RELATIVE DEFINABLE C^rG TRIVIALITY OF G INVARIANT PROPER DEFINABLE C^r FUNCTIONS

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Abstract. Let G be a compact definable C^r group and $1 \leq r < \infty$. Let X be an affine definable C^rG manifold and $X_1, \dots, X_k$ definable C^rG submanifolds of X such that $X_1, \dots, X_k$ are in general position in X. Suppose that $f : X \to \mathbb{R}$ is a G invariant proper surjective submersive definable C^r function such that for every $1 \leq i_1 < \cdots < i_s \leq k$, $f|X_{i_1} \cap \cdots \cap X_{i_s} : X_{i_1} \cap \cdots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion. We prove that there exists a definable C^rG diffeomorphism $h = (h', f) : (X; X_1, \dots, X_k) \to (X^*; X_1^*, \dots, X_k^*) \times \mathbb{R}$, where Z^*denotes $Z \cap f^{-1}(0)$ for a subset Z of X

Moreover we prove an equivariant definable C^∞ version under some conditions and its application.

1. Introduction[1]

M. Coste and M. Shiota [1] proved that a proper Nash surjective submersion f from an affine Nash manifold X to $\mathbb{R}$ is Nash trivial, namely there exist a point $a \in \mathbb{R}$ and a Nash map $h : X \to f^{-1}(a)$ such that $(h, f) : X \to f^{-1}(a) \times \mathbb{R}$ is a Nash diffeomorphism.

Let $\mathcal{M} = (\mathbb{R}, +\cdot, <, \dots)$ denote an o-minimal expansion of the standard structure $\mathcal{R} = (R, +\cdot, <)$ of the field $\mathbb{R}$ of real numbers. The term "definable" means "definable with parameters in $\mathcal{M}$". General references on o-minimal structures are [2], [5], see also [15]. The Nash category is a special case of the definable C^∞ category and it coincides with the definable C^∞ category based on $\mathcal{R}$ [16]. Further properties and constructions of them are studied in [3], [4], [6], [13] and there are uncountably many o-minimal expansions of $\mathcal{R}$ [14]. Equivariant definable C^r categories are studied in [7], [8], [9], [10]. Everything is considered in $\mathcal{M}$ and each manifold does not have boundary unless otherwise stated.

A map $\psi : M \to N$ between topological spaces is *proper* if for any compact set $C \subset N, \psi^{-1}(C)$ is compact.

Let X be a C^r manifold, $X_1, \dots, X_n$ C^r submanifolds of X and $r \geq 1$. We say that $\{X_i\}_{i=1}^n$ *are in general position in* X if for each $i \in I$ and $J \subset I - \{i\}$, X_i intersects transverse to $\cap_{j \in J} X_j$.

The following is an equivariant relative definable C^r version of [1].

Theorem 1.1. *Let G be a compact definable C^r group and $1 \leq r < \infty$. Let X be an affine definable C^rG manifold and $X_1, \dots, X_k$ definable C^rG submanifolds of X such that $X_1, \dots, X_k$ are in general position in X. Suppose that $f : X \to \mathbb{R}$ is a G invariant proper surjective submersive definable C^r function such that for every $1 \leq i_1 < \cdots < i_s \leq k$, $f|X_{i_1} \cap \cdots \cap X_{i_s} : X_{i_1} \cap \cdots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion. Then there exists*

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a definable C^rG diffeomorphism $h = (h', f) : (X; X_1, \dots, X_k) \to (X^*; X_1^*, \dots, X_k^*) \times \mathbb{R}$, where Z^* denotes $Z \cap f^{-1}(0)$ for a subset Z of X.

Let $X = \{(x, y)|y = 0\} \cup \{(x, y)|xy = 1\} \subset \mathbb{R}^2$ and $f : X \to \mathbb{R}, f(x, y) = x$. Then f is a surjective submersive polynomial map and it is not definably trivial. Thus even in the non-equivariant category, the proper condition in Theorem 1.1 is necessary.

Let $1 \le r < \infty$ and let $F : \mathbb{R} \to (-1, 1)$ be a definable C^r function such that $F(x) = x$ in a definable open neighborhood of 0, $F|(-\infty, -2] = -\frac{1}{2}$ and $F|[2, \infty) = \frac{1}{2}$. Suppose that $X = S^1 \times \mathbb{R} \subset \mathbb{R}^3, f : S^1 \times \mathbb{R} \to \mathbb{R}, f(x, y, t) = t, X_1 = \{(0, 1)\} \times \mathbb{R}$ and $X_2 = \{(x, y, t) \in S^1 \times \mathbb{R}|x = F(t), y = \sqrt{1 - x^2}\}$. Then X_1, X_2 are in general position in X, $f, f|X_1, f|X_2$ are proper surjective submersions and $f|X_1 \cap X_2 : X_1 \cap X_2 \to \mathbb{R}$ is not surjective. Since there exists no definable C^1 diffeomorphism $h : (h', f) : (X; X_1, X_2) \to (X^*; X_1^*, X_2^*) \times \mathbb{R}$, even in the non-equivariant category, the condition that every $f|X_{i_1} \cap \dots \cap X_{i_s} : X_{i_1} \cap \dots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion is necessary.

Let $f : U \to \mathbb{R}$ be a definable C^∞ function on a definable open subset $U \subset \mathbb{R}^n$. We say that f has *controlled derivatives* if there exist a definable continuous function $u : U \to \mathbb{R}$, real numbers $C_1, C_2, \dots$ and natural numbers $E_1, E_2, \dots$ such that $|D^\alpha f(x)| \le C_{|\alpha|} u(x)^{E_{|\alpha|}}$ for all $x \in U$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$, where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}}$ and $|\alpha| = \alpha_1 + \dots + \alpha_n$. We say that $\mathcal{M}$ has *piecewise controlled derivatives* if for every definable C^∞ function $f : U \to \mathbb{R}$ defined in a definable open subset U of $\mathbb{R}^n$, there exist definable open sets $U_1, \dots, U_l \subset U$ such that $\dim(U - \cup_{i=1}^l U_i) < n$ and each $f|U_i$ has controlled derivatives.

The following is an equivariant definable C^∞ version of Theorem 1.1.

Theorem 1.2. *Suppose that $\mathcal{M}$ is exponential, admits the C^∞ cell decomposition and has piecewise controlled derivatives. Let G be a compact definable C^∞ group, X an affine definable $C^\infty G$ manifold and $X_1, \dots, X_k$ definable $C^\infty G$ submanifolds of X such that $X_1, \dots, X_k$ are in general position in X. Suppose that $f : X \to \mathbb{R}$ is a G invariant proper surjective submersive definable C^∞ function such that for every $1 \le i_1 < \dots < i_s \le k$, $f|X_{i_1} \cap \dots \cap X_{i_s} : X_{i_1} \cap \dots \cap X_{i_s} \to \mathbb{R}$ is a proper surjective submersion. Then there exists a definable $C^\infty G$ diffeomorphism $h = (h', f) : (X; X_1, \dots, X_k) \to (X^*; X_1^*, \dots, X_k^*) \times \mathbb{R}$, where Z^* denotes $Z \cap f^{-1}(0)$ for a subset Z of X.*

Let G be a compact definable C^∞ group, X a noncompact definable $C^\infty G$ manifold and $X_1, \dots, X_k$ noncompact definable $C^\infty G$ submanifolds of X in general position in X. If $\mathcal{M}$ is exponential, admits the C^∞ cell decomposition and has piecewise controlled derivatives and X is affine, then by Proposition 3.2, we may assume that X is a bounded definable $C^\infty G$ submanifold of some representation Ω of G. We say that $(X; X_1, \dots, X_k)$ satisfies *the frontier condition* if each $\overline{X_i} - X_i$ is contained in $\overline{X} - X$, where $\overline{X_i}$ (resp. $\overline{X}$) denotes the closure of X_i (resp. X) in Ω. We say that $(X; X_1, \dots, X_k)$ is *simultaneously definably $C^\infty G$ compactifiable* if there exist a compact definable $C^\infty G$ manifold Y with boundary ∂Y, compact definable $C^\infty G$ submanifolds $Y_1, \dots, Y_k$ of Y with boundary $\partial Y_1, \dots, \partial Y_n$, respectively, and a definable $C^\infty G$ diffeomorphism $f : X \to Int\ Y$ such that for any i, $f(X_i) = Int\ Y_i$, each ∂Y_i is contained in ∂Y, and $Y_1, \dots, Y_k$ and ∂Y are in general position in Y. Here $Int\ Y$ (resp. $Int\ Y_i$) denotes the interior of Y (resp. Y_i).

As an application of Theorem 1.2, we have the following theorem.

Theorem 1.3. *Suppose that $\mathcal{M}$ is exponential, admits the C^∞ cell decomposition and has piecewise controlled derivatives. Let G be a compact definable C^∞ group, X a noncompact affine definable $C^\infty G$ manifold and $X_1, \dots, X_k$ noncompact definable $C^\infty G$ submanifolds of X in general position in X such that $(X; X_1, \dots, X_k)$ satisfies the frontier condition. Then $(X; X_1, \dots, X_k)$ is simultaneously definably $C^\infty G$ compactifiable.*

Theorem 1.3 is an equivariant relative definable version of [1] and an equivariant definable C^r version is proved in [11] when r is a natural number.

2. Proof of Theorem 1.1

Let r be a non-negative integer, ∞ or ω. A definable C^r manifold G is a *definable C^r group* if the group operations $G \times G \to G$ and $G \to G$ are definable C^r maps.

Let G be a definable C^r group. A *representation map* of G is a group homomorphism from G to some $O_n(\mathbb{R})$ which is a definable C^r map. A *representation* means the representation space of a representation map of G. In this paper, we assume that every representation of G is orthogonal. A *definable C^rG submanifold* of a representation Ω of G is a G invariant definable C^r submanifold of Ω. A *definable C^rG manifold* is a pair (X, ϕ) consisting of a definable C^r manifold X and a group action $\phi : G \times X \to X$ which is a definable C^r map. We simply write X instead of (X, ϕ). A definable C^rG manifold is *affine* if it is definably C^rG diffeomorphic to (definably G homeomorphic to if $r = 0$) a definable C^rG submanifold of some representation of G. Definable C^rG manifolds and affine definable C^rG manifolds are introduced in [9].

Let G be a definable C^r group, and X definable C^rG manifold and Y a definable C^r manifold. A G invariant definable C^r map $f : X \to Y$ is *definably C^rG trivial* if there exist a point $y \in Y$ and a definable C^rG map $h : X \to f^{-1}(y)$ such that $H = (h, f) : X \to f^{-1}(y) \times Y$ is a definable C^rG diffeomorphism.

The following is piecewise definable C^rG triviality of G invariant surjective submersive definable C^r maps [9].

Theorem 2.1 (1.1 [9])**.** *Let r be a natural number. Let G be a compact definable C^r group, X an affine definable C^rG manifold and Y a definable C^r manifold. Suppose that $f : X \to Y$ is a G invariant surjective submersive definable C^r map. Then there exists a finite decomposition $\{T_i\}$ of Y into definable C^r submanifolds of Y such that each $f|f^{-1}(T_i) : f^{-1}(T_i) \to T_i$ is definably C^rG trivial. If $\mathcal{M}$ admits the C^∞ (resp. C^ω) cell decomposition, the we can take $r = \infty$ (resp. ω).*

The following is existence of a definable C^rG tubular neighborhood of a definable C^rG submanifold of a representation of G.

Theorem 2.2 ([10], [8])**.** *Let r be a non-negative integer, ∞ or ω. Then every definable C^rG submanifold X of a representation Ω of G has a definable C^rG tubular neighborhood (U, θ_X) of X in Ω, namely U is a G invariant definable open neighborhood of X in Ω and $\theta_X : U \to X$ is a definable C^rG map with $\theta_X|X = id_X$.*

Proposition 2.3 (P4 [11])**.** *Let r be a natural number. Let Y, Z be affine definable C^rG manifolds, $Y_1, \dots, Y_k$ (resp. $Z_1, \dots, Z_k$) definable C^rG submanifolds of Y (resp. Z) in general position in Y (resp. Z). Suppose that $F : (\cup_{i=1}^k Y_i; Y_1, \dots, Y_k) \to (\cup_{i=1}^k Z_i; Z_1, \dots, Z_k)$ is a definable continuous G map. If each $F|Y_i$ is a definable C^rG map $(Y_i; Y_i \cap$*

$Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_k) \to (Z_i; Z_i \cap Z_1, \dots, Z_i \cap Z_{i-1}, Z_i \cap Z_{i+1}, \dots, Z_i \cap Z_k)$*, then there exist a* G *invariant definable open neighborhood* W *of* $\cup_{i=1}^n Y_i$ *in* Y *and a definable* C^rG *map* $H : (W; Y_1, \dots, Y_k) \to (Z; Z_1, \dots, Z_k)$ *such that* $H| \cup_{i=1}^k Y_i = F$*.*

Let $1 \le r < \infty$ and $Def^r(\mathbb{R}^n)$ denote the set of definable C^r functions on $\mathbb{R}^n$. For each $f \in Def^r(\mathbb{R}^n)$ and for each positive definable continuous function $\epsilon : \mathbb{R}^n \to \mathbb{R}$, the ϵ-neighborhood $N(f;\epsilon)$ of f in $Def^r(\mathbb{R}^n)$ is defined by $\{h \in Def^r(\mathbb{R}^n) || D^\alpha(h-f)| < \epsilon, \forall \alpha \in (\mathbb{N} \cup \{0\})^n, |\alpha| \le r\}$, where $\alpha = (\alpha_1, \dots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n, |\alpha| = \alpha_1 + \dots + \alpha_n$. We call the topology defined by these ϵ-neighborhoods the *definable* C^r *topology*. By taking the relative topology of the definable C^r topology of $\mathbb{R}^n$, we can define the *definable* C^r *topology* of a definable C^r submanifold X of $\mathbb{R}^n$.

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be definable C^r submanifolds. Note that if X is compact, then the definable C^r topology of the set of definable C^r maps from X to Y coincides the C^r Whitney topology of it [15].

Theorem 2.4 ([15])**.** *Let* X *and* Y *be definable* C^s *submanifolds of* $\mathbb{R}^n$ *and* $0 < s < \infty$*. Let* $f : X \to Y$ *be a definable* C^s *map. If* f *is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image), then an approximation of* f *in the definable* C^s *topology is an immersion (resp. a diffeomorphism, a diffeomorphism onto its image). Moreover if* f *is a diffeomorphism, then* $h^{-1} \to f^{-1}$ *as* $h \to f$*.*

Proof of Theorem 1.1. Since X is affine, we may assume that X is a definable C^rG submanifold of a representation Ω of G.

We first prove the case where $k = 0$. Applying Theorem 2.1, we have a partition $-\infty = a_0 < a_1 < a_2 < \dots < a_j < a_{j+1} = \infty$ of $\mathbb{R}$ and definable C^rG diffeomorphisms $w_i : f^{-1}((a_i, a_{i+1})) \to f^{-1}(y_i) \times (a_i, a_{i+1})$ with $f|f^{-1}((a_i, a_{i+1})) = p_i \circ w_i$, $0 \le i \le j$, where p_i denotes the projection $f^{-1}(y_i) \times (a_i, a_{i+1}) \to (a_i, a_{i+1})$ and $y_i \in (a_i, a_{i+1})$.

Now we prove that for each a_i with $1 \le i \le j$, there exist an open interval I_i containing a_i and a definable C^rG map $\pi_i : f^{-1}(I_i) \to f^{-1}(a_i)$ such that $F_i = (\pi_i, f) : f^{-1}(I_i) \to f^{-1}(a_i) \times I_i$ is a definable C^rG diffeomorphism. By Theorem 2.2, we have a definable C^rG tubular neighborhood $(U_i, \theta_{f^{-1}(a_i)})$ of $f^{-1}(a_i)$ in X. Since f is proper, there exists an open interval I_i containing a_i such that $f^{-1}(I_i) \subset U_i$. Note that if f is not proper, then such an open interval does not always exist. Hence shrinking I_i, if necessary, $F_i = (\pi_i, f) : f^{-1}(I_i) \to f^{-1}(a_i) \times I_i$ is the required definable C^rG diffeomorphism.

By the above argument, we have a finite family of $\{J_i\}_{i=1}^l$ of open intervals and definable C^rG diffeomorphisms $\phi_i : f^{-1}(J_i) \to f^{-1}(y_i) \times J_i$, $1 \le i \le l$, such that $y_i \in J_i$, $\cup_{i=1}^l J_i = \mathbb{R}$ and the composition of ϕ_i with the projection $f^{-1}(y_i) \times J_i$ onto J_i is $f|f^{-1}(J_i)$.

Now we glue these trivializations to get a global one. We can suppose that $i \ge 2$, $U_{i-1} \cap J_i = (a, b)$ and $\psi_{i-1} : f^{-1}(U_{i-1}) \to f^{-1}(y_1) \times U_{i-1}$ is a definable C^rG diffeomorphism with $f|f^{-1}(U_{i-1}) = proj_{i-1} \circ \psi_{i-1}$, where $U_{i-1} = \cup_{s=1}^{i-1} J_s$ and $proj_{i-1}$ denotes the projection $f^{-1}(y_1) \times U_{i-1} \to U_{i-1}$. Take $z \in (a, b) = U_{i-1} \cap J_i$. Then since $f^{-1}(y_1) \cong f^{-1}(z) \cong f^{-1}(y_i)$, $f^{-1}(y_1)$ is definably C^rG diffeomorphic to $f^{-1}(y_i)$. Hence we may assume that ϕ'_i is a definable C^rG diffeomorphism from $f^{-1}(J_i)$ to $f^{-1}(y_1) \times J_i$. Then we have a definable C^rG diffeomorphism

$$\psi_{i-1} \circ (\phi'_i)^{-1} : f^{-1}(y_1) \times (a, b) \to f^{-1}(y_1) \times (a, b), (x, t) \mapsto (q(x, t), t).$$

Take a C^r Nash function $u : \mathbb{R} \to \mathbb{R}$ such that $u = \frac{a+b}{2}$ on $(-\infty, \frac{3}{4}a + \frac{1}{4}b]$ and $u = id$ on $[\frac{1}{4}a + \frac{3}{4}b, \infty)$. Let

$$\tau : f^{-1}(y_1) \times (a,b) \to f^{-1}((a,b)), \tau(x,t) = \psi_{i-1}^{-1}(q(x,u(t)),t).$$

Then τ is a definable C^rG diffeomorphism such that $\tau = (\phi'_i)^{-1}$ if $\frac{1}{4}a + \frac{3}{4}b \le t \le b$ and $\tau = \psi_{i-1}^{-1} \circ (P \times id)$ if $a \le t \le \frac{3}{4}a + \frac{1}{4}b$, where $P : f^{-1}(y_1) \to f^{-1}(y_1), P(x) = q(x, \frac{a+b}{2})$. Thus we can define

$$\tilde{\psi}_i : f^{-1}(U_i) \to f^{-1}(y_1) \times U_i,$$

$$\tilde{\psi}_i(x) = \begin{cases} (P \times id)^{-1} \circ \psi_{i-1}(x), & f(x) \le \frac{3}{4}a + \frac{1}{4}b \\ \tau^{-1}(x), & \frac{3}{4}a + \frac{1}{4}b \le f(x) \le b \\ \phi_i(x), & f(x) > b \end{cases}.$$

Then $\tilde{\psi}_i$ is a definable C^rG diffeomorphism. Thus $\tilde{\psi}_l : X \to f^{-1}(y_1) \times \mathbb{R}$ is a definable C^rG diffeomorphism. Therefore we have the required a definable C^rG diffeomorphism $(H,f) : X \to X^* \times \mathbb{R}$.

We now prove the general case by induction on k.

Let $k \ge 1$. By the inductive hypothesis, for any i, there exists a definable C^rG diffeomorphism $h_i = (h'_i, f) : (X_i; X_i \cap X_1, \dots X_i \cap X_{i-1}, X_i \cap X_{i+1}, \dots, X_i \cap X_k) \to (X^*_1; X^*_i \cap X^*_1, \dots X^*_i \cap X^*_{i-1}, X^*_i \cap X^*_{i+1}, \dots, X^*_i \cap X^*_k) \times \mathbb{R}$. In particular $h'_1|X_2 \cap X_1 : (X_2 \cap X_1; X_2 \cap X_1 \cap X_3, \dots, X_2 \cap X_1 \cap X_k) \to (X^*_2 \cap X^*_1; X^*_2 \cap X^*_1 \cap X^*_3, \dots, X^*_2 \cap X^*_1 \cap X^*_k)$ is a definable C^rG map. By Theorem 2.2, we have a G invariant definable open neighborhood W_2 of $X_1 \cap X_2$ in X_2 and a definable C^rG map $\Phi_2 : (W_2; X_2 \cap X_1; X_2 \cap X_1 \cap X_3, \dots, X_2 \cap X_1 \cap X_k) \to (X^*_2; X^*_2 \cap X^*_1; X^*_2 \cap X^*_1 \cap X^*_3, \dots, X^*_2 \cap X^*_1 \cap X^*_k)$ such that $\Phi_2|X_2 \cap X_1 = h'_1|X_2 \cap X_1$. Take a G invariant definable open neighborhood $W'_2 \subset W_2$ of $X_1 \cap X_2$ in X_2 whose closure in X_2 is properly contained in W_1 and a G invariant definable C^r function $a : X_2 \to \mathbb{R}$ such that its support lies in W_2 and $a|W'_2 = 1$. By Theorem 2.2, we have a G invariant definable open neighborhood O of X^*_2 in Ω and a definable C^rG map $\theta_{X^*_2} : O \to X^*_2$ with $\theta|X^*_2 = id_{X^*_2}$.

Define

$$\Psi'_2(x) = \begin{cases} \theta_{X^*_2}((1-a(x))h'_2(x) + a(x)\Phi_2(x)), & x \in W_1 \\ h'_2(x), & x \in X_2 - W_1 \end{cases}.$$

Then $\Psi'_2 : (X_2; X_2 \cap X_1, \dots X_2 \cap X_k) \to (X^*_2; X^*_2 \cap X^*_1, \dots X^*_2 \cap X^*_k)$ is a definable C^rG map which is an approximation of h'_2. Thus h'_1 is extensible to a definable continuous G map $\tilde{\Psi}_2 : X_1 \cup X_2 \to X^*$ such that $\tilde{\Psi}_2|X_1$ and $\tilde{\Psi}_2|X_2$ are definable C^rG maps.

Repeating this process, we have a definable continuous G map $\Phi : (\cup_{i=1}^k X_i; X_1, \dots, X_k) \to (X^*; X^*_1, \dots, X^*_k)$ such that each $\Phi|X_i$ is a definable C^rG map which is an approximation of h'_i.

By Proposition 2.3, we have a G invariant definable open neighborhood U of $\cup_{i=1}^k X_i$ and a definable C^rG map $L : U \to X^*$ extending Φ.

Take a G invariant definable open neighborhood U' of $\cup_{i=1}^k X_i$ in X whose closure in X is properly contained in U and a G invariant definable C^r function $b : X \to \mathbb{R}$ such that its support lies in U and $b|U' = 1$. By Theorem 2.2, we have a G invariant definable open neighborhood V of X^* in Ω and a definable C^rG map $\theta_{X^*} : V \to X^*$ with $\theta_{X^*}|X^* = id_{X^*}$.

Define $h'(x) =$

$$\begin{cases} \theta_{X^*}((1-b(x))H(x)+b(x)L(x)), & x \in U \\ H(x), & x \in X-U \end{cases}.$$

Then $h' : (X; X_1, \dots, X_k) \to (X^*; X_1^*, \dots, X_k^*)$ is a definable $C^r G$ map. Thus $h = (h', f) : (X; X_1, \dots, X_k) \to (X^*; X_1^*, \dots, X_k^*) \times \mathbb{R}$ is a definable $C^r G$ map which is an approximation of (H, f). Therefore by Theorem 2.4, h is the required definable $C^r G$ diffeomorphism. □

3. Proof of Theorem 1.2 and 1.3

From now on we assume that $\mathcal{M}$ is exponential, admits the C^∞ cell decomposition and has piecewise controlled derivatives.

Theorem 3.1 (1.2 [12]). *Every definable closed subset of $\mathbb{R}^n$ is the zero set of a definable C^∞ function on $\mathbb{R}^n$.*

Proposition 3.2. *Let G be a compact definable C^∞ group and X a definable $C^\infty G$ manifold in a representation Ω of G. Then X is definably $C^\infty G$ imbeddable into $\Omega \times \mathbb{R}^2$ such that X is bounded and $\overline{X} - X$ consists of at most one point, where $\overline{X}$ denotes the closure of X.*

Proof. We may assume that X is noncompact. Then $\overline{X} - X$ is a G invariant closed definable subset of Ω. Let $\pi : \Omega \to \Omega/G \subset \mathbb{R}^s$ denote the orbit map. Then $i \circ \pi : \Omega \to \mathbb{R}^s$ is a proper polynomial map (see Section 4 [10]), where $i : \Omega/G \to \mathbb{R}^s$ denotes the inclusion. Hence $i \circ \pi | \overline{X} - X : \overline{X} - X \to \mathbb{R}^s$ is proper because $\overline{X} - X$ is closed in Ω. Thus $i \circ \pi(\overline{X} - X)$ $(=\pi(\overline{X} - X))$ is a definable closed subset of $\mathbb{R}^s$. Applying Theorem 3.1, there exists a definable C^∞ function $f : \mathbb{R}^s \to \mathbb{R}$ with $\pi(\overline{X} - X) = f^{-1}(0)$. Hence $F := f \circ \pi : \Omega \to \mathbb{R}$ is a G invariant definable C^∞ function with $\overline{X} - X = F^{-1}(0)$. Therefore replacing the graph of $1/F$ by X, we may assume that X is closed in $\Omega \times \mathbb{R}$. Applying the stereographic projection $s : \Omega \times \mathbb{R} \to S(\Omega \times \mathbb{R}^2)$, $s(X)$ satisfies our requirements, where $S(\Omega \times \mathbb{R}^2)$ denotes the unit sphere of $\Omega \times \mathbb{R}^2$. □

The proof of Proposition 3.2 proves the following two theorems and proposition.

Theorem 3.3. *Let G be a compact definable C^∞ group and Ω a representation of G. Every G invariant definable closed subset of Ω is the zero set of a G invariant definable C^∞ function on Ω.*

Theorem 3.4. *Let G be a compact definable C^∞ group and X an affine definable $C^\infty G$ manifold. Suppose that A, B are G invariant definable disjoint closed subsets of X. Then there exists a G invariant definable C^∞ function $f : X \to \mathbb{R}$ such that $f|A = 1$ and $f|B = 0$.*

Proposition 3.5. *Let G be a compact definable C^∞ group, X a noncompact affine definable $C^\infty G$ manifold and $X_1, \dots, X_n$ noncompact definable $C^r G$ submanifolds of X in general position in X such that $(X; X_1, \dots, X_n)$ satisfies the frontier condition. Then we may assume that X is a bounded definable $C^\infty G$ submanifold of some representation Ω of G such that $\overline{X_1} - X_1 = \cdots = \overline{X_n} - X_n = \overline{X} - X = \{0\}$, where $\overline{X}$ (resp. $\overline{X_i}$) denotes the closure of X (resp. X_i) in Ω.*

Using Theorem 3.4, a similar proof of P4 [11] proves the following proposition.

Proposition 3.6. *Let Y, Z be affine definable $C^\infty G$ manifolds, $Y_1, \dots, Y_k$ (resp. $Z_1, \dots, Z_k$) definable $C^\infty G$ submanifolds of Y (resp. Z) in general position in Y (resp. Z). Suppose that $F : (\cup_{i=1}^k Y_i; Y_1, \dots, Y_k) \to (\cup_{i=1}^k Z_i; Z_1, \dots, Z_k)$ is a definable continuous G map. If each $F|Y_i$ is a definable $C^\infty G$ map $(Y_i; Y_i \cap Y_1, \dots, Y_i \cap Y_{i-1}, Y_i \cap Y_{i+1}, \dots, Y_i \cap Y_k) \to (Z_i; Z_i \cap Z_1, \dots, Z_i \cap Z_{i-1}, Z_i \cap Z_{i+1}, \dots, Z_i \cap Z_k)$, then there exist a G invariant definable open neighborhood W of $\cup_{i=1}^n Y_i$ in Y and a definable $C^\infty G$ map $H : (W; Y_1, \dots, Y_k) \to (Z; Z_1, \dots, Z_k)$ such that $H| \cup_{i=1}^k Y_i = F$.*

Proof of Theorem 1.2. Using Theorem 3.4 and Proposition 3.6, a similar proof of Theorem 1.1 proves Theorem 1.2. □

Proof of Theorem 1.3. By Proposition 3.5, we may assume that X is a bounded definable $C^\infty G$ submanifold of a representation Ω of G such that $\overline{X_1} - X_1 = \cdots = \overline{X_n} - X_n = \overline{X} - X = \{0\}$.

Let $f : X \to \mathbb{R}, f(x) = ||x||^{-1}$, where $||x||$ denotes the standard norm of x in Ω. Since f is submersive and G invariant and by Theorem 2.1, there exist a sufficiently large positive number α and a definable $C^\infty G$ map $h_1 : f^{-1}((\alpha, \infty)) \to f^{-1}(\alpha)$ such that $h := (h_1, f) : f^{-1}((\alpha, \infty)) \to f^{-1}(\alpha) \times (\alpha, \infty)$ is a definable $C^\infty G$ diffeomorphism.

Let $f_i := f|X_i$. Since $(X; X_1, \dots, X_k)$ satisfies the frontier condition and $X_1, \dots, X_k$ are in general position in X, each $Y_i := f_i^{-1}((\alpha, \infty))$ is a definable $C^\infty G$ submanifold of $Y := f^{-1}((\alpha, \infty))$, $Y_1, \dots, Y_k$ are in general position in Y and for every $1 \le i_1 < \cdots < i_s \le k$, $f|Y_{i_1} \cap \cdots \cap Y_{i_s} : Y_{i_1} \cap \cdots \cap Y_{i_s} \to (\alpha, \infty)$ is a proper surjective submersion. Since (α, ∞) is definably C^∞ diffeomorphic to $\mathbb{R}$, there exists a G invariant surjective submersive definable C^∞ function $F : (Y; Y_1, \dots, Y_k) \to \mathbb{R}$ satisfying the conditions in Theorem 1.2.

Applying Theorem 1.2 to F, there exists a definable $C^\infty G$ diffeomorphism $(f^{-1}((\alpha, \infty)); f_1^{-1}((\alpha, \infty)), \dots, f_k^{-1}((\alpha, \infty))) \to (f^{-1}(\alpha); f_1^{-1}(\alpha), \dots, f_k^{-1}(\alpha)) \times \mathbb{R}$. Thus we have a definable $C^\infty G$ diffeomorphism $H : (f^{-1}((\alpha, \infty)); f_1^{-1}((\alpha, \infty)), \dots, f_k^{-1}((\alpha, \infty))) \to (f^{-1}(\alpha); f_1^{-1}(\alpha), \dots, f_k^{-1}(\alpha)) \times (\alpha, \infty)$. Since α is sufficiently large, $f^{-1}([0, \alpha + 1])$ is a compact definable $C^\infty G$ manifold with boundary $f^{-1}(\alpha + 1)$ and each $f_i^{-1}([0, \alpha + 1])$ is a compact definable $C^\infty G$ submanifold of $f^{-1}([0, \alpha + 1])$ with boundary $f_i^{-1}(\alpha + 1)$. Therefore using H and Theorem 3.4, $(X; X_1, \dots, X_k)$ is definably $C^\infty G$ diffeomorphic to $(f^{-1}([0, \alpha + 1); f_1^{-1}([0, \alpha + 1), \dots, f_k^{-1}([0, \alpha + 1)))$. □

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