STRONGLY NIP ALMOST REAL CLOSED FIELDS

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Abstract. The following conjecture is due to Shelah–Hasson: Any infinite strongly NIP field is either real closed, algebraically closed, or admits a non-trivial definable henselian valuation, in the language of rings. We specialise this conjecture to ordered fields in the language of ordered rings, which leads towards a systematic study of the class of strongly NIP almost real closed fields. As a result, we obtain a complete characterisation of this class.

Contents

1. Introduction

Let $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$ be the language of rings, $\mathcal{L}_{\mathrm{or}} = \mathcal{L}_{\mathrm{r}} \cup \{<\}$ the language of ordered rings and $\mathcal{L}_{\mathrm{og}} = \{+, 0, <\}$ the language of ordered groups. Throughout this work, we will abbreviate the $\mathcal{L}_{\mathrm{r}}$-structure of a field $(K, +, -, \cdot, 0, 1)$ simply by K, the $\mathcal{L}_{\mathrm{or}}$-structure of an ordered field $(K, +, -, \cdot, 0, 1, <)$ by $(K, <)$ and the $\mathcal{L}_{\mathrm{og}}$-structure of an ordered group $(G, +, 0, <)$ by G. Conjecture 1.1 below was suggested by Shelah in [19] and reformulated as follows in [4]. For the notions 'strongly NIP', 'dp-minimal' and other model theoretic terminology below, see Section 2.

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Conjecture 1.1. *Let K be an infinite strongly NIP field. Then K is either real closed, or algebraically closed, or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable*[1] *henselian valuation.*

In [8, Conjecture 1.3] the authors obtain a conjectural classification of strongly NIP fields in the language $\mathcal{L}_{\mathrm{r}}$ which is equivalent to Conjecture 1.1. In this paper, we study Conjecture 1.1 for ordered fields in the language $\mathcal{L}_{\mathrm{or}}$. Note that in ordered fields, henselian valuations are always convex:

Fact 1.2. [13, Lemma 2.1] *Let $(K, <)$ be an ordered field and v a henselian valuation on K. Then v is convex on $(K, <)$.*

We therefore specialise and enhance Conjecture 1.1 to ordered fields as follows.

Conjecture 1.3. *Let $(K, <)$ be a strongly NIP ordered field. Then K is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation.*

Note that there are ordered fields which admit a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation but are not NIP (see Remark 5.1 (2)). We can reformulate Conjecture 1.3 in terms of the model theoretically well-studied class of almost real closed fields (see [1] and also Definition 3.1).

Conjecture 1.4. *Any strongly NIP ordered field is almost real closed.*

In Section 2 we gather some general basic preliminaries. In Section 3 we briefly introduce the class of almost real closed fields and prove analogues in the language of ordered rings to known model theoretic results on almost real closed fields in the language of rings. Our investigation of strongly NIP ordered fields starts in Section 4. We first focus on dp-minimal ordered fields (which are, in particular, strongly NIP). By a careful analysis of the results of [12], we deduce in Proposition 4.4 that an ordered field is dp-minimal if and only if it is almost real closed with respect to some dp-minimal ordered abelian group G. Thereafter, we address the following query **(classification of strongly NIP ordered fields)**:

An ordered field is strongly NIP if and only if it is almost real closed with respect to some strongly NIP ordered abelian group G.

In Theorem 4.12 we show that an almost real closed field with respect some ordered abelian group G is strongly NIP if and only if G is strongly NIP. This settles the backward direction of the above query, and reduces its forward direction to Conjecture 1.4. In Section 5, we show that Conjecture 1.3 and Conjecture 1.4 are equivalent (see Theorem 5.4). We conclude in Section 6 by stating some open questions motivated by this work.[2]

[1] Throughout this work *definable* always means *definable with parameters*.

[2] A preliminary version of this work is contained in our arXiv preprint [14], which contains also a systematic study of $\mathcal{L}_{\mathrm{or}}$-definable henselian valuations in ordered fields as well as of the class of ordered fields which are dense in their real closure. This systematic study, of independent interest, will be the subject of a separate publication.

2. General Preliminaries

All notions on strongly NIP theories can be found in [20] and all notions on valued fields in [15, 5]. The set of natural numbers with 0 will be denoted by $\mathbb{N}_0$, the set of natural numbers without 0 by $\mathbb{N}$.

Let $\mathcal{L}$ be a language and T an $\mathcal{L}$-theory. We fix a monster model $\mathcal{M}$ of T. Let $\varphi(\underline{x};\underline{y})$ be an $\mathcal{L}$-formula. We say that φ **has the independence property (IP)** if there are $(\underline{a}_i)_{i\in\omega}$ and $(\underline{b}_J)_{J\subseteq\omega}$ in M such that $\mathcal{M} \models \varphi(\underline{a}_i;\underline{b}_J)$ if and only if $i \in J$. We say that the theory T **has IP** if there is some formula φ which has IP. If T does not have IP, it is called **NIP** (*not the independence property*). For an $\mathcal{L}$-structure $\mathcal{N}$, we also say that $\mathcal{N}$ **is NIP** if its complete theory $\mathrm{Th}(\mathcal{N})$ is NIP. A well-known example of an IP theory is the complete theory of the $\mathcal{L}_{\mathrm{r}}$-structure $(\mathbb{Z}, +, -, \cdot, 0, 1)$ (cf. [20, Example 2.4]). Since $\mathbb{Z}$ is parameter-free definable in the $\mathcal{L}_{\mathrm{r}}$-structure $\mathbb{Q}$ (cf. [17, Theorem 3.1]), also the complete $\mathcal{L}_{\mathrm{r}}$-theory of $\mathbb{Q}$ has IP.

Let $A \subseteq M$ be a set of parameters, Δ a set of $\mathcal{L}$-formulas and $(J, <)$ a linearly ordered set. A sequence $S = (a_j \mid j \in J)$ in M is **Δ-indiscernible over** A if for every $k \in \mathbb{N}$, any increasing tuples $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$ in J, any formula $\varphi(x_1, \ldots, x_k; \underline{y}) \in \Delta$ and any tuple $\underline{b} \in A$, we have $\mathcal{M} \models \varphi(a_{i_1}, \ldots, a_{i_k}; \underline{b}) \leftrightarrow \varphi(a_{j_1}, \ldots, a_{j_k}; \underline{b})$. The sequence S is called **indiscernible over** A if it is Δ-indiscernible over A for any set of $\mathcal{L}$-formulas Δ. A family of sequences $(S_t \mid t \in X)$ is called **mutually indiscernible over** A if for each $u \in X$, the sequence S_u is indiscernible over $A \cup \bigcup_{t\in X\setminus\{u\}} S_t$.

Let p be a partial n-type over a set $A \subseteq M$. We define the **dp-rank of** p **over** A as follows: Let κ be a cardinal. The dp-rank of p over A is less than κ (in symbols, $\text{dp-rk}(p, A) < \kappa$) if for every family $(S_t \mid t < \kappa)$ of mutually indiscernible sequences over A and any $\underline{b} \in M^n$ realising p in $\mathcal{M}$, there is some $t < \kappa$ such that S_t is indiscernible over $A \cup \{b_1, \ldots, b_n\}$. The theory T is called **strongly NIP** if it is NIP and $\text{dp-rk}(\{x = x\}, \emptyset) < \aleph_0$, where $\{x = x\}$ is the partial type over $\emptyset$ only consisting of the formula $x = x$. The theory T is called **dp-minimal** if it is NIP and $\text{dp-rk}(\{x = x\}, \emptyset) = 1$. Again, we call an $\mathcal{L}$-structure $\mathcal{N}$ strongly NIP (respectively dp-minimal) if $\mathrm{Th}(\mathcal{N})$ is strongly NIP (respectively dp-minimal).

Any reduct of a strongly NIP structure is strongly NIP (cf. [18, Claim 3.14, 3)]) and any reduct of a dp-minimal structure is dp-minimal (cf. [16, Observation 3.7]). Since any weakly o-minimal theory is dp-minimal (cf. [2, Corollary 4.3]), we obtain the following hierarchy (in particular, any divisible ordered abelian group and any real closed field are strongly NIP):

o-minimal $\rightarrow$ weakly o-minimal $\rightarrow$ dp-minimal $\rightarrow$ strongly NIP $\rightarrow$ NIP

Let K be a field and v a valuation on K. We denote the **valuation ring** of v in K by $\mathcal{O}_v$, the **valuation ideal**, i.e. the maximal ideal of $\mathcal{O}_v$, by $\mathcal{M}_v$, the **ordered value group** by vK and the **residue field** $\mathcal{O}_v/\mathcal{M}_v$ by

Kv. For $a \in \mathcal{O}_v$ we also denote $a + \mathcal{M}_v$ by $\overline{a}$. For an ordered field $(K, <)$ a valuation is called **convex** (in $(K, <)$) if the valuation ring $\mathcal{O}_v$ is a convex subset of K. In this case, the relation $\overline{a} < \overline{b} :\Leftrightarrow \overline{a} \neq \overline{b} \wedge a < b$ defines an order relation on Kv making it an ordered field.

Let $\mathcal{L}_{\mathrm{vf}} = \mathcal{L}_{\mathrm{r}} \cup \{\mathcal{O}_v\}$ be the **language of valued fields**, where $\mathcal{O}_v$ stands for a unary predicate. Let $(K, \mathcal{O}_v)$ be a valued field. An atomic formula of the form $v(t_1) \geq v(t_2)$, where t_1 and t_2 are $\mathcal{L}_{\mathrm{r}}$-terms, stands for the $\mathcal{L}_{\mathrm{vf}}$-formula $t_1 = t_2 = 0 \vee (t_2 \neq 0 \wedge \mathcal{O}_v(t_1/t_2))$. Thus, by abuse of notation, we also denote the $\mathcal{L}_{\mathrm{vf}}$-structure $(K, \mathcal{O}_v)$ by (K, v). Similarly, we also call $(K, <, v)$ an ordered valued field. We say that a valuation v is $\mathcal{L}$-definable for some language $\mathcal{L} \in \{\mathcal{L}_{\mathrm{r}}, \mathcal{L}_{\mathrm{or}}\}$ if its valuation ring is an $\mathcal{L}$-definable subset of K.

For any ordered abelian groups G_1 and G_2, we denote the **lexicographic sum** of G_1 and G_2 by $G_1 \oplus G_2$. This is the abelian group $G_1 \times G_2$ with the lexicographic ordering $(a, b) < (c, d)$ if $a < c$, or $a = c$ and $b < d$.

Let K be a field and v, w be valuations on K. We write $v \leq w$ if and only if $\mathcal{O}_v \supseteq \mathcal{O}_w$. In this case we say that w is **finer than** v and v is **coarser than** v. Note that $\leq$ defines an order relation on the set of convex valuations of an ordered field. We call two elements $a, b \in K$ **archimedean equivalent** (in symbols $a \sim b$) if there is some $n \in \mathbb{N}$ such that $|a| < n|b|$ and $|b| < n|a|$. Let $G = \{[a] \mid a \in K^\times\}$ the set of archimedean equivalence classes of $K^\times$. Equipped with addition $[a] + [b] = [ab]$ and the ordering $[a] < [b] :\Leftrightarrow a \not\sim b \wedge |b| < |a|$, the set G becomes an ordered abelian group. Then $v : K^\times \to G$ defines a convex valuation on K. This is called the **natural valuation** on K and denoted by v_{nat}.

Let $(k, <)$ be an ordered field and G an ordered abelian group. We denote the **ordered Hahn field** with coefficients in k and exponents in G by $k((G))$. We denote an element $s \in k((G))$ by $s = \sum_{g \in G} s_g t^g$, where $s_g = s(g)$ and t^g is the characteristic function on G mapping g to 1 and everything else to 0. The ordering on $k((G))$ is given by $s > 0 :\Leftrightarrow s(\min \operatorname{supp} s) > 0$, where $\operatorname{supp} s = \{g \in G \mid s(g) \neq 0\}$ is the **support of** s. Let $v_{\min}$ be the valuation on $k((G))$ given by $v_{\min}(s) = \min \operatorname{supp} s$ for $s \neq 0$. Note that $v_{\min}$ is convex and henselian. Note further that if k is archimedean, then $v_{\min}$ coincides with v_{nat}.

We will repeatedly use the Ax–Kochen–Ershov principle for ordered fields (cf. [6, Corollary 4.2(iii)]).

Fact 2.1 (Ax–Kochen–Ershov principle)**.** *Let $(K, <, v)$ and $(L, <, w)$ be two henselian ordered valued fields. Then $(Kv, <) \equiv (Lw, <)$ and $vK \equiv wL$ if and only if $(K, <, v) \equiv (L, <, w)$.*

3. Almost Real Closed Fields

Algebraic and model theoretic properties of the class of almost real closed fields in the language $\mathcal{L}_{\mathrm{r}}$ have been studied in [1], in particular [1, Theorem 4.4] gives a complete characterisation of $\mathcal{L}_{\mathrm{r}}$-definable henselian valuations. In the following, we will prove some useful properties of almost real closed fields in the language $\mathcal{L}_{\mathrm{or}}$.

Definition 3.1. Let $(K, <)$ be an ordered field, G an ordered abelian group and v a henselian valuation on K. We call K an **almost real closed field (with respect to v and G)** if Kv is real closed and $vK = G$.

Depending on the context, we may simply say that $(K, <)$ is an almost real closed field without specifying the henselian valuation v or the ordered abelian group $G = vK$.

Remark 3.2. In [1], almost real closed fields are defined as pure fields which admit a henselian valuation with real closed residue field. However, any such field admits an ordering, which is due to the Baer–Krull Representation Theorem (cf. [5, p. 37 f.]). We consider almost real closed fields as ordered fields with a fixed order.

Due to Fact 1.2 and the following fact, we do not need to make a distinction between convex and henselian valuations in almost real closed fields.

Fact 3.3. [1, Proposition 2.9] *Let $(K, <)$ be an almost real closed field. Then any convex valuation on $(K, <)$ is henselian.*

[1, Proposition 2.8] implies that the class of almost real closed fields in the language $\mathcal{L}_{\mathrm{r}}$ is closed under elementary equivalence. We can easily deduce that this also holds in the language $\mathcal{L}_{\mathrm{or}}$.

Proposition 3.4. *Let $(K, <)$ be an almost real closed field and let $(L, <) \equiv (K, <)$. Then $(L, <)$ is an almost real closed field.*

Proof. Since $L \equiv K$, we obtain by [1, Proposition 2.8] that L admits a henselian valuation v such that Lv is real closed. Hence, $(L, <)$ is almost real closed. $\square$

Corollary 3.5. *Let $(K, <)$ be an ordered field. Then $(K, <)$ is almost real closed if and only if $(K, <) \equiv (\mathbb{R}(\!(G)\!), <)$ for some ordered abelian group G.*

Proof. The forward direction follows from the Ax–Kochen–Ershov Principle. The backward direction is a consequence of Proposition 3.4. $\square$

Corollary 3.6. *Let $(K, <)$ be an almost real closed field and G an ordered abelian group. Then $(K(\!(G)\!), <)$ is almost real closed.*

Proof. Let v be a henselian valuation on K such that Kv is real closed. By the Ax–Kochen–Ershov principle, we have $(K(\!(G)\!), <, v_{\min}) \equiv (\mathbb{R}(\!(vK)\!)(\!(G)\!), <, v_{\min})$. Now $(\mathbb{R}(\!(vK)\!)(\!(G)\!), <) \cong (\mathbb{R}(\!(G \oplus vK)\!), <)$, which is an almost real closed field. By Corollary 3.5, $(K(\!(G)\!), <)$ is almost real closed. $\square$

4. Strongly NIP Ordered Fields

In this section we will study the class of strongly NIP ordered fields in the light of Conjecture 1.3 and Conjecture 1.4. A special class of strongly NIP ordered fields are dp-minimal ordered fields. These are fully classified in [12]. In Proposition 4.4 below we show that our query (see p. 2) holds for dp-minimal ordered fields. An ordered group G is called **non-singular** if G/pG is finite for all prime numbers p.

Fact 4.1. [12, Proposition 5.1] *An $\aleph_1$-saturated ordered abelian group G is dp-minimal if and only if it is non-singular.*

Fact 4.2. [12, Theorem 6.2] *An ordered field $(K,<)$ is dp-minimal if and only if there exists a non-singular ordered abelian group G such that $(K,<) \equiv (\mathbb{R}((G)),<)$.*

Lemma 4.3. *Let $(K,<)$ be a dp-minimal almost real closed field with respect to some henselian valuation v. Then vK is dp-minimal.*

Proof. Since Kv is an ordered field, it is not separably closed. Thus, by [10, Theorem A], v is definable in the Shelah expansion $(K,<)^{\mathrm{Sh}}$ (cf. [10, Section 2]) of $(K,<)$. By [16, Observation 3.8], also $(K,<)^{\mathrm{Sh}}$ is dp-minimal, whence the reduct (K,v) is dp-minimal. Since Kv is real closed, $Kv^{\times}/(Kv^{\times})^n$ is finite for all $n \in \mathbb{N}$. Hence, by [12, Proposition 6.1] also vK is dp-minimal. □

Proposition 4.4. *Let $(K,<)$ be an ordered field. Then $(K,<)$ is dp-minimal if and only if it is almost real closed with respect to a dp-minimal ordered abelian group.*

Proof. Suppose that $(K,<)$ is almost real closed with respect to a dp-minimal ordered abelian group G. By Fact 4.1, an $\aleph_1$-saturated elementary extension G_1 of G is non-singular. By the Ax–Kochen–Ershov Principle, we have $(K,<) \equiv (\mathbb{R}((G_1)),<)$, which is dp-minimal by Fact 4.2. Hence, $(K,<)$ is dp-minimal.

Conversely, suppose that $(K,<)$ is dp-minimal. By Fact 4.2, $(K,<) \equiv (\mathbb{R}((G)),<)$ for some non-singular ordered abelian group G. Since $(\mathbb{R}((G)),<)$ is almost real closed, by Proposition 3.4 also $(K,<)$ is almost real closed with respect to some henselian valuation v. By Lemma 4.3, also vK is dp-minimal, as required. □

As a result, we obtain a characterisation of dp-minimal archimedean ordered fields.

Corollary 4.5. *Let $(K,<)$ be a dp-minimal archimedean field. Then K is real closed.*

Proof. The only archimedean almost real closed fields are the archimedean real closed fields. Thus, by Proposition 4.4, any archimedean dp-minimal ordered field is real closed.

□

We now turn to strongly NIP almost real closed fields. Our aim is to obtain a characterisation of strongly NIP almost real closed fields (see Theorem 4.12). We have seen in Proposition 4.4 that every almost real closed field with respect to a dp-minimal ordered abelian group is dp-minimal. We obtain a similar result for almost real closed fields with respect to a strongly NIP ordered abelian group. The following two facts will be exploited.

Fact 4.6. [8, p. 2] *Let K be a perfect field. Suppose that there exists a henselian valuation v on K such that the following hold:*

(1) v is defectless.

(2) The residue field Kv is either an algebraically closed field of characteristic p or elementarily equivalent to a local field of characteristic 0.

(3) The ordered value group vK is strongly NIP.

(4) If $\mathrm{char}(Kv) = p \neq \mathrm{char}(K)$, then $[-v(p), v(p)] \subseteq pvK$.

Then K is strongly NIP.

Fact 4.7. [7, Theorem 1] *Let G be an ordered abelian group. Then the following are equivalent:*

(1) G is strongly NIP.

(2) G is elementarily equivalent to a lexicographic sum of ordered abelian groups $\bigoplus_{i\in I} G_i$, where for every prime p, we have $|\{i \in I \mid pG_i \neq G_i\}| < \infty$, and for any $i \in I$, we have $|\{p \text{ prime} \mid [G_i : pG_i] = \infty\}| < \infty$.

Lemma 4.8. *Let G be a strongly NIP ordered abelian group. Then the ordered Hahn field $(\mathbb{R}((G)), <)$ is strongly NIP.*

Proof. If $K = \mathbb{R}((G))$ is real closed, then we are done. Otherwise, let v be the natural valuation on K. We will first verify that v satisfies conditions (1)–(4) of Fact 4.6. Condition (4) is trivially satisfied; (2) and (3) hold by assumption. The valuation v is defectless if every finite extension (L, v) over (K, v) is defectless. Since this always holds in the characteristic 0 case, (1) is satisfied.

Now K is ac-valued with angular component map $\mathrm{ac} : K \to \mathbb{R}$ given by $\mathrm{ac}(s) = s(v(s))$ for $s \neq 0$ and $\mathrm{ac}(0) = 0$ (cf. [3, Section 5.4 f.]). Following the argument of [8, p. 2], we obtain that (K, v, ac) is a strongly NIP ac-valued field. Since $\mathbb{R}$ is closed under square roots for positive element, for any $a \in K$ we have $a \geq 0$ if and only if the following holds in K:

$$\exists y\; y^2 = \mathrm{ac}(a).$$

Hence, the order relation $<$ is definable in (K, v, ac). We obtain that $(K, <)$ is strongly NIP. □

Proposition 4.9. *Let $(K, <)$ be an almost real closed field with respect to a strongly NIP ordered abelian group and let G be strongly NIP ordered abelian group. Then $(K((G)), <)$ is a strongly NIP ordered field.*

Proof. Let H be a strongly NIP ordered abelian group such that $(K, <)$ is almost real closed with respect to H. As in the proof of Corollary 3.6

we have that $(K((G)), <) \equiv (\mathbb{R}((G \oplus H)), <)$. Since G and H are strongly NIP, also $G \oplus H$ is strongly NIP by Fact 4.7. Hence, by Lemma 4.8, also $(K((G)), <)$ is strongly NIP. □

Corollary 4.10. *Let $(K, <)$ be an almost real closed with respect to a henselian valuation v such that vK is strongly NIP. Then $(K, <)$ is strongly NIP.*

Proof. This follows immediately from Proposition 4.9 by setting $G = \{0\}$ and $H = vK$. □

For the proof of Theorem 4.12, we need one further result on general strongly NIP ordered fields, which will also be used for the proof of Theorem 5.4.

Proposition 4.11. *Let $(K, <)$ be a strongly NIP ordered field and let v be a henselian valuation on K. Then also $(Kv, <)$ and vK are strongly NIP.*

Proof. Arguing as in the proof of Lemma 4.3, we obtain that v is definable in $(K, <)^{\mathrm{Sh}}$. Now $(K, <)^{\mathrm{Sh}}$ is also strongly NIP (cf. [16, Observation 3.8]), whence $(K, <, v)$ is strongly NIP. By [19, Observation 1.4 (2)][3], any structure which is first-order interpretable in $(K, <, v)$ is strongly NIP. Hence, also $(Kv, <)$ and vK are strongly NIP. □

We obtain from Corollary 4.10 and Proposition 4.11 the following characterisation of strongly NIP almost real closed fields.

Theorem 4.12. *Let $(K, <)$ be an almost real closed field with respect to some ordered abelian group G. Then $(K, <)$ is strongly NIP if and only if G is strongly NIP.*

Remark 4.13. Fact 4.7 and Theorem 4.12 give us following complete characterisation of strongly NIP almost real closed fields: An almost real closed field $(K, <)$ is strongly NIP if and only if it is elementarily equivalent to some ordered Hahn field $(\mathbb{R}((G)), <)$ where G is a lexicographic sum as in Fact 4.7 (2).

5. Equivalence of Conjectures

Recall our two main conjectures.

Conjecture 1.3. *Let $(K, <)$ be a strongly NIP ordered field. Then K is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation.*

Conjecture 1.4. *Any strongly NIP ordered field is almost real closed.*

In this section, we will show that Conjecture 1.3 and Conjecture 1.4 are equivalent (see Theorem 5.4).

[3]We thank Yatir Halevi for pointing out this reference to us.

Remark 5.1. (1) An ordered field is real closed if and only if it is o-minimal. Hence, for any real closed field K, if $\mathcal{O} \subseteq K$ is a definable convex ring, its endpoints must lie in $K \cup \{\pm\infty\}$. This implies that any definable convex valuation ring must already contain K, i.e. is trivial. Thus, the two cases in the consequence of Conjecture 1.3 are exclusive.

(2) Recall from Section 2 that the field $\mathbb{Q}$ is not NIP. By [9, Theorem 4], the henselian valuation $v_{\min}$ is $\mathcal{L}_{\mathrm{r}}$-definable in $\mathbb{Q}((\mathbb{Z}))$. Hence, by Proposition 4.11 $(\mathbb{Q}((\mathbb{Z})), <)$ is an example of an ordered field which is not real closed, admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable henselian valuation but is not strongly NIP.

Lemma 5.2 and Lemma 5.3 below are used in the proof of Theorem 5.4. For the first result, we adapt [8, Lemma 1.9] to the context of ordered fields.

Lemma 5.2. *Assume that any strongly NIP ordered field is either real closed or admits a non-trivial henselian valuation*[4]*. Let $(K,<)$ be a strongly NIP ordered field. Then $(K,<)$ is almost real closed with respect to the canonical valuation, i.e. the finest henselian valuation on K.*

Proof. Let $(K,<)$ be a strongly NIP ordered field. If K is real closed, we can take the natural valuation. Otherwise, by assumption, the set of non-trivial henselian valuations on K is non-empty. Let v be the canonical valuation on K. By Proposition 4.11, $(Kv,<)$ is strongly NIP. Note that Kv cannot admit a non-trivial henselian valuation, as otherwise this would induce a non-trivial henselian valuation on K finer than v. Hence, by assumption, Kv must be real closed. □

The next result is obtained from a slight adjustment of the proof of [8, Fact 1.8].

Lemma 5.3. *Let $(K,<)$ be a strongly NIP ordered field which not real closed but is almost real closed with respect to a henselian valuation v. Then there exists a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian coarsening of v.*

Proof. By Proposition 4.11, $vK = G$ is strongly NIP. Since K is not real closed, G is non-divisible. By [7, Proposition 5.5], any henselian valuation with non-divisible value group on a strongly NIP field has a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian coarsening. Hence, there is a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian coarsening u of v. □

Theorem 5.4. *Conjecture* 1.3 *and Conjecture* 1.4 *are equivalent.*

Proof. Assume Conjecture 1.4, and let $(K,<)$ be a strongly NIP ordered field which is not real closed. Then $(K,<)$ admits a non-trivial henselian valuation v. By Lemma 5.3, it also admits an non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation. Now assume Conjecture 1.3. Let $(K,<)$ be strongly NIP ordered field. By Lemma 5.2, K is almost real closed with respect to the canonical valuation v. □

[4]Note that this valuation does not necessarily have to be $\mathcal{L}_{\mathrm{or}}$-definable

As a final observation, we will give two further equivalent formulations of Conjecture 1.4 which follow from results throughout this work.

Observation 5.5. *The following are equivalent:*

(1) Any strongly NIP ordered field $(K,<)$ is almost real closed.

(2) For any strongly NIP ordered field $(K,<)$, the natural valuation v_{nat} on K is henselian.

(3) For any strongly NIP ordered valued field $(K,<,v)$, whenever v is convex, it is already henselian.

Proof. (1) implies (3) by Fact 3.3. Suppose that (3) holds and let $(K,<)$ be strongly NIP. By [10, Proposition 4.2], any convex valuation is definable in the Shelah expansion $(K,<)^{\mathrm{Sh}}$, whence $(K,<,v_{\mathrm{nat}})$ is a strongly NIP ordered valued field. By assumption, v_{nat} is henselian on K, which implies (2). Finally, suppose that (2) holds. Let $(K,<)$ be a strongly NIP ordered field and $(K_1,<)$ an $\aleph_1$-saturated elementary extension of $(K,<)$. Then $K_1 v_{\mathrm{nat}} = \mathbb{R}$. By assumption, v_{nat} is henselian on K_1, whence $(K_1,<)$ is almost real closed. By Proposition 3.4, also $(K,<)$ is almost real closed. □

6. Open Questions

We conclude with open questions connected to results throughout this work. Conjecture 1.4 for archimedean fields states that any strongly NIP archimedean ordered field is real closed, as the only archimedean almost real closed fields are the real closed ones. Corollary 4.5 shows that any dp-minimal archimedean ordered fields is real closed. We can ask whether the same holds for all strongly NIP ordered fields.

Question 6.1. *Let $(K,<)$ be a strongly NIP archimedean ordered field. Is K necessarily real closed?*

It is shown in [14] that any almost real closed field which is not real closed cannot be dense in its real closure. Thus, any dp-minimal ordered field which is dense in its real closure is real closed. Moreover, if Conjecture 1.4 is true, then, in particular, a strongly NIP ordered field which is not real closed cannot be dense in its real closure.

Question 6.2. *Let $(K,<)$ be a strongly NIP ordered field which is dense in its real closure. Is $(K,<)$ real closed?*

Note that Question 6.2 is more general than Question 6.1, as a positive answer to Question 6.2 would automatically tell us that any archimedean ordered field is real closed (since every archimedean field is dense in its real closure).

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