# ON LOVELY PAIRS OF GEOMETRIC STRUCTURES

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Abstract. We study the theory of lovely pairs of geometric structures, in particular o-minimal structures. We characterize "linear" theories in terms of properties of the corresponding theory of the lovely pair. For o-minimal theories, we use Peterzil-Starchenko's trichotomy theorem to characterize for sufficiently general points, the local geometry around it in terms of the thorn U rank of its type.

## 1. Introduction

This paper brings together results on dense pairs by van den Dries [9] and lovely pairs of rank one simple theories developed by Vassiliev [24].

In [24] the second author of this paper studies lovely pairs of an SU-rank one simple theory $T$ and, provided $T$ eliminates the quantifier $\exists^\infty$, shows that the theory $T_P$ of lovely pairs of $T$ exists and it is simple. In this paper we start with *geometric theories* , i.e. theories whose models are geometric structures, that is, models where acl satisfies the exchange property and that eliminate the quantifier $\exists^\infty$. We show that the theory of lovely pairs of models of a geometric theory $T$ exists; that is, we note that lovely pairs exist, and prove that any two lovely pairs of models of such a theory $T$ are elementarily equivalent, and that the saturated models of their common theory $T_P$ are again lovely pairs.

In [24], Vassiliev characterizes the geometry associated to a rank one structure in terms of the properties of the corresponding pair. We follow the ideas from [24] and in Section 4 we study the relations between geometric structures with a "linear" geometry and model theoretic properties of the corresponding pair. We prove:

**Theorem 1** Let $T$ be a geometric theory. Then the following are equivalent.

(i) $\mathrm{acl} = \mathrm{acl}_P$ in $T_P$ (on the home sort)

(ii) For some (any) lovely pair $(M, P)$ of models of $T$, the localization of the pregeometry $(M, \mathrm{acl}_L)$ at $P(M)$ is modular.

(iii) For any two sets $A$ and $B$ in a model of $T$ there is $C \mathop{\smile\!\!\!\!\!\mid}_{\emptyset} AB$ such that $A \mathop{\smile\!\!\!\!\!\mid}_{\mathrm{acl}(AC)\cap \mathrm{acl}(BC)} B$.

(iv) for any $a, b, c_1, \ldots, c_n$ in a model of $T$, if $a \in \mathrm{acl}(b, c_1, \ldots, c_n)$, then there is $\vec{u} \mathop{\smile\!\!\!\!\!\mid}_{\emptyset} ab\vec{c}$ such that $a \in \mathrm{acl}(bd\vec{u})$ for some $d \in \mathrm{acl}(\vec{c}\vec{u})$.

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In [9] van den Dries studies dense pairs of o-minimal theories that expand the theory of ordered abelian groups, generalizing the classical work of Robinson on the theory of real closed fields with a predicate for a real dense closed subfield [23]. He shows that the theory of dense pairs is complete and gives a description of definable sets. It is well known that dense o-minimal theories eliminate the quantifier $\exists^\infty$ and that the algebraic closure in models of such a theory satisfy the exchange principle, that is, they are geometric structures. In this paper we show that the theory of lovely pairs of models of o-minimal theories expanding the theory of ordered abelian groups agrees with the corresponding theory of dense pairs. Part of the goals of this paper is to extend the description of definable sets provided in [9] to the larger class of all lovely pairs of dense o-minimal structures (see section 5).

Berenstein, Ealy and Gunaydin showed in [5] that the theory of dense pairs of o-minimal theories that expand the theory of ordered abelian groups is super-rosy of rank $\leq \omega$. The tools used in the proof depended mainly on the description of definable sets given by van den Dries in [9]. Since such a description can be extended to the larger class of lovely pairs of dense o-minimal theories, the proof found in [5] can be adapted to show that the theory of any lovely pairs of a dense o-minimal theory is super-rosy of rank $\leq \omega$. A more general result was proved recently by Boxall [2]; he showed that for any rosy theory of thorn rank one (with elimination of $\exists^\infty$), the corresponding theory of lovely pairs is super-rosy of rank $\leq \omega$.

Finally, following ideas of Buechler and Vassiliev [3, 24], combined with the trichotomy theorem by Peterzil-Starchenko, we study the relation between the rank of a generic type and the local geometry of the underlying o-minimal structure:

**Theorem 2** Let $M$ be an o-minimal structure whose theory extends DLO, let $P(M) \preceq M$ and assume that $(M, P(M))$ is a lovely pair.

(1) If $a \in M$ is trivial, $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) \leq 1$ $(= 1$ iff $a \notin \mathrm{dcl}(\emptyset))$.
(2) If $a \notin P(M)$ is non-trivial, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) \geq 2$.
(3) If $M$ has global addition (i.e. expands the theory of ordered abelian groups) and does not interpret an infinite field, then $(M, P)$ has þ-rank 2.
(4) If $M$ induces the structure of an o-minimal expansion of a real closed field in a neighborhood of $a \notin P(M)$, then $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a)) = \omega$.

This paper is organized as follows. In Section 2 we study the theory $T_P$ of lovely pairs associated to a geometric theory $T$. In section 3 we characterize the definable sets of such pairs. In section 4 we discuss linearity in the context of geometric structures and prove Theorem 1. In section 5 we generalize van den Dries' description of definable sets in dense pairs to the class of lovely pairs of o-minimal structures extending DLO. Finally in section 6 we show Theorem 2.

We assume throughout this paper that the reader is familiar with the basic ideas of rosy theories presented in [19], [1]. We follow the notation from [5], we write capital letters such as $C, D, X, Y$ for definable sets and sometimes we write $C_{\vec{b}}$ to emphasize that $C$ is definable over $\vec{b}$. We may write $\vec{b} \in C_{\vec{y}}$ to mean that $\vec{b}$ is a tuple of the same arity as $\vec{y}$ whose components belong to $C$.

## 2. Lovely pairs of geometric structures

We begin by translating to the setting of geometric structures, the definitions used by Vassiliev in [24]. Let $T$ be a complete theory in a language $\mathcal{L}$ such that for any model $M \models T$, the algebraic closure satisfies the Exchange Property and that eliminates the quantifier $\exists^\infty$ (see [17, Def. 2.1]). We call such a theory *geometric* (see [13]). Examples includes rosy rank one theories that eliminate $\exists^\infty$ such as strongly minimal theories, $SU$-rank one simple theories and dense o-minimal theories; as well as more geometric structures such as the p-adics. We will assume, to simplify the presentation of the results, that $T$ eliminates quantifiers in the language $\mathcal{L}$. Let $P$ be a new unary predicate and let $\mathcal{L}_P = \mathcal{L} \cup \{P\}$. Let $T'$ be the $\mathcal{L}_P$-theory of all structures $(M, P)$, where $M \models T$ and $P(M)$ is an $\mathcal{L}$-algebraically closed subset of $M$. Let $T_{pairs}$ be the theory of elementary $T$-pairs, that is, the theory of structures of the form $(M, P(M))$ where $P(M) \preceq M$ and $M \models T$.

**Notation 2.1.** *Let $(M, P(M)) \models T'$ and let $A \subset M$. We write $P(A)$ for $P(M) \cap A$.*

**Notation 2.2.** *Throughout this section* independence *means* acl*-independent, where* acl *stands for the algebraic closure in the sense of $\mathcal{L}$. We write* $\mathrm{tp}(\vec{a})$ *for the $\mathcal{L}$-type of $a$ and* dcl *for the definable closure in the language $\mathcal{L}$. Similarly we write* $\mathrm{dcl}_P, \mathrm{acl}_P, \mathrm{tp}_P(\vec{a})$ *for the definable closure, the algebraic closure and the type in the language $\mathcal{L}_P$. For $A \subset B$ sets and $q \in S_n(B)$, we say that $q$ is* free *over $A$ or that $q$ is a free extension of $q \restriction_A$ if for any (all) $\vec{c} \models q$, $\vec{c}$ is independent from $B$ over $A$.*

**Definition 2.3.** We say that a structure $(M, P(M))$ is a *lovely pair of models* of $T$ if

(1) $(M, P(M)) \models T'$
(2) (Density/coheir property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in P(M)$ such that $a \models q$.
(3) (Extension property) If $A \subset M$ is algebraically closed and finite dimensional and $q \in S_1(A)$ is non-algebraic, there is $a \in M$, $a \models q$ and $a \notin \mathrm{acl}(A \cup P(M))$.

Lovely pairs of geometric structures had been previously studied, from the perspective of fusions, by Martin Hils [14].

**Lemma 2.4.** *Let $(M, P(M)) \models T'$. Then $(M, P(M))$ is a lovely pair of models of $T$ if and only if:*

(2') *(Generalized density/coheir property) If $A \subset M$ is finite dimensional and $q \in S_n(A)$ is free over $P(A)$, then there is $\vec{a} \in P(M)^n$ such that $\vec{a} \models q$.*
(3') *(Generalized extension property) If $A \subset M$ is finite dimensional and $q \in S_n(A)$, then there is $\vec{a} \in M^n$ realizing $q$ such that* $\mathrm{tp}(\vec{a}/A \cup P(M))$ *is free over $A$.*

*Proof.* We prove (2') and leave (3') to the reader. Let $\vec{b} \models q$, we may write $\vec{b} = (b_1, \ldots, b_k, b_{k+1}, \ldots, b_n)$ and we may assume that $b_1, \ldots, b_k$ are $\mathrm{acl}(A)$-independent and $b_{k+1}, \ldots, b_n \in \mathrm{acl}(A, b_1, \ldots, b_k)$. Since $q$ is free over $P(A)$, we have that $b_{k+1}, \ldots, b_n \in \mathrm{acl}(P(A), b_1, \ldots, b_k)$. Since $(M, P(M))$ is a lovely pair, applying $k$ times the density property we can find $a_1, \ldots, a_k \in P(M)$ such that

$$\mathrm{tp}(a_1, \ldots, a_k/\mathrm{acl}(A)) = \mathrm{tp}(b_1, \ldots, b_k/\mathrm{acl}(A)).$$

Now let $a_{k+1}, \dots, a_n \in M$ be such that $\mathrm{tp}(a_1, \dots, a_n/A) = \mathrm{tp}(b_1, \dots, b_n/A)$. Then $a_{k+1}, \dots, a_n \in \mathrm{acl}(P(A), a_1, \dots, a_k)$ and since $P(M)$ is algebraically closed we get $\vec{a} = (a_1, \dots, a_n) \in P(M)$. $\square$

The previous lemma shows that we could follow the approach from [4] and define, for $\kappa \geq |T|^+$, the class of $\kappa$-lovely pairs, as the pairs satisfying condition (1) together with the the clauses (2') and (3') above replacing the condition $A \subset M$ *is finite dimensional* by $A \subset M$ *of cardinality* $< \kappa$.

Note that if $(M, P(M))$ is a lovely pair, the extension property implies that $M$ is $\aleph_0$-saturated. If $(M, P(M))$ is a $\kappa$-lovely pair, the extension property implies that $M$ is $\kappa$-saturated and that $M \setminus P(M)$ is non-empty. Assume now that $T$ is an o-minimal theory extending DLO and that $(M, P(M))$ is a lovely pair of models of $T$. Let $a, b \in M$ be such that $a < b$; then the partial type $a < x < b$ is non-algebraic and by the density property it is realized in $P(M)$. Thus, the density property implies that $P(M)$ is dense in $M$.

**Lemma 2.5.** *Any lovely pair of models of $T$ is an elementary $T$-pair.*

*Proof.* We apply the Tarski-Vaught test. Let $(M, P(M))$ be a lovely $T$-pair, let $\varphi(x, \vec{y})$ be an $\mathcal{L}$-formula and let $\vec{b} \in P(M)_{\vec{y}}$. Assume that there is $a \in M$ such that $M \models \varphi(a, \vec{b})$. If $a$ is algebraic over $\vec{b}$, since $P(M)$ is algebraically closed, we get $a \in P(M)$. If $a$ is not algebraic over $\vec{b}$, the type $\mathrm{tp}(a/\vec{b})$ is not algebraic and by the density property there is $a' \in P(M)$ such that $a' \models \mathrm{tp}(a/\vec{b})$; in particular, $M \models \varphi(a', \vec{b})$. $\square$

We follow now section 3 of [4]. The existence of $\kappa$-lovely pairs follows from [4, Lemma 3.5]. The proof presented there does not use the Independence Theorem, in fact, it only uses transitivity and the existence of non-forking extensions. Exchanging the word (non-forking) independence for algebraic independence gives a proof in our setting.

**Definition 2.6.** Let $A$ be a subset of a lovely pair $(M, P(M))$ of models of $T$. We say that $A$ is *$P$-independent* if $A$ is independent from $P(M)$ over $P(A)$.

**Lemma 2.7.** *Let $(M, P(M))$ and $(N, P(N))$ be lovely pairs of models of $T$. Let $\vec{a}$, $\vec{b}$ be finite tuples of the same length from $M$, $N$ respectively, which are both $P$-independent. Assume that $\vec{a}$, $\vec{b}$ have the same quantifier free $\mathcal{L}_P$-type. Then $\vec{a}$, $\vec{b}$ have the same $\mathcal{L}_P$-type.*

*Proof.* Let $f$ be a partial $\mathcal{L}_P$-isomorphism sending the tuple $\vec{a}$ to the tuple $\vec{b}$. It suffices to show that for any $\vec{c} \in N^n$, we can find a partial isomorphism $g$ extending $f$ whose domain includes $\vec{c}$. Replacing $\vec{c}$ for a longer tuple if necessary, we may assume that $\vec{a}\vec{c}$ is $P$-independent. Let $\vec{c}_1 = P(c)$ and let $\vec{c}_2$ be the remaining part of $\vec{c}$. Let $p = \mathrm{tp}(\vec{c}_1/\vec{a})$, since $\vec{c}_1 \in P(M)$ and $\vec{a}$ is $P$-independent, we get that $\vec{c}_1$ is independent from $\vec{a}$ over $P(\vec{a})$. Let $p' = f(\mathrm{tp}(\vec{c}_1/\vec{a}))$, which is a type over $\vec{b}$. Since $\vec{c}_1 \downarrow_{P(\vec{a})} \vec{a}$, we get that $p'$ is free over $P(\vec{b})$ and by the generalized density property we can find $\vec{d}_1 \in P(M)$ such that $\vec{d}_1 \models p'$. In particular, $\mathrm{qftp}_P(\vec{c}_1, \vec{a}) = \mathrm{qftp}_P(\vec{d}_1, \vec{b})$. Let $\hat{f}$ be a partial $\mathcal{L}_P$-isomorphism sending the tuple $\vec{c}_1\vec{a}$ to the tuple $\vec{d}_1\vec{b}$. Now let $q = \mathrm{tp}(\vec{c}_2/\vec{a}\vec{c}_1)$ and let $q' = \hat{f}(\mathrm{tp}(\vec{c}_2/\vec{c}_1\vec{a}))$, which is a type over $\vec{d}_1\vec{b}$. By the generalized extension property there is $\vec{d}_2 \models q'$ such that $\vec{d}_2 \downarrow_{\vec{b}\vec{d}_1} P(M)\vec{b}\vec{d}_1$.

**Claim.** $P(\vec{d}_2) = \emptyset$

Otherwise there is $d \in P(\vec{d}_2)$ and thus $d \in \mathrm{acl}(P(\vec{b})\vec{d}_1)$, so $\hat{f}^{-1}(d) \in \mathrm{acl}(P(\vec{a})\vec{c}_1)$ and we get that $P(\vec{c}_2) \neq \emptyset$, a contradiction.

Thus $\mathrm{qftp}_P(\vec{c}_1\vec{c}_2, \vec{a}) = \mathrm{qftp}_P(\vec{d}_1\vec{d}_2, \vec{b})$

□

The previous result has the following consequence:

**Corollary 2.8.** *All lovely pairs of models of $T$ are elementarily equivalent.*

We write $T_P$ for the common complete theory of all lovely pairs of models of $T$.

To axiomatize $T_P$ we follow the ideas of [24, Prop 2.15]. Here we use for the first time that $T$ eliminates $\exists^\infty$. Recall that whenever $T$ eliminates $\exists^\infty$ the expression *the formula $\varphi(x, \vec{b})$ is nonalgebraic* is first order.

**Theorem 2.9.** *Assume $T$ eliminates $\exists^\infty$. Then the theory $T_P$ is axiomatized by:*

(1) $T'$
(2) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \in P))$.
(3) *For all $\mathcal{L}$-formulas $\varphi(x, \vec{y})$, $m \in \omega$, and all $\mathcal{L}$-formulas $\psi(x, z_1, \ldots, z_m, \vec{y})$ such that for some $n \in \omega$ $\forall \vec{z}\forall \vec{y}\exists^{\leq n} x\psi(x, \vec{z}, \vec{y})$ (so $\psi(x, \vec{y}, \vec{z})$ is always algebraic in $x$)*
$\forall \vec{y}(\varphi(x, \vec{y})$ *nonalgebraic* $\implies \exists x(\varphi(x, \vec{y}) \wedge x \notin P) \wedge$
$\forall w_1 \ldots \forall w_m \in P \neg\psi(x, w_1, \ldots, w_m, \vec{y}))$

*Furthermore, if $(M, P(M)) \models T_P$ is $|T|^+$-saturated, then $(M, P(M))$ is a lovely pair.*

The second scheme of axioms corresponds to the density property and the third scheme to the extension property.

*Proof.* Let $T_0$ be the theory axiomatized by the scheme of axioms described above.

**Claim** Any lovely $T$-pair is a model of $T_0$.

Let $(M, P(M))$ be a lovely $T$-pair. Clearly it is a model of $T'$. Now let $\varphi(x, \vec{y})$ be a formula, let $\vec{b} \in M_{\vec{y}}$ and assume that $\varphi(x, \vec{b})$ is non-algebraic. Let $B = \mathrm{acl}(\vec{b})$ and let $p(x)$ be a non algebraic $\mathcal{L}$-type over $B$ extending $\varphi(x, \vec{b})$. Since $(M, P(M))$ is a lovely pair, by the density property $p(x)$ is realized in $P(M)$ and thus the second axiom holds. Now assume that $\psi(x, \vec{z}, \vec{y})$ is a formula such that there is $n$ with the property that for all $\vec{z}$, $\vec{y}$ there are at most $n$ realizations of $\psi(x, \vec{z}, \vec{y})$. Let $\varphi(x, \vec{y})$ be a formula and $\vec{b} \in M_{\vec{y}}$ be such that $\varphi(x, \vec{b})$ is non-algebraic. Let $B = \mathrm{acl}(\vec{b})$ and let $p(x)$ be a non algebraic $\mathcal{L}$-type over $B$ extending $\varphi(x, \vec{b})$. By the extension property there is $c \in M$ realizing $p$ and independent from $P(M)$ over $B$. For $\vec{d} \in P(M)_{\vec{z}}$, $c$ is not algebraic over $\vec{d}\vec{b}$, so $M \models \neg\psi(c, \vec{d}, \vec{b})$ and the third axiom holds.

**Claim** Any $|T|^+$-saturated model of $T_0$ is a lovely pair.

Let $(M, P(M)) \models T_0$ be $|T|^+$-saturated and let $A \subset M$ be algebraically closed and finite dimensional. Let $p(x)$ be a non-algebraic $\mathcal{L}$-type over $A$. First consider the $\mathcal{L}_P$ partial type $p(x) \wedge P(x)$. By the second axiom this partial type is finitely realizable and by $|T|^+$-saturation it is realized in $(M, P(M))$. Thus $(M, P(M))$ satisfies the density property. Now consider the partial type $p(x) \cup \{\forall \vec{w} \in P \neg\psi(x, \vec{w}, \vec{a}) : \psi$ is as in (3), $\vec{a} \in A_{\vec{y}}\}$. By the third axiom this type is finitely realizable in

$(M, P(M))$ and by $|T|^+$-saturation it is realized in $(M, P(M))$. Thus $(M, P(M))$ satisfies the extension property. □

We now compare lovely pairs with the dense pairs studied by van den Dries in [9]. We start by recalling some definitions from that paper:

Assume that $\mathcal{L} = \{<, 0, 1, +, -, \dots\}$ and that $T$ is an o-minimal $\mathcal{L}$-theory that extends the theory of ordered abelian groups with a positive element 1.

**Definition 2.10.** A *dense pair* is an elementary pair (so $P(M) \preceq M$) such that $P(M) \neq M$ and $P(M)$ is dense in $M$.

Note that such a theory $T$ extends DLO so any lovely $T$-pair $(M, P(M))$ is a dense pair. It is proved in [9, Theorem 2.5] that the common theory of dense pairs is complete, and thus it coincides with $T_P$. Thus, the study of $T_P$ can be seen as a generalization of van den Dries' work on dense pairs of o-minimal structures.

## 3. Definable sets

Fix $T$ a geometric theory and let $(M, P(M)) \models T_P$. Our next goal is to obtain a description of definable subsets of $M$ and $P(M)$ in the language $\mathcal{L}_P$.

We start by considering the $\mathcal{L}_P$-definable subsets of $M$; we follow the ideas from [4, Corollary 3.11]. We will extend the language adding new relation symbols. Let $\mathcal{L}'_P$ be $\mathcal{L}_P$ together with new relation symbols $R_\varphi(\vec{y})$ for each $\mathcal{L}$-formula $\varphi(\vec{x}, \vec{y})$. Let $T'_P$ be the theory $T_P$ together with the sentences $\forall \vec{y}(R_\varphi(\vec{y}) \leftrightarrow \exists \vec{x}(P(\vec{x}) \wedge \varphi(\vec{x}, \vec{y})))$. Since $T_P$ is a complete theory so is $T'_P$. We will show that $T'_P$ has quantifier elimination. We should point out that this result is also proved in [9, Theorem 2.5] for dense pairs of o-minimal structures that extends the theory of ordered abelian groups.

**Lemma 3.1.** *Let $(M, P(M))$, $(N, P(N))$ be lovely pairs. Let $\vec{a}$, $\vec{b}$ be tuples of the same arity from $M$, $N$ respectively. Then the following are equivalent:*

(1) *$\vec{a}$, $\vec{b}$ have the same quantifier-free $\mathcal{L}'_P$-type.*
(2) *$\vec{a}$, $\vec{b}$ have the same $\mathcal{L}_P$-type.*

*Proof.* Clearly (2) implies (1). Assume (1). Since $\mathcal{L}$ has quantifier elimination, $\mathrm{tp}(\vec{a}) = \mathrm{tp}(\vec{b})$. Since the algebraic closure has finite character, there is $A \subset P(M)$ finite such that $\vec{a}$ is independent from $P(M))$ over $A$. Let $q(\vec{z}, \vec{a})$ be the $\mathcal{L}$-type of $A$ over $\vec{a}$. Since the quantifier free $\mathcal{L}'_P$-type of $\vec{a}$ agrees with the quantifier free $\mathcal{L}'_P$-type of $\vec{b}$, $q(\vec{z}, \vec{b})$ is a free extension of $\mathrm{tp}(A)$. Since $(N, P(N))$ is a lovely pair, by the generalized density property $q(\vec{z}, \vec{b})$ is realized in $P(N)$, say by $B$.

**Claim** $\vec{b}$ is free from $P(N)$ over $B$.

Say $\vec{b} = (b_1, \dots, b_n)$ and assume that for some $k \leq n$, $(b_1, \dots, b_k)$ are $B$-independent and $\vec{b} \in \mathrm{acl}(B, b_1, \dots, b_k)$. If the claim does not hold, $\dim(\vec{b}/B \cup P(N)) < k$ say $b_k \in \mathrm{acl}(b_1, \dots, b_{k-1}, B, P(N))$. Let $d_1, \dots, d_m \in P(N)$ such that $b_k \in \mathrm{acl}(b_1, \dots, b_{k-1}, B, d_1, \dots, d_m)$. Since the quantifier free $\mathcal{L}'_P$ type of $\vec{a}, A$ agrees with the quantifier free $\mathcal{L}'_P$ type of $\vec{b}, B$, there are $c_1, \dots, c_m \in P(M)$ such that $a_k \in \mathrm{acl}(a_1, \dots, a_{k-1}, A, d_1, \dots, d_m)$, a contradiction.

Also note that $\vec{a}A$, $\vec{b}B$ have the same quantifier free $\mathcal{L}_P$-type, so the result follows from Lemma 2.7. □

**Corollary 3.2.** *The theory $T'_P$ admits quantifier elimination.*

Now we are interested in the $\mathcal{L}_P$-definable subsets of $P(M)$. For this material we follow the presentation from [9, Theorem 2].

**Lemma 3.3.** *Let $(M_0, P(M_0)) \preceq (M_1, P(M_1))$ and assume that $(M_1, P(M_1))$ is $|M_0|$-saturated. The $M_0$ (seen as a subset of $M_1$) is a $P$-independent set.*

*Proof.* Assume not. Then there are $a_1, \ldots, a_n \in M_0 \setminus P(M_0)$ such that $a_n \in \operatorname{acl}(a_1, \ldots, a_{n-1}, P(M_1))$ and $a_n \notin \operatorname{acl}(a_1, \ldots, a_{n-1}, P(M_0))$. Let $\varphi(x, \vec{y}, \vec{z})$ be a formula and $\vec{b} \in P(M_1)_{\vec{z}}$ be a tuple such that

$$\varphi(a_n, a_1, \ldots, a_{n-1}, \vec{b}) \wedge \exists^{\leq n} x \varphi(x, a_1, \ldots, a_{n-1}, \vec{b})$$

holds. Since $(M_0, P(M_0)) \preceq (M_1, P(M_1))$ there is $\vec{b}' \in P(M_0)_{\vec{y}}$ such that

$$\varphi(a_n, a_1, \ldots, a_{n-1}, \vec{b}') \wedge \exists^{\leq n} x \varphi(x, a_1, \ldots, a_{n-1}, \vec{b}')$$

holds, so $a_n \in \operatorname{acl}(a_1, \ldots, a_{n-1}, P(M_0))$, a contradiction. $\square$

**Proposition 3.4.** *Let $(M, P(M))$ be a lovely pair and let $Y \subset P(M)^n$ be $\mathcal{L}_P$-definable. Then there is $X \subset M^n$ $\mathcal{L}$-definable such that $Y = X \cap P(M)^n$.*

*Proof.* Let $(M_1, P(M_1)) \succeq (M, P(M))$ be $\kappa$-saturated where $\kappa > |M| + |L|$ and let $\vec{a}, \vec{b} \in P(M_1)^n$ such that $\operatorname{tp}(\vec{a}/M) = \operatorname{tp}(\vec{b}/M)$. We will prove that $\operatorname{tp}_P(\vec{a}/M) = \operatorname{tp}_P(\vec{b}/M)$ and the result will follow by compactness. Since $\vec{a}, \vec{b} \in P(M_1)^n$, we get by lemma 3.3 that $M\vec{a}$, $M\vec{b}$ are $P$-independent sets and thus by Lemma 2.7 we get $\operatorname{tp}_P(\vec{a}/M) = \operatorname{tp}_P(\vec{b}/M)$. $\square$

Definable equivalence relations in $T_P$ are studied by Boxall in [2].

When $T$ is an $SU$-rank one theory, the theory $T_P$ also eliminates the quantifier $\exists^\infty$.

**Question 3.5.** *Does $T_P$ eliminate the quantifier $\exists^\infty$?*

We provide a positive answer when $T$ is an o-minimal extension of DLO in Corollary 5.6

## 4. Linearity and the geometric properties of the pair

Our next goal is to investigate the connection between the properties of the theory $T_P$ and the geometry associated to the base theory $T$. Our goal is to generalize (at least partially) the following result from [24] (Theorem 5.13).

**Fact 4.1.** *Let $T$ be a supersimple SU-rank 1 theory (with quantifier elimination). Then the following are equivalent:*

(i) $\operatorname{acl} = \operatorname{acl}_P$ *in $T_P$ (on the home sort)*
(ii) *$T_P$ has SU-rank $\leq 2$ ($= 2$ iff $T$ has non-trivial geometry)*
(iii) *For some (any) lovely pair $(M, P)$ of models of $T$, the localization of the pregeometry $(M, \operatorname{acl}_L)$ at $P(M)$ is modular.*
(iv) *$T$ is linear (meaning the canonical base of any plane curve has SU-rank $\leq 1$)*
(v) *$T_P$ is model complete.*

In the SU-rank 1 case, linearity is in fact equivalent to 1-basedness: for any two sets $A$ and $B$, $A \mathop{\smile\!\!\!\!\!\!|}_{\operatorname{acl}^{eq}(A)\cap \operatorname{acl}^{eq}(B)} B$, or equivalently, for any set $A$ and a tuple $\vec{a}$, $cb(\vec{a}/A) \in \operatorname{acl}^{eq}(\vec{a})$. Condition (ii) and (iv) have no natural analogue for lovely pairs of geometric structures. Even if we assume that $T$ is a þ-rank one theory, there is no notion of canonical base, and thus we cannot expect a direct generalization of the above theorem.

**Remark 4.2.** *If $T$ is a þ-rank 1 theory (eliminating $\exists^\infty$) with almost canonical bases, as defined in* [20] *(for each type $q(x, A)$ over an algebraically closed set $A$, there is the smallest algebraically closed subset of $A$ over which $q$ does not þ-fork), then one can define the 1-basedness and linearity as in the SU-rank 1 case, and the equivalence of conditions (i), (iii), (iv) and (v) in fact 4.1 still holds in this context.*

We will explore the relation between conditions (i), (ii), (iii) and (v) for geometric structures. We also add another two equivalent conditions which could be taken as the new definitions of one-basedness and linearity in the absence of canonical bases. Then we study the special case when $T$ is a rank one rosy theory. Most of the proof is a direct generalization of the proof of Fact 4.1, but we will recall some of the steps if necessary.

**Theorem 4.3.** *Let $T$ be a geometric theory. Then the following are equivalent.*

(i) $\mathrm{acl} = \mathrm{acl}_P$ *in $T_P$ (on the home sort)*

(ii) *For some (any) lovely pair $(M, P)$ of models of $T$, the localization of the pregeometry $(M, \mathrm{acl}_L)$ at $P(M)$ is modular.*

(iii) *For any two sets $A$ and $B$ in a model of $T$ there is $C \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} AB$ such that $A \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(AC)\cap\mathrm{acl}(BC)} B$.*

(iv) *For any $a, b, \vec{c}$ in a model of $T$, if $a \in \mathrm{acl}(b, \vec{c})$, then there is $\vec{u} \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} ab\vec{c}$ such that $a \in \mathrm{acl}(bd\vec{u})$ for some $d \in \mathrm{acl}(\vec{c}\vec{u})$.*

*Proof.* The proof of $(i \to ii)$ and $(ii \to i)$ is the same as the proof of $(i \to iii)$ and $(iii \to i)$ in Fact 4.1.

$(ii \to iii)$ Embed $AB$ into a lovely pair $(M, P)$ so that $AB \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} P(M)$. Take $C = P(M)$.

$(iii \to iv)$ By (iii), there is a set $U$ such that $ab\vec{c} \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} U$ such that

$$ab \mathop{\smile\!\!\!\!\!\!|}_{\mathrm{acl}(abU)\cap acl(\vec{c}U)} \vec{c}.$$

If either $a$ or $b$ is in $\mathrm{acl}(\vec{c}U)$, or $a \in \mathrm{acl}(b)$, then the conclusion of (iv) follows immediately. Suppose neither $a$ nor $b$ is in $\mathrm{acl}(\vec{c}U)$ and $a$ and $b$ are not interalgebraic. Then $ab$ is not independent from $\vec{c}$ over empty set, and thus there is a non-algebraic $d \in \mathrm{acl}(abU) \cap \mathrm{acl}(\vec{c}U)$. Suppose $d \in \mathrm{acl}(bU)$. Then $b \in \mathrm{acl}(dU) \subset \mathrm{acl}(\vec{c}U)$, a contradiction. Thus $d \notin \mathrm{acl}(bU)$, and by exchange, $a \in \mathrm{acl}(bdU)$. Now, $d \in \mathrm{acl}(\vec{c}U)$, and we can assume that $U$ is a finite tuple. This gives us the desired $\vec{u}$.

$(iv \to ii)$ Let $(M, P)$ be any lovely pair of models of $T$. We claim that the quotient pregeometry $(M, \mathrm{acl}(- \cup P(M)))$ is projective, i.e. for any $a, b, c_1, \ldots, c_n \in M$, if $a \in \mathrm{acl}(b\vec{c}P(M))$, then there is $d \in \mathrm{acl}(\vec{c}P(M))$ such that $a \in \mathrm{acl}(bdP(M))$. By enlarging $\vec{c}$ with elements of $P(M)$ if necessary, we may assume that $a \in \mathrm{acl}(b\vec{c})$. Now, let $\vec{u} \in M$ be as in (iv). Since $\vec{u} \mathop{\smile\!\!\!\!\!\!|}_{\emptyset} ab\vec{c}$, we may assume, by the coheir property, that $\vec{u} \in P(M)$, and thus there is $d \in \mathrm{acl}(\vec{c}\vec{u}) \subset \mathrm{acl}(\vec{c}P(M))$ such that $a \in \mathrm{acl}(bd\vec{u}) \subset \mathrm{acl}(bdP(M))$, as needed. Now, for any pregeometry, modularity is equivalent to projectivity, and thus (ii) holds. □

We will refer to a geometric theory satisfying the equivalent conditions above as *linear*. This agrees with the terminology in the simple case, and as we point out later,in the o-minimal case. Note that linearity is weaker than local modularity: we localize at a set of large cardinality to obtain modularity. There are examples of linear SU-rank 1 and o-minimal structures which are not locally modular.

Note that the proof of $(v \to iv)$ in Fact 4.1 (Theorem 5.13 in [24]) actually shows $(v \to iii)$. The proof is still valid in the context of geometric structures, and thus we have:

**Proposition 4.4.** *Let $T$ be a geometric theory (with quantifier elimination). Then if $T_P$ is model complete, then $T$ is linear.*

**Definition 4.5.** Let $(M, P) \models T_P$ and let $A \subset M$. We call $\mathrm{acl}(A \cup P(M))$ the *small closure* of $A$.

Note that the geometry of $M$ is linear if scl is modular. Following the proof in [24], we get the following description of the quotient geometry (i.e. the geometry of the small closure on the home sort) and the geometry of the base theory in the linear case.

**Proposition 4.6.** *Suppose $T$ satisfies the equivalent conditions of theorem 4.3 above, and that $(M, P)$ is a lovely pair of models of $T$. Then*

1. *The associated geometry of $(M, \mathrm{acl}(- \cup P(M))$ is a disjoint union of projective geometries over division rings and/or a trivial geometry.*
2. *The associated geometry of $(M, acl)$ is a disjoint union of "subgeometries" of projective geometries over division rings.*

We now concentrate on rank one rosy theories that eliminate $\exists^\infty$. The first ingredient to understand lovely pairs in this setting is the following result of G. Boxall (generalizing previous work of the second author [24]):

**Fact 4.7.** *(Boxall [2]) Suppose $T$ is a þ-rank 1 theory that eliminates $\exists^\infty$. Then $T_P$ is superrosy of þ-rank $\leq \omega$. Moreover:*

1. *Any definable "large" set in a lovely pair $(M, P)$ (i.e. a set definable over $A$ such that it has a realization in $M \setminus \mathrm{acl}(P(M) \cup A)$) does not þ-divide over $\emptyset$.*
2. *Any infinite definable subset of $P(M)$ does not þ-divide over $\emptyset$. In particular, $P(M)$ has þ-rank 1 in $(M, P)$.*

The following proposition generalizes the direction $(i \to ii)$ in the Fact 4.1.

**Proposition 4.8.** *Let $T$ be a theory of þ-rank one eliminating $\exists^\infty$. If $T$ is linear, then $T_P$ has þ-rank $\leq 2$.*

*Proof.* We follow the proof of $(i \to ii)$ in Fact 4.1. Let $(M, P)$ be a lovely pair and assume that $acl_P = \mathrm{acl}$ in $(M, P)$. Let $A \subset B \subset M$ and $a \in acl(AP(M)) \setminus \mathrm{acl}(B)$. By Fact 4.7(1), it suffices to show that $tp_P(a/B)$ does not þ-fork over $A$. Let $\vec{b} = (b_1, \ldots, b_n) \in P(M)^n$ be a minimal tuple in $P(M)$ such that $a \in \mathrm{acl}(A\vec{b})$. Then $b_1, \ldots, b_{n-1}$ are acl-independent over $Aa$. Since $\mathrm{acl}_P = acl$, we can find $b_1' \ldots b_{n-1}' \models \mathrm{tp}_P(b_1 \ldots b_{n-1}/Aa)$ acl-independent over $Ba$. Take $b_n'$ such that $b_1' \ldots b_n' \models \mathrm{tp}_P(b_1 \ldots b_n/Aa)$. Then $b_n' \in P(M)$ and $a \in \mathrm{acl}(Ab_1' \ldots b_n')$. Note that $b_1', \ldots, b_n'$ are acl-independent over $B$, since otherwise $b_n' \in \mathrm{acl}(b_1' \ldots b_{n-1}'B)$, and thus $a \in \operatorname{acl}(b'_1 \dots b'_{n-1}B)$ as well, contradicting the choice of $b'_1, \dots, b'_{n-1}$ and the fact that $a \notin \operatorname{acl}(B)$.

Thus $a \in \operatorname{acl}(Ab'_1 \dots b'_n)$, where $b'_1, \dots, b'_n \in P(M)$ and are acl-independent (and thus $\operatorname{acl}_P$-independent) over $B$. By Fact 4.7(2) $P(M)$ has þ-rank 1, so $tp_P(b'_1 \dots b'_n/B)$ does not þ-fork over $\emptyset$. Thus $tp_P(a/B)$ does not þ-fork over $A$, as needed. $\square$

**Question 4.9.** *Does the converse of Proposition 4.8 hold?*

The main obstacle for answering the question above is understanding þ-forking in the pair. In particular:

**Question 4.10.** *Let $T$ be a theory of þ-rank one. Let $(M, P)$ be a lovely pair of models of $T$ and assume that there are $A \subset B \subset M$ and $a \in M$ such that $a \in \operatorname{scl}(B) \setminus \operatorname{scl}(A)$. Does $\operatorname{tp}_P(a/B)$ þ-fork over $A$?*

## 5. More on definable sets: the o-minimal case

Fix $T$ an o-minimal theory that expands DLO. In particular, $T$ eliminates the quantifier $\exists^\infty$.

**Definition 5.1.** Let $(M, P(M))$ be a lovely pair of models of $T$. An $\mathcal{L}_P$-definable set $D \subset M^k$ is *small* if and only if there is some $m$, and an $\mathcal{L}$-definable function $f : M^m \to M^k$ such that $D \subset f(P(M)^m)$. Let $F$ be a cell and let $S \subset F$ be definable. We say $S$ is *large in* $F$ if $F \setminus S$ is small. A definable subset $D \subset M^k$ is *basic small* if it is small and of the form $\exists y_1 \in P \dots \exists y_n \in P\varphi(\vec{x}, \vec{y})$, where $\varphi(\vec{x}, \vec{y})$ is an $\mathcal{L}$-formula.

The definition above is what is called $P(M)$*-bound* in [5] and it turns out to be equivalent to the notion of small set from [5] (see Corollary 2.16). Note that if $D_1, D_2 \subset M^k$ are small their union is also small. Note that by the extension property no open interval is small.

We need to refine the description of $\mathcal{L}_P$-definable subsets of $M$ that we obtained in the previous section. In particular, we want to generalize Theorem 4 of [9] to general lovely pairs of o-minimal structures. We will follow the strategy from [9] and we start by reproving Lemma 4.3 of [9]. The proof we present is the one given in [9], we include it for completeness.

**Lemma 5.2.** *Let $X \subset M$ be small. Then $X$ is a finite union of sets of the form $f(P(M)^m \cap E)$ where $E$ is an $\mathcal{L}$-definable open cell in $M^m$ and $f : E \to M$ is $\mathcal{L}$-definable and continuous.*

*Proof.* Since $X$ is small, $X \subset f(P(M)^m)$ for some $\mathcal{L}$-definable function $f$ from $M^m$ into $M$. Thus we may write $X = f(X')$ for some $\mathcal{L}_P$-definable set $X' \subset P(M)^m$. By Proposition 3.4 we have $X' = P(M)^m \cap Y$ for some $\mathcal{L}$-definable $Y \subset M^m$. The rest of the proof is by induction on $m$. The case $m = 0$ is trivial, as $X$ is either empty or a single point. So assume the result holds for values lower than $m$ and we will prove it for $m$. We can subdivide $Y$ into smaller cells $E$ so that $f \restriction_E$ is continuous. If $E$ is an open cell in $M^m$ we get the conclusion of the lemma. If $E$ is not open and $\dim(E) = d < m$, there are indices $1 \le i_1 < i_2 < \cdots < i_d \le m$ such that the projection map $\pi : M^m \to M^d$, $\pi(x_1, \dots, x_n) = (x_{i_1}, \dots, x_{i_d})$ is homeomorphism from $E$ onto the open cell $E' = \pi(E)$ of $M^d$. Let $\mu : M^d \to M^m$ be a definable map such that $\mu(\pi(x)) = x$ for all $x \in E$. Then $f(P(M)^m \cap E) =$

$(f\circ\mu)(P(M)^d\cap E'\cap\mu^{-1}(P(M)^m))$ and by Proposition 3.4 there is an $\mathcal{L}$-definable set $F'\subset E'$ such that $P(M)^d\cap E'\cap\mu^{-1}(P(M)^m)=P(M)^d\cap F'$. By the induction hypothesis, $f(P(M)^m\cap E)=(f\circ\mu)(P(M)^d\cap F')$ is of the desired form. $\square$

**Lemma 5.3.** *Let $C\subset M^k$ be a cell. Then there is a partition $C_1,\dots,C_n$ of $C$ into cells such that $C_i\cap P(M)^k$ is either empty or a dense subset of $C_i$.*

*Proof.* The proof is by induction on $k$. The result is clear for $k=0$. Assume now that the result holds for values smaller than or equal to $k$ and we will prove it for $k+1$. First assume that $C$ is the set of realizations of the formula $f(y_1,\dots,y_k)<x<g(y_1,\dots,y_k)$ for $\vec{y}$ in a cell $D$ and $f$, $g$ continuous functions. By induction hypothesis we need to consider two cases. If $D\cap P(M)^k$ is dense in $D$, then $C\cap P(M)^{k+1}$ is dense in $C$. If $D\cap P(M)^k$ is empty, then so is $C\cap P(M)^{k+1}$.

Now assume that $C$ is of the form $x=f(y_1,\dots,y_k)$ for $\vec{y}$ in a cell $D$ and $f$ a continuous function. Then there is $d\leq k$ and there are indices $1\leq i_1<i_2<\dots<i_d\leq k+1$ such that the projection map $\pi:M^{k+1}\to M^d$, $\pi(x_1,\dots,x_{k+1})=(x_{i_1},\dots,x_{i_d})$ is homeomorphism from $C$ onto the open cell $C'=\pi(C)$ of $M^d$. Let $\mu:M^d\to M^m$ be a definable map such that $\mu(\pi(x))=x$ for all $x\in C$. Note that $\mu$ is a definable function. Then $P(M)^{k+1}\cap C=\mu(P(M)^d\cap C'\cap\mu^{-1}(P(M)^{k+1}))$ and by Proposition 3.4 there is an $\mathcal{L}$-definable set $F\subset C'$ such that $P(M)^d\cap C'\cap\mu^{-1}(P(M))=P(M)^d\cap F$. By the induction hypothesis we can find a finite partition $F$ into cells $\{F_j:j\leq n_1\}$ such that either $F_j\cap P(M)^d=\emptyset$ or $F_j\cap P(M)^d$ is dense in $F_j$. Furthermore, we can extend the partition $\{F_j:j\in J\}$ to a partition $\{C'_i:i\leq n_2\}$ of $C'$ with the same properties. Since $\mu$ is a homeomorphism, $\{\mu(C'_j):j\in J\}$ forms a partition of $C$ into cells. Let $C_j=\mu(C'_j)$. Note that if $C_k\cap P(M)^{k+1}\neq\emptyset$, then $\pi(C_k)\cap P(M)^d\cap\mu^{-1}(P(M)^{k+1})\neq\emptyset$, so $\pi(C_k)=F_j$ for some $j$ such that $F_j\cap P(M)^d$ is dense in $F_j$. Then $\mu(F_j\cap P(M)^d)$ is a dense subset of $C_j$. Since $P(M)^d\cap F_j\subset P(M)^d\cap C'\cap\mu^{-1}(P(M)^{k+1})$, $\mu(F_j\cap P(M)^d)\subset P(M)^{k+1}$, so $C_j\cap P(M)^{k+1}$ is a dense subset of $C_j$. $\square$

Now we generalize Lemma 2.15 from [5]:

**Proposition 5.4.** *Let $D\subset M$ be definable in $(M,P(M))$ over $\vec{d}$. Then there is a partition $-\infty=a_0<\dots<a_n=\infty$ and basic small dense sets $S_1,\dots,S_n$ such that $D\cap[a_{i-1},a_i]$ is either contained in the set $S_i$ or contains the set $S_i^c\cap[a_{i-1},a_i]$, and each $S_i$ is defined from $\vec{d}$.*

*Proof.* We first show the result for sets $D$ defined by formulas of the form

$$\exists y_1\dots\exists y_n P(y_1)\wedge\dots\wedge P(y_n)\wedge\varphi(y_1,\dots,y_n,x),$$

where $\varphi(y_1,\dots,y_n,x)$ is a cell.

Assume the cell defined by $\varphi(y_1,\dots,y_n,x)$ is of the form $f(y_1,\dots,y_n)<x<g(y_1,\dots,y_n)$ for $\vec{y}$ in a cell $C$ and $f$, $g$ continuous functions. Then, by Lemma 5.3, after subdividing $C$ if necessary, we obtain two cases. If $P(M)^n\cap C$ is empty, then $D$ is empty. If $P(M)^n\cap C$ is dense in $C$, then $D$ is an open interval.

Now assume that the cell defined by $\varphi(y_1,\dots,y_n,x)$ is of the form $x=f(y_1,\dots,y_n)$ for $\vec{y}$ in a cell $C$ and $f$ is a continuous function, which is either constant, strictly increasing or strictly decreasing. As above, after subdividing $C$ if necessary, we obtain the following cases. If $P(M)^n\cap C$ is empty, then $D$ is empty. If $P(M)^n\cap C$ is dense in $C$ and $f$ is constant, then $D$ is a point. If $f$ is strictly monotone, then $D$ is a dense small subset of $f(C)$.

Clearly if the conclusion of the Proposition holds for a set $D$, then it also holds for the complement of $D$. By Corollary 3.2 and cell decomposition, it remains to see what happens with intersections. Assume that $D_1$, $D_2$ are definable over $\vec{d}$ and that there is a partition $-\infty = a_0 < \cdots < a_n = \infty$ and basic small dense sets $S_{11}, \ldots, S_{1n}$, $S_{21}, \ldots, S_{2n}$ as prescribed by the Proposition for $D_1$, $D_2$ respectively. If $D_1 \cap [a_{i-1}, a_i] \subset S_{i1}$, then $(D_1 \cap D_2) \cap [a_{i-1}, a_i] \subset S_{i1}$. On the other hand, if $D_1 \cap [a_{i-1}, a_i] \supset S_{i1}^c \cap [a_{i-1}, a_i]$, $D_2 \cap [a_{i-1}, a_i] \supset S_{i2}^c \cap [a_{i-1}, a_i]$, then $D_1 \cap D_2 \cap [a_{i-1}, a_i] \supset (S_{i1} \cup S_{i2})^c \cap [a_{i-1}, a_i]$. □

**Proposition 5.5.** *If $X \subset M$ is $\mathcal{L}_P$-definable and small, then there is a partition $-\infty = b_0 < b_1 < \cdots < b_{k+1} = \infty$ of $M$ such that for each $i = 0, \ldots, k$, either $X \cap (b_i, b_{i+1}) = \emptyset$, or $X \cap (b_i, b_{i+1})$ as well as $(b_i, b_{i+1}) \setminus X$ are dense in $(b_i, b_{i+1})$. If $X \subset M$ is $\mathcal{L}_P$-definable then there is a partition $-\infty = b_0 < b_1 < \cdots < b_{k+1} = \infty$ of $M$ such that for each $i = 0, \ldots, k$, either $X \cap (b_i, b_{i+1}) = \emptyset$, or $X \cap (b_i, b_{i+1}) = (b_i, b_{i+1})$ or $X \cap (b_i, b_{i+1})$ as well as $(b_i, b_{i+1}) \setminus X$ are dense in $(b_i, b_{i+1})$.*

*Proof.* Let $X \subset M$ be small. By Lemma 5.2 we can write $X$ as a finite union of sets $f(P(M)^m \cap E)$ where $E \subset M^m$ is an open cell and $f$ is $\mathcal{L}$-definable continuous function. If $X$ is a single point there is nothing to prove, so we may assume that $f(E)$ is an interval possibly with endpoints. The set $f(P(M)^m \cap E)$ is dense in $f(E)$ and by the extension property $f(E) \setminus f(P(M)^m \cap E)$ is also dense in $f(E)$. The second part of the Proposition follows from the first part and from Proposition 5.4. □

As in [9, Corollary 4.5] we get from the previous results that $T_P$ eliminates the quantifier $\exists^\infty$.

**Corollary 5.6.** *Let $S \subset M^{m+n}$ be $\mathcal{L}_P$-definable in $(M, P(M))$ and assume that for each $\vec{a} \in M^m$ the fiber $S_{\vec{a}} = \{\vec{y} \in M^n \colon (\vec{a}, \vec{y}) \in S\}$ if finite. Then there is a natural number $k$ such that for all $\vec{a} \in M^m$, $|S_{\vec{a}}| \leq k$.*

*Proof.* It suffices to prove the property for the case $n = 1$. By Proposition 5.4 an $\mathcal{L}_P$-definable subset of $M$ is finite if and only if it is discrete. If the sets $S_{\vec{a}}$ are not uniformly bounded, by compactness in an elementary extension there is a set $S_{\vec{b}}$ which is infinite. Since being discrete is an elementary property, $S_{\vec{b}}$ can be chosen to be discrete, a contradiction. □

Dolich, Miller and Steinhorn showed [7] that whenever $T$ extends the theory is an expansion of an o-minimal ordered group, $T_P$ has o-minimal open core. Their proof uses a criterion that depends on the existence of a global group operation.

**Question 5.7.** *If $T$ is o-minimal, does $T_P$ have o-minimal open core?*

## 6. More on Geometry: the o-minimal case

Here we again fix an o-minimal theory $T$, expanding DLO. Our goal in this subsection is to study, for $(M, P(M))$ a lovely pair and $a \in M$, the relation between properties of the pair and the local $\mathcal{L}$-structure that $M$ induces on a neighborhood of $a$. A key tool in this section is the Trichotomy Theorem of Peterzil and Starchenko [21, 22]. We recall from definitions and results from [21]:

**Definition 6.1.** Let $M$ be an o-minimal structure and let $a \in M$. We say that $a$ is *non-trivial* if there is an infinite open interval $I$ containing $a$ and a definable continuous function $F\colon I\times I\to M$ such that $F$ is strictly monotone in each variable. A point which is not non-trivial is called *trivial*. Now assume that $(G,+,0)\subset M$ is a convex type-definable ordered group and that $p>0$ belongs to $G$. Then the structure $([-p,p],<,+,0)$ is called a *group interval*.

**Fact 6.2.** *(Trichotomy Theorem) Let $M$ be an $\omega_1$-saturated structure. Given $a\in M$ one and only one of the following holds:*

(1) *$a$ is trivial.*
(2) *the structure that $M$ induces in some convex neighborhood of $a$ is an ordered vector space over a division ring. Furthermore, there is a closed interval containing $a$ on which a group interval is definable.*
(3) *The structure that $M$ induces on some open interval around $a$ is an o-minimal expansion of a real closed field.*

We start with relating thorn-forking and small sets:

**Lemma 6.3.** *Let $M$ be an o-minimal structure and assume that $(M,P)\models T_P$ is sufficiently saturated. Let $\varphi(x,\vec{b})$ be a formula that thorn-forks over $\emptyset$. Then $\varphi(x,\vec{b})$ defines a small set.*

*Proof.* Now assume for a contradiction that $\varphi(x,\vec{b})$ is not a small set. By Proposition 5.4 there is some open interval $I_{\vec{b}}$ such that $D_{\vec{b}}$ is large in $I_{\vec{b}}$. Suppose that $\theta(\vec{y},\vec{c})$ is such that for any $\vec{b}_1,\dots,\vec{b}_k$ different realizations of $\theta(\vec{y},\vec{c})$, one has

$$D_{\vec{b}_1}\cap\cdots\cap D_{\vec{b}_k}=\emptyset.$$

**Claim** $J:=I_{\vec{b}_1}\cap\cdots\cap I_{\vec{b}_k}=\emptyset$.

Otherwise $\mathbb{J}$ is an open interval $(d_1,d_2)$. Let $S_{\vec{b}}$ be a small set such that $D_{\vec{b}}=I_b\setminus S_{\vec{b}}$. Then $(D_{\vec{b}_1}\cap\cdots\cap D_{\vec{b}_k})\cap(d_1,d_2)=J\setminus(S_{\vec{b}_1}\cup\cdots\cup S_{\vec{b}_k})\neq\emptyset$ by the extension property.

Thus, if $\psi(x,\vec{b})$ defines $I_{\vec{b}}$, we see that $\psi(x,\vec{b})$ also þ-divides. But since intervals are $\mathcal{L}$-definable, this contradicts Fact 4.7. □

We begin with analysing the þ-rank around trivial points.

**Lemma 6.4.** *Let $a\in M$ be such that the structure induced by $M$ on $a$ is trivial in the sense of Peterzil-Starchenko. Let $b_1,\dots b_n\in M$ and assume that $a\in\operatorname{dcl}(b_1,\dots,b_n)$. Then there is $i\le n$ such that $a\in\operatorname{dcl}(b_i)$.*

*Proof.* We may reduce the problem to $n=2$. Assume, in order to get a contradiction, that there are $b,c\in M$ are such that $a\in\operatorname{dcl}(b,c)\setminus(\operatorname{dcl}(b)\cup\operatorname{dcl}(c))$. By the exchange property, it is clear that $c\in\operatorname{dcl}(a,b)\setminus(\operatorname{dcl}(a)\cup\operatorname{dcl}(b))$. Let $f(x,y)$ be a $\emptyset$-definable function such that $c=f(a,b)$. Consider now $f(x,b)$. Since $T$ is o-minimal and $c\notin\operatorname{dcl}(b)$, by the Monotonicity Theorem [10] $f(x,b)$ is continuous and monotone in a neighborhood $(a_1,a_2)$ of $a$. By reducing the interval $(a_1,a_2)$ if necessary, we may assume that $\dim(a_1,a_2/\{a,b\})=2$. Without loss of generality, we may assume that $f(x,b)$ is increasing. Since $b\notin\operatorname{dcl}(a,a_1,a_2)$, there is an open neighborhood $(b_1,b_2)$ around $b$ such that for all $b'\in I$, $f(x,b')\colon(a_1,a_2)\to M$ is continuous and increasing. In a similar way, after possibly reducing $(a_1,a_2)$ and $(b_1,b_2)$, we may assume that $f(a',y)\colon(b_1,b_2)\to M$ is continuous and monotone for all $a'\in(a_1,a_2)$. By Lemma 2.16 [10], we get that $f(x,y)\colon(a_1,a_2)\times(b_1,b_2)\to M$ is continuous.

Finally, using similar ideas as above and reducing $(a_1, a_2)$ further if necessary, we may assume there is a continuous monotone function $h(y, c)\colon (a_1, a_2) \to (b_1, b_2)$ defined over $c$. Then the function $f(x, h(y, c)) : (a_1, a_2) \times (a_1, a_2) \to M$ is continuous and monotone on each variable. This contradicts the triviality of $a$. $\square$

We are ready to prove our first result:

**Proposition 6.5.** *Suppose $(M, P)$ be a lovely pair of models of an o-minimal theory $T$. Let $a \in M$ and assume that the structure induced by $M$ on $a$ is trivial in the sense of Peterzil-Starchenko sense. Then $U^{\text{þ}}(\mathrm{tp}(a)) \leq 1$.*

*Proof.* If $a \in \mathrm{scl}(\emptyset)$ then by Lemma 6.4 there is $b \in P(M)$ such that $a \in \mathrm{dcl}(b)$. By Fact 4.7, þ-$rk(P(M)) = 1$ and we get $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 1$. So assume that $a \notin \mathrm{scl}(\emptyset)$ and that $B \subset M$ is such that $\mathrm{tp}(a/B)$ þ-forks over $\emptyset$. Then by Lemma 6.3, $a \in \mathrm{scl}(B)$, so $a \in \mathrm{dcl}(B \cup P(M))$. Since $M$ is trivial in a neighborhood of $a$ and $a \notin P(M)$ by Lemma 6.4 we get that $a \in \mathrm{dcl}(B)$ so $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a/B)) = 0$ and $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a)) \leq 1$. $\square$

Now we find lower bounds on the rank of non-trivial elements in the pair.

**Proposition 6.6.** *Suppose $(M, P)$ is a lovely pair of models of an o-minimal theory $T$, and $a \in M \backslash P(M)$ is non-trivial. Then we have $U^{\text{þ}}(tp(a)) \geq 2$.*

*Proof.* In this case, by [21], in $M$ there is a definable group interval $(I, +, <)$ of an ordered divisible abelian group $(G, +, <)$, where $I = (-q, q)$ and contains $a$. Although the group $G$ may not be definable in $M$, any “linear equation" is definable in $M$. Namely, if $\lambda_1, \ldots, \lambda_n \in \mathbb{Q}$, not all equal to zero, then the equation

$$\lambda_1 x_1 + \ldots + \lambda_n x_n = 0$$

is definable for $x_1, \ldots, x_n \in I$, even if $\lambda_i x_i$ is not in $I$ for some $i$. Indeed, the equation is equivalent to

$$\frac{\lambda_1}{|\lambda_1| + \ldots + |\lambda_n|} x_1 + \ldots + \frac{\lambda_n}{|\lambda_1| + \ldots + |\lambda_n|} x_n = 0,$$

which is definable in $(I, +, <)$.

Adding $q$ as a constant, we may assume that $I$ is $\emptyset$-definable (in $T$). We can also assume that $a > 0$. Let $\sigma > 0$ be such that $a + \sigma \in I$. Take $c \in (a, a + \frac{\sigma}{2})$ such that $c$ a generic element of $I \cap P(M)$, and let $e = 2c - a$. Then $e \in I$ and $e \notin P(M)$. We claim that $tp(a/e)$ þ-forks over $\emptyset$. Let $E(x, y)$ be defined by

$$x = y \vee (x, y \in I \wedge \exists c_1, c_2 \in P \cap I \quad x - y + c_2 - c_1 = 0)\,.$$

Note that for $b, b' \in I$, $E(b, b')$ means that $b - b' = c_1 - c_2$ for some $c_1, c_2 \in P \cap I$, where the difference is taken in $G$, and may not actually be in $I$. But as noted above, $x - y + c_1 - c_2 = 0$ is definable in $(I, +, <)$. We claim that $E$ is an $L_P$-definable equivalence relation. To check transitivity, let $b, b', b'' \in I$ be distinct, and $c_1, c_2, c_3, c_4 \in P \cap I$, such that

$$b - b' = c_1 - c_2$$

and

$$b' - b'' = c_3 - c_4.$$

By density of $P$ in $M$, we may assume that $c_1 = b + \varepsilon$, $c_2 = b' + \varepsilon$, $c_3 = b' + \delta$, $c_4 = b'' + \delta$ for arbitrarily small $\varepsilon, \delta > 0$. Working in the vector space, we have:

$$b - b'' = (b - b') + (b' - b'') = (c_1 - c_2) + (c_3 - c_4) = c_1 - (c_4 + c_2 - c_3).$$

Note that $c_2 - c_3 = \varepsilon - \delta$ can be made small enough so that $d = c_4 + c_2 - c_3 \in I$. Now, $x - c_2 + c_3 - c_4 = 0$ is definable in $(I, +, <)$, and thus $d \in \operatorname{dcl}(c_2, c_3, c_4)$, hence $d \in P(M)$. Thus $b - b'' = c_1 - d$ with $c_1, d \in P \cap I$, which shows $E(b, b'')$.

Let $\phi(x, y/E)$ be the formula saying that $x \in I$ and for some $y' \in I$ in the $E$-class of $y$, we have $\dfrac{x + y'}{2} \in P$.

**Claim.** If for some $a, b_1, b_2 \in I$, we have $\dfrac{a + b_1}{2} \in P(M)$ and $\dfrac{a + b_2}{2} \in P(M)$, then $E(b_1, b_2)$.

*Proof of the Claim:* Let $c_1 = \dfrac{a + b_1}{2}$ and $c_2 = \dfrac{a + b_2}{2}$. Note that $c_1, c_2 \in I \cap P(M)$. Working in the abelian group, we have $b_1 - b_2 = 2c_1 - 2c_2$. By density of $P(M)$, we can choose $\varepsilon > 0$ such that $b_2 + \varepsilon \in I \cap P(M)$. Taking $\varepsilon$ small enough, we may also assume that $b_1 + \varepsilon \in I$. Now, working in the abelian group again, we have:

$$(b_1 + \varepsilon) - (b_2 + \varepsilon) = b_1 - b_2 = 2c_1 - 2c_2,$$

and thus

$$b_1 + \varepsilon = (b_2 + \varepsilon) + 2c_1 - 2c_2.$$

Since $b_1 + \varepsilon, b_2 + \varepsilon, c_1, c_2 \in I$, we conclude, as above, that $b_1 + \varepsilon \in \operatorname{dcl}(b_2 + \varepsilon, c_1, c_2)$ and therefore $b_1 + \varepsilon \in P(M)$. Thus

$$b_1 - b_2 = (b_1 + \varepsilon) - (b_2 + \varepsilon),$$

where $b_1 + \varepsilon, b_2 + \varepsilon \in I \cap P$, which means $E(b_1, b_2)$.

Thus for any two distinct $b_1/E, b_2/E \models \operatorname{tp}(e/E)$, $\phi(x, b_1/E) \wedge \phi(x, b_2/E)$ is inconsistent. This witnesses þ-forking of $tp_P(a/e)$. Since $a \notin \operatorname{acl}_P(e)$, we have $U^{þ}(tp_P(a)) \geq 2$.

$\square$

**Proposition 6.7.** *Let $(M, P)$ be a lovely pair of models of an o-minimal theory, let $a \in M$ and assume that the structure induced in an open interval around $a$ is an o-minimal expansion of a real closed field defined over some finite set $A$. Then whenever $a \notin \operatorname{scl}(A)$, $U^{þ}(\operatorname{tp}_P(a/A)) = \omega$.*

*Proof.* By Fact 4.7 $\mathrm{U}^{þ}(\operatorname{tp}_P(a/A)) \leq \omega$.

To show the other direction, let us assume that $a \notin \operatorname{scl}(A)$ and we show that for every $n \geq 0$, there exists $B \supset A$ such that $\mathrm{U}^{þ}(\operatorname{tp}_P(a/B)) = n$. Let $I = (a_1, a_2)$ be the underlying set for the field, we may assume that $a_1, a_2 \in A$. Let $c_1, \ldots, c_n \in I$ be such that $c_1 \notin \operatorname{scl}(a, A)$, $c_2 \notin \operatorname{scl}(a, A, c_1), \ldots, c_n \notin \operatorname{scl}(a, A, c_1, \ldots, c_{n_1})$ (these elements exist by the extension property). Now let $g_1, \ldots, g_n \in P(M) \cap I$ be non-algebraic elements which are independent from each other and independent from $a, A, c_1, \ldots, c_n$ (these elements exist by the density property).

**Claim** $g_i \in \operatorname{dcl}(c_1 g_1 + \cdots + c_n g_n, c_1, \ldots, c_n, A)$ for $i \leq n$.

Consider the equation $c_1 x_1 + \cdots + c_n x_n = c_1 g_1 + \cdots + c_n g_n$. If the equation has a solution $(g'_1, \ldots, g'_n)$ in $(P(M) \cap I)^n$ different from $(g_1, \ldots, g_n)$ we get $c_1(g_1 - g'_1) + \cdots + c_n(g_n - g'_n) = 0$ and $g_j - g'_j \neq 0$ for some $j \leq n$. Then $c_j \in \operatorname{scl}(A, c_1, \ldots, c_{j-1}, c_{j+1}, \ldots, c_n)$ and this is a contradiction. Thus $(g_1, \ldots, g_n)$ is the unique solution of the equation in $(P(M) \cap I)^n$, which proves the claim.

Let $d = a + c_1g_1 + \cdots + c_ng_n$ and $B = A \cup \{d, c_1, \ldots, c_n\}$. Then $a$ and $c_1g_1 + \cdots + c_ng_n$ are interdefinable over $B$ and by the claim both these elements are interdefinable with $\{g_1, \ldots, g_n\}$ over $B$. Thus $\mathrm{U}^{\text{þ}}(\mathrm{tp}(a/B)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}(g_1, \ldots, g_n/B))$. On the other hand, $a \notin \mathrm{scl}\{c_1, \ldots, c_n, A\}$, so $d \notin \mathrm{scl}\{c_1, \ldots, c_n, A\}$ and

$$d \mathop{\smile\!\!\!\!\!\!\;|}^{\text{þ}} \{c_1, \ldots, c_n, g_1, \ldots, g_n\} \cup A.$$

This implies that $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(g_1, \ldots, g_n/B)) = \mathrm{U}^{\text{þ}}(\mathrm{tp}_P(g_1, \ldots, g_n/\{c_1, \ldots, c_n\} \cup A)) = n$ and thus $\mathrm{U}^{\text{þ}}(\mathrm{tp}_P(a/B)) = n$ as we wanted. $\square$

We now turn our attention to the linear case, aiming at proving that the $U^{\text{þ}}$-rank is $\leq 2$. We need an extra assumption in order to study the structure: the existence of a global addition operation. First we prove the following lemma.

**Lemma 6.8.** *Let $(V, +, \lambda(x))_{\lambda \in D}$ be an ordered vector space over an ordered division ring $D$, and let $U$ be a convex neighborhood of $0$ in $V$. Then the pregeometry induced by* dcl *(equivalently, linear span) on $U$ is projective (modular).*

*Proof.* Suppose $u \in dcl(v, w_1, \ldots, w_n)$, where $u, v, w_1, \ldots, w_n \in U$. We need to find $w \in \mathrm{dcl}(w_1, \ldots, w_n) \cap U$ such that $u \in \mathrm{dcl}(v, w)$. Now, $u = \beta v + \lambda_1 w_1 + \ldots + \lambda_n w_n$, and we may assume that not all $\lambda_i$ are equal to 0. Let $\lambda = |\lambda_1| + \ldots + |\lambda_n|$ and let

$$w = \frac{\lambda_1}{\lambda} w_1 + \ldots + \frac{\lambda_n}{\lambda} w_n.$$

Then $w \in U$, $w \in \mathrm{dcl}(w_1, \ldots, w_n)$ and $u = \beta v + \lambda w \in dcl(v, w)$, as needed. $\square$

**Proposition 6.9.** *Suppose $T$ is an o-minimal expansion of an ordered divisible abelian group, which is linear in the sense of the trichotomy theorem (no definable field). Then $T$ is linear as a geometric theory.*

*Proof.* Suppose $(M, P)$ is an $\omega^+$-saturated lovely pair of models of $T$. We will prove that the geometry of localization of $M$ at $P$ is projective. It suffices to show that for dcl-independent $a_1, \ldots, a_n, b \in (M, P)$ if $c \in dcl(b\vec{a})$, then there is $d \in \mathrm{dcl}(\vec{a}P)$ such that $c \in \mathrm{dcl}(bdP)$.

So let $a_1, \ldots, a_n, b$ be dcl-independent and $c = f(b, \vec{a})$ for some $\mathcal{L}$-definable function $f(x, \vec{y})$. We may assume that $f$ is continuous at $(b, \vec{a})$. Note that the expanded structure $(M, b\vec{a})$ is still o-minimal and linear in the sense of the trichotomy theorem. Thus on some convex neighborhood $U$ of 0 the structure induced by $b\vec{a}$-definable relations is that of a vector space over a division ring. Choose $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon) \subset U$ and $\varepsilon \notin dcl_L(b\vec{a})$. By lemma 6.8, the pregeometry induced by $\mathrm{dcl}(-, b\vec{a})$ on $U$ is projective (modular), and we have: whenever $u, v, \vec{w} \in (-\varepsilon, \varepsilon)$ and $u \in dcl(v, \vec{w}, b, \vec{a})$, there is $r$ $(\in U)$ such that $r \in dcl(\vec{w}, b, \vec{a})$ and $u \in dcl(v, r, b, \vec{a})$.

Consider $g(y, \vec{z}, b, \vec{a}) = f(y+b, z_1+a_1, \ldots, z_n+a_n) - f(b, \vec{a})$. Since $f$ is continuous at $(b, \vec{a})$ there is $0 < \rho \leq \epsilon$ such that if $v, \vec{w} \in (-\rho, \rho)$ then $g(y, \vec{z}, b, \vec{a}) \in (-\epsilon, \epsilon)$. Thus for $v, \vec{w} \in (-\rho, \rho)$ we have $g(v, \vec{w}, b, \vec{a}) = h(v, r, b, \vec{a})$ where $r = s(\vec{w}, b, \vec{a})$ for some $\mathcal{L}$-definable functions $h$ and $s$. We may assume that $\rho$ is independent from $a_1, \ldots, a_n, b$. By compactness, there are $\mathcal{L}$-definable functions $h_1, \ldots, h_m$ and $s_1, \ldots, s_k$ such that for any $v, \vec{w} \in (-\rho, \rho)$ $g(v, \vec{w}, b, \vec{a}) = h_i(v, s_j(\vec{w}, b\vec{a}), b, \vec{a})$ for some $i \leq m$, $j \leq k$.

So $b, \vec{a}$ satisfy

$$\theta(y, \vec{x}, \rho) = \forall y', \vec{x}' \in (-\rho, \rho) \bigvee_{i \leq m, j \leq k} g(y', \vec{x}', y, \vec{x}) = h_i(y', s_j(\vec{x}', y, \vec{x}), y, \vec{x}).$$

Since $\rho, a_1, \ldots, a_n, b$ are $\mathrm{dcl}_L$-independent, there is $\delta > 0$ such that such that for any $b^*, a_1^*, \ldots, a_n^*$ such that $|a_i^* - a_i| < \delta$, $|b^* - b| < \delta$ we have $\models \theta(b^*, \vec{a}^*, \rho)$. We may also assume that $\delta < \rho$. By density, choose $b^*, \vec{a}^* \in P$. Thus

$$\forall y', \vec{x}' \in (-\rho, \rho) \bigvee_{i \leq m, j \leq k} g(y', \vec{x}', b^*, \vec{a}^*) = h_i(y', s_j(\vec{x}', b^*, \vec{a}^*), b^*, \vec{a}^*).$$

Let $v = b - b^*, w_i = a_i - a_i^*$. Then $v, w_i \in (-\rho, \rho)$. Then

$$g(v, \vec{w}, b^*, \vec{a}^*) = h_i(v, s_j(\vec{w}, b^*, \vec{a}^*), b^*, \vec{a}^*),$$

for some $i, j$, and thus

$$g(b - b^*, a_1 - a_1^*, \ldots, a_n - a_n^*, b^*, \vec{a}^*) =$$

$$f(b - b^* + b^*, a_1 - a_1^* + a_1^*, \ldots, a_n - a_n^* + a_n^*) - f(b^*, \vec{a}^*) =$$

$$h_i(b - b^*, s_j(a_1 - a_1^*, \ldots, a_n - a_n^*, b^*, \vec{a}^*), b^*, \vec{a}^*).$$

Hence

$$c = f(b, \vec{a}) = f(b^*, \vec{a}^*) + h_i(b - b^*, s_j(a_1 - a_1^*, \ldots, a_n - a_n^*, b^*, \vec{a}^*), b^*, \vec{a}^*).$$

Let $d = s_j(a_1 - a_1^*, \ldots, a_n - a_n^*, b^*, \vec{a}^*)$. Then $d \in dcl(\vec{a}P)$ and $c \in dcl(bdP)$, as needed. $\square$

**Remark 6.10.** *Given any $u, v \in M$ $\mathrm{dcl}_L$-independent over $P$,*

$$u + v \in dcl(uvP) \backslash (dcl(uP) \cup dcl(vP)).$$

*Thus in the quotient geometry closure of any two points contains a third one. Thus the quotient geometry (geometry of the small closure) is a single projective geometry over a division ring.*

We are ready to summarize the results from this section:

**Theorem 6.11.** *Let $M$ be an o-minimal structure whose theory extends DLO, let $P(M) \preceq M$ and assume that $(M, P(M))$ is a lovely pair.*

1. *If $a \in M$ is trivial, $U^{þ}(\mathrm{tp}_P(a)) \leq 1$ ($= 1$ iff $a \notin \mathrm{dcl}(\emptyset)$).*
2. *If $a \notin P(M)$ is non-trivial, then $U^{þ}(\mathrm{tp}_P(a)) \geq 2$.*
3. *If $M$ has global addition (i.e. expands the theory of ordered abelian groups) and does not interpret an infinite field, then $(M, P)$ has þ-rank 2.*
4. *If $M$ induces the structure of an o-minimal expansion of a real closed field in a neighborhood of $a \notin P(M)$, then $U^{þ}(\mathrm{tp}_P(a)) = \omega$.*

*Proof.* (1) By Proposition 6.5.
(2) By Proposition 6.6.
(3) By Propositions 6.9 and 4.3.
(4) By Proposition 6.7. $\square$

Now, we will give an example of a lovely pair in the trivial case.

**Lemma 6.12.** *The structure $(\mathbb{R}, <, \mathbb{Q})$ is a lovely pair.*

*Proof.* We first show that the Density property holds. Let $A \subset R$ be finite, say $A = \{a_1, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$ and let $q \in S_1(A)$ be non-algebraic. Then $q$ is describing an open interval, either $(-\infty, a_1)$, $(a_i, a_{i+1})$ for some $i$, or $(a_k, \infty)$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ there is $c \in P(\mathbb{R}) = \mathbb{Q}$ such that $c \models q$.

Now we show that the Extension property holds. Let $A \subset R$ be finite, say $A = \{a_1, \ldots, a_k\}$ with $a_1 < a_2 < \cdots < a_k$ and let $q \in S_1(A)$ be non-algebraic. Then $q$ describes an open interval with endpoints in the set $A$. Since $\mathbb{R} \setminus (A \cup \mathbb{Q})$ is dense in $\mathbb{R}$, we can find a realization of $q$ in $\mathbb{R} \setminus (A \cup \mathbb{Q})$. ☐

It is easy to check that the pair $(\mathbb{R}, <, \mathbb{Q})$ is an expansion of $(\mathbb{R}, <)$ with a generic predicate (in the sense of Chatzidakis, Pillay [6]). It is proved in [6, Corollary 2.6 part 3] that for such expansions, the algebraic closure in the extended language $\mathcal{L}_p$ coincides with the algebraic closure in the language $\mathcal{L}$. In particular, algebraic independence inside the structure $(\mathbb{R}, <, \mathbb{Q})$ satisfies the usual properties of an independence relation for *real elements.* On the other side, Sergio Fratarcangeli showed in [12] that expansions of o-minimal structures with a generic predicate eliminate imaginaries. Thus algebraic independence inside the structure $(\mathbb{R}, <, \mathbb{Q})$ defines an independence relation that extends to an independence relation for all elements in $(\mathbb{R}, <, \mathbb{Q})^{eq}$ and thus $T_P$ is rosy and $\mathrm{acl}_L$-independence coincides with thorn-forking independence in the sense of $T_P$. Furthermore þ-rank$(Th((\mathbb{R}, <, \mathbb{Q}))) = 1$, as we expected from Theorem 6.11.

Note that Proposition 4.8 and Theorem 6.11(4) show that our notion of linearity defined in the context of geometric structures, in the o-minimal case implies linearity in the sense of Trichotomy (non-definability of a field, or equivalently, the CF property from [18]), and by Proposition 6.9, the two notions coincide for expansions of ordered divisible abelian groups. Note that in [18], theories of o-minimal groups satisfying the CF property were called linear, which agrees with our terminology. The following is proved in [18, Theorem 1.3]:

**Fact 6.13.** *Any linear o-minimal theory of a (divisible ordered abelian) group is a reduct of a theory of an ordered vector space over an ordered division ring (possibly with constants). Conversely, any such reduct is linear.*

Here $T$ being a reduct of $T'$ means that any definable relation in $T$ is definable in $T'$. Note that a similar connection with vector spaces (but on the level of associated geometry) holds in the general case of geometric structures, as shown in Proposition 4.3 and Remark 6.10.

The following example of a reduct of an ordered vector space from [18, Example 4.5] illustrates the difference between the (local) modularity and linearity, and shows how taking the quotient over a dense substructure leads to modularity.

**Example 6.14.** Let $\mathcal{R} = (\mathbb{R}, +, <, f|_{(-1,1)})$ where $f$ is defined by $f(x) = \pi x$. Clearly, $f|_{(-1,1)}$ can be extended to all of $\mathcal{R}$ by $f(x) = nf\left(\frac{x}{n}\right)$ for $x \in (-n, n)$, however this extension is not uniformly definable, and thus in a $\omega^+$-saturated model $\mathcal{R}^*$ of $T = Th(\mathcal{R})$, we cannot define $f(x)$ for "infinite" elements. As the theory of a reduct of a vector space over $\mathbb{Q}(\pi)$, $T$ is a linear (CF) theory, but is not modular (or even locally modular). It is also shown in [20] that $T$ does not have almost canonical bases.

The non-modularity of $(\mathcal{R}^*, \mathrm{dcl})$ can be witnessed by considering

$$a = f(b + c_1) + c_2,$$

where $b, c_1$ are infinite elements such that $b + c_1 \in (-1, 1)$, and $b$, $c_1$ and $c_2$ are independent.While $a \in \mathrm{dcl}(b, c_1, c_2)$, there is no $c \in \mathrm{dcl}(c_1, c_2)$ such that $a \in \mathrm{dcl}(b, c)$.

Suppose now, that $b, c_1, c_2$ are also independent over $P(\mathcal{R}^*)$. By density, we can take $c_1' \in P(\mathcal{R}^*)$ such that $c_1 - c_1' \in (-1, 1)$ and we still have $b + c_1' \in (-1, 1)$. Then $a = f(b + c_1' + c_1 - c_1') + c_2 = f(b + c_1') + f(c_1 - c_1') + c_2$. Now,

$$c = f(c_1 - c_1') + c_2 \in \mathrm{dcl}(c_1 c_2 P(\mathcal{R}^*)),$$

and $a \in \mathrm{dcl}(bcP(\mathcal{R}^*))$. Thus taking a quotient over $P$ "removes" this particular non-modularity.

In [8], the geometry of a nontrivial linear Lascar strong type $D$ of SU-rank 1 in a simple theory has been extended to a projective geometry over a division ring, "recovering" the missing points by adding canonical bases of surfaces in $D^3$. In the absence of canonical bases, one can still recover the projective geometry over division ring, by taking a quotient over $P$ in a lovely pair. Another possible approach is to go beyond first order, by adding quotients by equivalence relations defined by infinite disjunctions (see [16]), e.g. considering the E-class of $c_1 c_2$ above, where

$$E(x_1 x_2, x_1' x_2') = \bigvee_{n=1}^{\infty} x_1 - x_1' \in (-n, n) \wedge n \cdot f\left(\frac{x_1 - x_1'}{n}\right) = x_2 - x_2'.$$

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