structures of locally C^rG manifolds*, Bull. Fac. Ed. Wakayama Univ. Natur. Sci. **56** (2006), 1–12.
[12] T. Kawakami, *Relative Definable C^rG triviality of G invariant proper definable C^r functions*, preprint.
[13] F. Kutzschebauch *On the uniqueness of the real analytic G-structure on a smooth proper G-manifold for G a Lie group*, Manuscripta Math. **90** (1996), 17-22.
[14] M. Shiota, *Abstract Nash manifolds*, Proc. Amer. Math. Soc. **96** (1986), 155-162.
[15] M. Shiota, *Approximation theorems for Nash mappings and Nash manifolds*, Trans. Amer. Math. Soc. **293** (1986), 319-337.

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