HYPERFIELDS, TRUNCATED DVRS, AND VALUED FIELDS

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Abstract. For any two complete discrete valued fields K_1 and K_2 of mixed characteristic with perfect residue fields, we show that if the n-th valued hyperfields of K_1 and K_2 are isomorphic over p for each $n \geq 1$, then K_1 and K_2 are isomorphic. More generally, for $n_1, n_2 \geq 1$, if n_2 is large enough, then any homomorphism, which is over p, from the n_1-th valued hyperfield of K_1 to the n_2-th valued hyperfield of K_2 can be lifted to a homomorphism from K_1 to K_2. We compute such n_2 effectively, which depends only on the ramification indices of K_1 and K_2. Moreover, if K_1 is tamely ramified, then any homomorphism over p between the first valued hyperfields is induced from a unique homomorphism of valued fields. Using this lifting result, we deduce a relative completeness theorem of AKE-style in terms of valued hyperfields.

We also study some relationships between valued hyperfields, truncated discrete valuation rings, and complete discrete valued fields of mixed characteristic. For a prime number p and a positive integer e and for large enough n, we show that a certain category of valued hyperfields is equivalent to the category of truncated discrete valuation rings of length n and the ramification indices e having perfect residue fields of characteristic p. Furthermore, in the tamely ramified case, we show that a subcategory of this category of valued hyperfields is equivalent to the category of complete discrete valued rings of mixed characteristic $(0, p)$ having perfect residue fields.

1. Introduction

Our main object in this article is a valued hyperfield. A hyperfield is a field-like algebraic structure whose addition is multivalued, and a valued hyperfield is a hyperfield equipped with a valuation. A typical example is the quotient of a valued field by a multiplicative subgroup of the form $1 + \mathfrak{m}^n$ for the maximal ideal $\mathfrak{m}$ of the valuation ring, which is called the n-th valued hyperfield of a valued field. In this article, we consider a lifting problem of homomorphisms of the n-th valued hyperfields to homomorphisms of valued fields(See Corollary 3.11 and Theorem 3.18). Also we study relationships between certain categories of valued hyperfields, truncated discrete valuation rings, and discrete valued fields of mixed characteristic(See Theorem 4.8 and Theorem 4.10). At last, we prove a relative completeness theorem of AKE-style in terms of valued hyperfields(See Theorem 5.8).

M. Krasner in [11] introduced a notion of valued hyperfield and used it to do a theory of limits of local fields. In [7], P. Deligne did the theory of limits of local fields in a different way by defining a notion of triple, which consists of truncated discrete valuation rings and some additional data. Typical examples of a valued

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hyperfield and a truncated discrete valuation ring are the n-th valued hyperfield and the n-th residue ring of a valued field respectively, where the n-th residue ring is a quotient of a valuation ring by the n-th power of the maximal ideal. J. Tolliver in [17] showed that discrete valued hyperfields and triples are essentially same, stated by P. Deligne in [7] without a proof. In [14], W. Lee and the author showed that given complete discrete valued fields of mixed characteristic with perfect residue fields, any homomorphism between the n-th residue rings of the valued fields is lifted to a homomorphism between the valued fields for large enough n. From this, we ask the following question.

Question 1.1. *Let K_1 and K_2 be discrete valued fields of mixed characteristic $(0,p)$ having perfect residue fields. Suppose the n-th valued hyperfields of K_1 and K_2 are isomorphic for every $n \geq 1$. Then are K_1 and K_2 are isomorphic? Moreover, is there $N > 0$ such that if the N-th valued hyperfiels of K_1 and K_2 are isomorphic, are K_1 and K_2 isomorphic?*

The n-th valued hyperfield $\mathcal{H}_n(K) := K/(1+\mathfrak{m}^n)$ of a valued field K contains automatically information of the multiplicative group $K^\times/(1+\mathfrak{m}^n)$. By the multivalued addition of $\mathcal{H}_n(K)$, it also contains information of the n-residue ring of K too(See Fact 2.19). We compute the multivalued addition of the n-th valued hyperfields rather explicitly, where the multivalued addition of given two elements in $\mathcal{H}_n(K)$ is given as a ball and we can compute its center and its radius of the ball explicitly(See Lemma 3.1). In [4] and [12], it was considered some structures, called *amc-structures*, consisted with information of the n-th residue rings and the multiplicative groups $K^\times/(1+\mathfrak{m}^n)$. For $n \geq 1$, the n-th amc structure of a valued field K is a triple $K_n := (R_K^{2n}, G_K^n, \Theta_n)$, where $R_K^n := R/t^{2(n-1)}\mathfrak{m}$, $G_K^n := K^\times/(1+t^{n-1}\mathfrak{m})$, and a map $\Theta_n : \{x \in R_K^{2n} : \ x^2 \neq 0\} \to G_K^n, a + t^{2n}\mathfrak{m} \mapsto a(1+t^n\mathfrak{m})$. Here, R is the valuation ring of K, and $t = 1$ if the residue field is of characteristic 0 and $t = p$ if the residue field is of characteristic $p > 0$. In [4], S. A. Basrab and F. V. Kuhlmann showed that for a valued field K, and for henselian valued fields L and F which are algebraic extension of K, if L_n and F_n are isomorphic over K_n for each n, then L and F are isomorphic over K(See [4, Corollary 1.4]). Moreover, if L and F are finite extension of K, then it is enough to consider whether L_n and F_n are isomorphic over K_n for large enough n. In [12], F. V. Kuhlmann showed that if L and F are additionally tame extensions of K, then L and F are isomorphic over K if L_1 and F_1 are isomorphic over K_1(See [12, Lemma 3.1]).

We also study relationships between discrete valued hyperfields, truncated discrete valuation rings, and complete discrete valued fields of mixed characteristic. Fix a prime number p and a positive integer e. Let $\mathcal{C}_{p,e}$ be the category of complete discrete valuation rings of mixed characteristic $(0,p)$ having perfect residue fields and ramification index e. Let $\mathcal{R}_{p,e}^n$ be the category of truncated discrete valuation rings of length n with perfect residue fields such that p is in the e-th power of the maximal ideal but not in the $e+1$-th power of the maximal ideal.

Fact 1.2. [14, Theorem 4.7, Example 3.7(2)] *For large enough n, there is a lifting functor $\mathrm{L} : \mathcal{R}_{p,e}^n \to \mathcal{C}_{p,e}$ satisfying several natural conditions. But two categories $\mathcal{C}_{p,e}$ and $\mathcal{R}_{p,e}^n$ are not equivalent.*

We ask a question analogous to Fact 1.2 for valued hyperfields.

Problem 1.3. *Fix a prime number p and a positive integer e. Let n be a positive integer. Find a suitable category $\mathcal{H}_{p,e}^n$ of valued hyperfields whose has a lifting* *functor* $\mathrm{L}^{\mathcal{H}} : \mathcal{H}_{p,e}^n \to \mathcal{C}_{p,e}$ *satisfying some natural conditions, and which makes* $\mathcal{H}_{p,e}^n$ *and* $\mathcal{C}_{p,e}$ *equivalent.*

At last, we concern a question of relative completeness of AKE-style in terms of hyper valuedfields. S. A. Basarab in [3] showed that the theory of finitely ramified henselian valued fields of mixed characteristic is determined by the theory of the n-th residue ring for each n and the theory of value group.

Fact 1.4. [3, Corollay 3.1] *Let* K_1 *and* K_2 *be finitely ramified henselian valued fields of mixed characteristic. Let* $R_{1,n}$ *and* $R_{2,n}$ *be the* n*-th residue rings of* K_1 *and* K_2 *respectively. Let* Γ_1 *and* Γ_2 *be the value groups of* K_1 *and* K_2*. Then the following are equivalent:*

(1) $K_1 \equiv K_2$.
(2) $R_{1,n} \equiv R_{2,n}$ *for every* $n \geq 1$ *and* $\Gamma_1 \equiv \Gamma_2$.

Question 1.5. *Let* K_1 *and* K_2 *be finitely ramified henselian valued fields mixed characteristic. Let* $\mathcal{H}_n(K_1)$ *and* $\mathcal{H}_n(K_2)$ *be the* n*-th valued hyperfields rings of* K_1 *and* K_2 *respectively. If* $\mathcal{H}_n(K_1) \equiv \mathcal{H}_n(K_2)$ *for every* $n \geq 1$*,* $K_1 \equiv K_2$*?*

In Section 2, we recall basic notations and facts. In Section 3, we answer Question 1.1 positively. We also compute such N effectively. To lift a homomorphism of the n-th valued hyperfields, we consider homomorphisms of valued hyperfields which are over p. In Subsection 3.1, it is enough to check whether there is an isomorphism over p between two first valued hyperfields in the tamely ramified case. More precisely, any homomorphism over p between the first valued hyperfields is induced from a unique homomorphism between given tamely ramified complete discrete valued fields. In Subsection 3.2, we compute such N in Question 1.1 depending only on ramification indices of given complete discrete valued fields. To compute N effectively, we first show that a homomorphism over p of the n-th valued gives a unique homomorphism between Witt rings of given complete discrete valued fields of mixed characteristic with perfect residue fields. Using the structure theorem of totally ramified extension and Krasner's lemma, we compute N effectively, which is depending only on the ramification index. In general, we can not drop the condition of being over p. In Section 4, we suggest a suitable category $\widehat{\mathcal{H}}_{p,e}^n$ of valued hyperfields whose morphisms are isometric homomorphism, and show that there is a lifting functor $\mathrm{L}^{\mathcal{H}} : \widehat{\mathcal{H}}_{p,e}^n \to \mathcal{C}_{p,e}$ satisfying proper conditions for large enough n. Actually, we show that this category $\widehat{\mathcal{H}}_{p,e}^n$ and the category $\mathcal{R}_{p,e}^n$ are equivalent for large enough n. This essentially comes from the result of J. Tolliver in [17]. We also show that the a subcategory $\mathcal{H}_{p,e}^n$ of $\widehat{\mathcal{H}}_{p,e}^n$ whose morphisms are over p is equivalent to $\mathcal{C}_{p,e}$ for every $n > e$ if p does not divide e. In Section 5, using lifting result of homomorphisms of n-th valued hyperfields in Section 3, we give a positive answer of Question 1.5 for the case of perfect residue fields. In this case, it is enough to check whether N-th valued hyperfields are elementary equivalent for large enough N. Specially, if K_1 and K_2 are tamely ramified and the first valued hyperfields are elementary equivalent, then K_1 and K_2 are elementary equivalent.

2. Preliminaries

We introduce basic notations and terminologies which will be used in this paper. We denote a valued field by a tuple $(K, R(K), \mathfrak{m}_K, \nu_K, k(K), \Gamma_K)$ consisting of the following data : K is the underlying field, $R(K)$ is the valuation ring, $\mathfrak{m}_K$ is the maximal ideal of $R(K)$, ν_K is the valuation map, k_K is the residue field, and Γ_K is the value group considering as an additive group. If there is no confusion, we omit K. Hereafter, the full tuple $(K, R, \mathfrak{m}, \nu, k, \Gamma)$ will be abbreviated in accordance with the situational need for the components. For $\gamma \in \Gamma$, we write $\Gamma_{*\gamma} := \{x \in \Gamma |\ x * \gamma\}$ for $* \in \{\geq, >\}$ and $\mathfrak{m}^\gamma := \{x \in K |\ \nu(x) \in \Gamma_{>\gamma}\}$. Note that for each $\gamma \in \Gamma_{\geq 0}$, $1+\mathfrak{m}^\gamma$ is a multiplicative subgroup of $K^\times$. For $\gamma \in \Gamma_{\geq 0}$, let $R_\gamma := R/\mathfrak{m}^\gamma$. We recall the definitions of ramified valued fields.

Definition 2.1. *We say that a valued field (K, ν, Γ) is* (absolutely) unramified *if $char(k) = 0$, or $char(k) = p$ and $\nu(p)$ is the minimal positive element in Γ for $p > 0$. We say (K, ν) is* (absolutely) ramified *if it is not absolutely unramified.*

Definition 2.2. *Let (K, ν, Γ) be a valued field whose residue field has prime characteristic p.*

(1) *We say (K, ν, Γ) is* (absolutely) finitely ramified *if the set $\{\gamma \in \Gamma |\ 0 < \gamma \leq \nu(p)\}$ is finite. The cardinality of $\{\gamma \in \Gamma |\ 0 < \gamma \leq \nu(p)\}$ is called* the (absolute) ramification index *of (K, ν), denoted by $e(K, \nu)$ or $e(R)$. If K or ν is clear from context, we write $e(K)$ or e for $e(K, \nu)$. For $x \in R$, we write $e_\nu(x) := |\{\gamma \in \Gamma |\ 0 < \gamma \leq \nu(x)\}|$. If there is no confusion, we write $e(x)$ for $e_\nu(x)$*

(2) *Let (K, ν, Γ) be finitely ramified. If p does not divide $e_\nu(p)$, we say (K, ν) is* (absolutely) tamely ramified. *Otherwise, we say (K, ν) is* (absolutely) wildly ramified.

Note that a discrete valued field having a residue field of characteristic $p > 0$ is finitely ramified. We say that a discrete valued field (K, ν, Γ) with the residue field having characteristic $p > 0$ is *normalized* if Γ is a subgroup of $\mathbb{R}$ and $\nu(p) = 1$. From now on, we mean a homomorphism between valued fields is an isometric homomorphism, where a field homomorphism $f : K_1 \to K_2$ is called *isometric* if for $a, b \in K_1$,

$$\nu_{K_1}(a) < \nu_{K_1}(b) \Leftrightarrow \nu_{K_2}(f(a)) < \nu_{K_2}(f(b)).$$

We recall some facts on the Witt ring and Teichmüller representatives of complete discrete valued fields of mixed characteristic(c.f. [16]).

Fact 2.3.

(1) *Let $W(k_1)$ and $W(k_2)$ be Witt rings of perfect residue fields k_1 and k_2 of characteristic $p > 0$ respectively. Suppose that there is a homomorphism $\phi : k_1 \longrightarrow k_2$. Then there is a unique lifting homomorphism $g : R_1 \longrightarrow R_2$ such that g induces ϕ.*

(2) *Let R be a complete discrete valuation ring of characteristic 0 with perfect residue field k of characteristic p and corresponding valuation ν. Then $\mathrm{W}(k)$ can be embedded as a subring of R and R is a free $\mathrm{W}(k)$-module of rank $\nu(p)$. Moreover, $R = \mathrm{W}(k)[\pi]$ where π is a uniformizer of R.*

(3) *Let A be a ring that is Hausdorff and complete for a topology defined by a decreasing sequence $\mathfrak{a}_1 \supset \mathfrak{a}_2 \supset \ldots$ of ideals such that $\mathfrak{a}_n \cdot \mathfrak{a}_m \subset \mathfrak{a}_{n+m}$. Assume that the residue ring $A_1 = A/\mathfrak{a}_1$ is a perfect field of characteristic p. Then:*

   (a) *There exists one and only one system of representatives $h : A_1 \longrightarrow A$ which commutes with p-th powers: $h(\lambda^p) = h(\lambda)^p$. This system of representatives is called the set of Teichmüller representatives.*

(b) *In order that $a \in A$ belong to $S = h(A_1)$, it is necessary and sufficient that a be a p^n-th power for all $n \geq 0$.*
(c) *This system of representatives is multiplicative which means*

$$h(\lambda\mu) = h(\lambda)h(\mu)$$

for all $\lambda, \mu \in A_1$.
(d) *S contains 0 and 1.*
(e) *$S \setminus \{0\}$ is a subgroup of the unit group of A.*

Next we introduce a notion of valued hyperfields.

Definition 2.4. [11, Definition 1.2 and 1.4]

(1) *A* hyeprfield *is an algebraic structure $(H, +, \cdot, 0, 1)$ such that $(H^\times, \cdot, 1)$, where $H^\times := H \setminus \{0\}$, is an abelian group and there is a multivalued operation $+ :\ H \times H \to 2^H$ for the power set 2^H of H satisfying the followings:*
  (a) *$0 \cdot \alpha = 0$ for all $\alpha \in H$.*
  (b) *(Associative) $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ for all $\alpha, \beta, \gamma \in H$.*
  (c) *(Commutative) $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in H$.*
  (d) *(Distributive) $(\alpha + \beta) \cdot \gamma \subset \alpha \cdot \gamma + \beta \cdot \gamma$ for all $\alpha, \beta, \gamma \in H$.*
  (e) *(Identity) $\alpha + 0 = \{\alpha\}$ for all $\alpha \in H$.*
  (f) *(Inverse) For any $\alpha \in H$, there is a unique $-\alpha \in H$ such that $0 \in \alpha + (-\alpha)$.*
  (g) *For all $\alpha, \beta, \gamma \in H$, $\alpha \in \beta + (-\gamma)$ if and only if $\beta \in \alpha + \gamma$.*

(2) *A* valued hyperfield *is a hyperfield $(H, +, \cdot, 0, 1)$ equipped with a map ν from H to $\Gamma \cup \{\infty\}$ for an ordered abelian group Γ such that*
  (a) *For $\alpha \in H$, $\nu(\alpha) = \infty$ if and only if $\alpha = 0$;*
  (b) *For all $\alpha, \beta \in H$, $\nu(\alpha \cdot \beta) = \nu(\alpha) + \nu(\beta)$;*
  (c) *For all $\alpha, \beta \in H$, $\nu(\alpha + \beta) \geq \min\{\nu(\alpha), \nu(\beta)\}$;*
  (d) *For all $\alpha, \beta \in H$, $\nu(\alpha + \beta)$ consists of single element unless $0 \in \alpha + \beta$; and*
  (e) *There is $\rho_H \in \Gamma$ such that either $\alpha + \beta$ is a closed ball of radius $\rho_H + \min\{\nu(\alpha), \nu(\beta)\}$ for all $\alpha, \beta \in H$, or $\alpha + \beta$ is a open ball of radius $\rho_H + \min\{\nu(\alpha), \nu(\beta)\}$ for all $\alpha, \beta \in H^\times$.*

For $B \subset H$ and $\alpha \in H$, define $\alpha + B := \bigcup_{\beta \in B} \alpha + \beta$ (*). The associativity of $+$ means that given $\alpha, \beta, \gamma \in H$, we have $(\alpha + \beta), (\beta + \gamma) \subset H$ and $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ in the sense of $(*)$. We say that H is discrete *if Γ is a discrete subgroup of $\mathbb{R}$.*

For $\alpha_0, \ldots, \alpha_k \in H$, we write $\sum^{\mathcal{H}} \alpha_i$ for $(\alpha_0 + \cdots + \alpha_k) \subset H$. Since the multivalued operation $+$ is associative, the notion of $\sum^{\mathcal{H}}$ is well-defined.

Definition 2.5. *Let $(H_i, +_i, \cdot_i, 0_i, 1_i, \nu_i)$ be a valued hyperfield for $i = 1, 2$. A map f from H_1 to H_2 is called a* homomorphsim *if the followings hold:*

(1) *$f(0_1) = 0_2$ and $f(1_2) = 1_2$.*
(2) *$f(\alpha \cdot_1 \beta) = f(\alpha) \cdot_2 f(\beta)$ for all $\alpha, \beta \in H_1$.*
(3) *$f(\alpha +_1 \beta) \subset f(\alpha) +_2 f(\beta)$ for all $\alpha, \beta \in H_1$.*
(4) *For all $\alpha, \beta \in H_1$,*

$$\nu_1(\alpha) \leq \nu_1(\beta) \Leftrightarrow \nu_2(f(\alpha_1)) \leq \nu_2(f(\beta)).$$

Let $\operatorname{Hom}(H_1, H_2)$ be the set of homomorphism from H_1 to H_2.

Note that the definition of homomorphisms in Definition 2.5 is weaker than one in [17, Definition 1.5]. We call homomorphisms in [17, Definition 1.5] an *isometric homomorphism*.

Definition 2.6. [17, Definition 1.5] *Let* $(H_i, +_i, \cdot_i, 0_i, 1_i, \nu_i)$ *be a valued hyperfield for* $i = 1, 2$. *Suppose* $\nu_1(H_1^\times) = \nu_2(H_2^\times)$. *A map* f *from* H_1 *to* H_2 *is called an* isometric homomorphism *if the followings hold:*

1. $f(0_1) = 0_2$ *and* $f(1_2) = 1_2$.
2. $f(\alpha \cdot_1 \beta) = f(\alpha) \cdot_2 f(\beta)$ *for all* $\alpha, \beta \in H_1$.
3. $f^{-1}(\alpha +_2 \beta) = f^{-1}(\alpha) +_1 f^{-1}(\beta)$ *for all* $\alpha, \beta \in f(H_1)$.
4. *For all* $\alpha \in H_1$, $\nu_1(\alpha) = \nu_2(f(\alpha))$.
5. $f^{-1}(1_2)$ *is a ball.*

Let $\mathrm{Iso}(H_1, H_2)$ *be the set of isometric homomorphisms from* H_1 *to* H_2. *Note that* (3) *implies that for* $\alpha, \beta \in H_1$ *and* $f \in \mathrm{Hom}(H_1, H_2)$, $f(\alpha +_1 \beta) \subset f(\alpha) +_2 f(\beta)$ *so that* $\mathrm{Iso}(H_1, H_2) \subset \mathrm{Hom}(H_1, H_2)$.

Definition 2.7. *Let* $\gamma \in \Gamma_{\geq 0}$. *The* γ*-hyperfield of* K *is a hyperfield* $(K/(1 + \mathfrak{m}^\gamma), +_{\mathcal{H}}, \cdot_{\mathcal{H}})$, *denoted by* $\mathcal{H}_\gamma(K)$, *such that for each* $a(1+\mathfrak{m}^\gamma), b(1+\mathfrak{m}^n) \in K/1+\mathfrak{m}^n$,

1. $a(1 + \mathfrak{m}^\gamma) \cdot_{\mathcal{H}} b(1 + \mathfrak{m}^\gamma) := ab(1 + \mathfrak{m}^\gamma)$,
2. $a(1 + \mathfrak{m}^\gamma) +_{\mathcal{H}} b(1 + \mathfrak{m}^\gamma) := \{(x + y)(1 + \mathfrak{m}^\gamma) |\ x \in a(1 + \mathfrak{m}^\gamma), y \in b(1 + \mathfrak{m}^\gamma)\}$, *and*

Conventionally, we write 0 *for* $0(1 + \mathfrak{m}^\gamma)$ *and* 1 *for* $1(1 + \mathfrak{m}^\gamma)$. *The valuation* ν *of* K *induces a map* $\nu_{\mathcal{H}}$ *on* $K/(1 + \mathfrak{m}^\gamma)$ *sending* $\nu_{\mathcal{H}}(a(1 + \mathfrak{m}^\gamma))$ *to* $\nu(a)$. *We call* $(K/(1 + \mathfrak{m}^\gamma), +_{\mathcal{H}}, \cdot_{\mathcal{H}}, \nu_{\mathcal{H}})$ *the* valued γ-hyperfield. *For* $\gamma \leq \lambda \in \Gamma_{>0}$, *we have that* $(1+\mathfrak{m}^\lambda) \subset (1+\mathfrak{m}^\gamma)$ *and it induces a projection* $\mathcal{H}_\gamma^\lambda :\ \mathcal{H}_\lambda(K) \to \mathcal{H}_\gamma(K), a(1+\mathfrak{m}^\lambda) \mapsto a(1 + \mathfrak{m}^\gamma)$. *We write* $\mathcal{H}_\gamma : K \to \mathcal{H}_\gamma(K), x \mapsto [x]_\gamma$.

For $A \subset K$, let $\mathcal{H}_\gamma(A) := \{a(1 + \mathfrak{m}^\gamma) :\ a \in A\}$ and for $a \in K$, we write $[a]_n$ for $a(1 + \mathfrak{m}^\gamma) \in \mathcal{H}_\gamma(K)$. If γ is obvious, we write $[a]$ for $[a]_\gamma$. Note that the valuation $\nu_{\mathcal{H}} : \mathcal{H}_\gamma^\times(K) \to \Gamma$ is a group epimorphism with the kernel $\mathcal{H}_\gamma(R^\times)$.

Remark 2.8. *The quotient group* $\mathcal{H}_\gamma^\times(K)/\mathcal{H}_\gamma(R^\times)$ *is an ordered group isomorphic to* Γ.

Definition 2.9. *Let* K_1 *and* K_2 *be valued fields whose residue fields are of characteristic* $p > 0$. *Let* $\mathcal{H}_\gamma(K_1)$ *and* $\mathcal{H}_\lambda(K_2)$ *be valued hyper fields of* K_1 *and* K_2 *respectively. We say a homomorphism* $f : \mathcal{H}_\gamma(K_1) \to \mathcal{H}_\lambda(K_2)$ *is* over p *if* $f([p]) = [p]$.

Note that there is an isometric isomorphism which is not over p(See Example 4.5). In Section 4, we see how two sets of isometric homomorphisms and of isometric homomorphisms over p are different(See Theorem 4.8 and Theorem 4.11).

Notation 2.10. *Let* (K, ν) *be a finitely ramified valued field and let* π *be a uniformizer of* K. *Let* $n \geq m$ *be positive integers. We write* $\mathfrak{m}^n := \mathfrak{m}^{\nu(\pi^{n-1})}$. *We write* $R_n(K) := R(K)/\mathfrak{m}^n$ *and* $\mathcal{H}_n(K) := K/(1 + \mathfrak{m}^n)$. *We call* $R_n(K)$ the n-th residue ring *of* K *and* $\mathcal{H}_n(K)$ the n-th valued hyperfield *of* K. *We write* $[x]_n := x(1 + \mathfrak{m}^n)$ *for* $x \in K$. *We denote* $\mathcal{H}_m^n$ *and* $\mathcal{H}_m$ *for* $\mathcal{H}_{\nu(\pi^{m-1})}^{\nu(\pi^{n-1})}$ *and* $\mathcal{H}_{\nu(\pi^{m-1})}$ *respectively.*

We now recall some results on hyperfields and (Deligne's) triples in [17].

Definition 2.11. [17, Definition 1.8] *A* truncated discrete valuation ring, *in short a truncated DVR, is a principal Artinian local ring. Let* R *be a truncated DVR of* *length l. For $x \in R$, we define $\nu_R(x) = \sup\{i \in \mathbb{N} | \ x \in \mathfrak{m}_R^i\}$, where $\mathfrak{m}_R$ is the maximal ideal of R. Then $\nu_R(R) = \{0, 1, \dots, l-1\} \cup \{\infty\}$, and $\nu_R(x) = \infty$ if and only if $x = 0$.*

Definition 2.12. [17, Definition 1.9] *A* (Deligne's) triple *(R, M, ϵ) consists of a truncated DVR R, a free R-module M of rank 1, and a R-module homomorphism $\epsilon : \ M \to R$ whose image is the maximal ideal of R.*

Definition 2.13. [17, Definition 1.10] *Let $T_1 = (R_1, M_1, \epsilon_1)$ and $T_2 = (R_2, M_2, \epsilon_2)$ be triples. A morphism of triples $(r, f, \eta) : T_1 \to T_2$ consists of a homomorphism $f : R_1 \to R_2$, an integer r, called the ramification index, and an R_1-module homomorphism $\eta : M_1 \to M_2^{\otimes r}$ such that*

- *$f \circ \epsilon_1 = \epsilon_2 \circ \eta$; and*
- *η induces a R_2-module isomorphism from $M_1 \otimes_{R_1} R_2$ to $M_2^{\otimes r}$.*

The composition of morphisms (r_1, f_2, η_1) and (r_2, f_2, η_2) of triples is given by

$$(r_1, f_2, \eta_1) \circ (r_2, f_2, \eta_2) = (r_1 r_2, f_1 \circ f_2, \eta_1^{\otimes r_2} \circ \eta_2).$$

Remark 2.14. [17, Remark 4.5] *Let $T = (R, M, \epsilon)$ be a triple. Let M be a free R-module of rank 1 and Π be a generator. For each $k \in \mathbb{Z}$, the tensor power $M^{\otimes k}$ is a well-defined R-module of rank 1. More precisely,*

$$M^{\otimes k} = \begin{cases} R(\Pi^{\otimes k}) & \text{if } k > 0 \\ R & \text{if } k = 0 \\ \mathrm{Hom}_R(M^{\otimes(-k)}, R) = R(\Pi^{\otimes k}) & \text{if } k < 0 \end{cases},$$

where for $k < 0$, $\Pi^{\otimes k} \in \mathrm{Hom}_R(M^{\otimes(-k)}, R)$ is a unique homomorphism sending $\Pi^{\otimes(-k)}$ to 1. Define a map $\nu_T : \bigcup_{k \in \mathbb{Z}} M^{\otimes k} \to \mathbb{Z} \cup \{\infty\}$, called a valuation map *of T, as follows: For $x = r\Pi^{\otimes k} \in M^{\otimes k}$, $\nu_T(x) = \nu_R(r) + k$.*

Notation 2.15. *Let (H, ν) be a discrete valued hyperfield. Denote $\theta_H := \min\{\nu(x) | \ \nu(x) > 0, \ x \in H\}$.*

Remark/Definition 2.16. [17, Remark 4.1] *Let H be a discrete valued hyperfield. There is a positive integer l such that $\rho_H = l\theta_H$. Such l is called the* length *of H, denoted by $l(H)$.*

Example 2.17. *For a discrete valued field K, we have that $l(\mathcal{H}_n(K)) = n$.*

Definition 2.18. *Let (H, ν) be a discrete valued hyperfield. For $\eta \in \mathbb{R}$, define an equivalence relation $\equiv_\eta$ on H as follows: For $\alpha, \beta \in H$, $\alpha \equiv_\eta \beta$ if and only if $\nu(\alpha - \beta) \geq \eta$. For $\alpha \in H$, we write $[\alpha]_\eta$ for the $\equiv_\eta$-class of α. If η is obvious, we omit it. Denote $\mathcal{O}_H := \{\alpha \in H | \ \nu(\alpha) > 0\}$ and $\mathfrak{m}_H := \{\alpha \in H | \ \nu(\alpha) \geq \theta_H\}$.*

Fact 2.19. [17, Sections 4 and 5] *Let H be a discrete valued hyperfield. Let $R = \mathcal{O}_H / \equiv_{\rho_H}$, $M = \mathfrak{m}_H / \equiv_{\rho_H + \theta_H}$, and $\epsilon : \ M \to R$, $[\alpha]_{\rho_H + \theta_H} \mapsto [\alpha]_{\rho_H}$ for $\alpha \in H$. Then,*

(1) $\mathrm{Tr}(H) = (R, M, \epsilon)$ is a triple, and
(2) R is of length $l(H)$ and the maximal ideal $\mathfrak{m}_R$ is $M / \equiv_{\rho_H}$.

We write $R_l(H)$ for R. If H or l is obvious, we write R_l or R for $R_l(H)$. For discrete valued hyperfields H' and H", and for $f \in \mathrm{Iso}(H, H')$ and $g \in \mathrm{Iso}(H', H'')$, we have that

(1) $\mathrm{Tr}(f)$ *and* $\mathrm{Tr}(g)$ *are homomorphisms between* $\mathrm{Tr}(H)$*,* $\mathrm{Tr}(H')$*, and* $\mathrm{Tr}(H'')$ *respectively, and*
(2) $\mathrm{Tr}(g \circ f) = \mathrm{Tr}(g) \circ \mathrm{Tr}(f)$.

Remark/Definition 2.20. *Let* H *be a discrete valued hyperfield and* $\mathrm{Tr}(H) = (R, M, \epsilon)$*. The set* $\mathcal{O}_H / \equiv_{\theta_H}$ *forms a field and it is isomorphic to* $R/\mathfrak{m}_R$*. The field* $\mathcal{O}_H / \equiv_{\theta_H}$ *is called the* residue field *of* H*, denoted by* $k(H)$*. Moreover, for each* $g \in \mathrm{Iso}(H_1, H_2)$*, it induces a homomorphism* $k(g) : k(H_1) \to k(H_2)$*.*

Proof. To show $\mathcal{O}_H / \equiv_{\theta_H}$ forms a field, mimic the proof of [17, Lemma 4.2]. And consider a map $f : \mathcal{O}_H / \equiv_{\theta_H} \to R$ sending $[\alpha]_{\theta_H}$ to $[\alpha]_{\rho_H} + \mathfrak{m}_R$. Since $\theta_H \le \rho_H$, f is well-defined and it induces an isomorphism. Let H_1 and H_2 be discrete valued hyperfield and let $g : H_1 \to H_2$ be an isometric homomorphism. Then it induces $\mathrm{Tr}(g) : \mathrm{Tr}(H_1) \to \mathrm{Tr}(H_2)$, which induces a homomorphism $k(g)$ from $k(H_1)$ to $k(H_2)$. □

Remark 2.21. [17, Section 6] *Let* $T = (R, M, \epsilon)$ *be a triple with the valuation map* ν_T*. Define* $\mathrm{U}(T) := \{0\} \cup \bigcup_{i \in \mathbb{Z}} \{x \in M^{\otimes i} | \ \nu_T(x) = i\}$*. Then* $\mathrm{U}(T)$ *is a discrete valued hyperfield. Moreover, if* $T = \mathrm{Tr}(H)$ *for a discrete valued hyperfield* H*, then* $\mathrm{U}(T) \cong H$ *by an isometric isomorphism after rescaling* $\nu_T(\Pi) = \theta_H$ *for a generator* Π *of* M*. Moreover the assignments* Tr *and* U *are funtorial. For discrete valued hyperfields* H_1 *and* H_2*, each* $f \in \mathrm{Iso}(H_1, H_2)$ *induces a morphism* $\mathrm{Tr}(f) \in \mathrm{Hom}(\mathrm{Tr}(H_1), \mathrm{Tr}(H_2))$*. For triples* T_1 *and* T_2*, each* $g \in \mathrm{Hom}(T_1, T_2)$ *induces a morphism* $\mathrm{U}(g) \in \mathrm{Iso}(\mathrm{U}(T_1), \mathrm{U}(T_2))$*.*

Example 2.22. [17, Example 4.9] *Let* K *be a discrete valued field. Let* $\mathcal{H}_n(K)$ *be the* n*-th valued hyperfield of* K *for a positive integer* n*. Then* $\mathrm{Tr}(\mathcal{H}_n(K)) \cong (R(K)/\mathfrak{m}_K^n, \mathfrak{m}_K/\mathfrak{m}_K^{n+1}, \epsilon)$*, where* $\mathfrak{m}_K$ *is the maximal ideal of the valuation ring* $R(K)$ *and the map* ϵ *is induced by the inclusion* $\mathfrak{m}_K \subset R(K)$*.*

We introduce some algebraic and model theoretic structural theorems in terms of the n-th valued rings for finitely ramified valued fields.

Remark/Definition 2.23. *Let* R *be a complete discrete valuation ring of mixed characteristic. Let* π *be a uniformizer of* R *and* ν *corresponding valuation of* R*. Let* L *and* K *be the fraction fields of* R *and* $\mathrm{W}(k)$ *respectively. We denote the maximal number*

$$\max\left\{\nu\big(\pi - \sigma(\pi)\big) : \sigma \in \mathrm{Hom}_K(L, L^{alg}), \sigma(\pi) \neq \pi\right\}$$

by $M(R)_\pi$ *or* $M(L)_\pi$*. This* $M(R)_\pi$ *does not depend on the choice of* π *and we write* $M(R) = M(L)$*.*

Let (F, ν_F) *be a complete discrete valuation ring of mixed characteristic having the same ramification index with* L*. Suppose there is an homomorphism from* L *to* F*. Then* $M(F) = M(L)$

Remark 2.24. [14, Theorem 3.9] *Let* (K_1, ν_1) *and* (K_2, ν_2) *be complete discrete valued fields of mixed characteristic* $(0, p)$*. Then we have that*

$$M(K_1) e_{\nu_1}(p) e_{\nu_2}(p) \le e_{\nu_2}(p)(1 + e_{\nu_1}^2(p)).$$

To determine whether two complete discrete valued fields of mixed characteristic with perfect residue fields are isomorphic, it is enough to check whether the n-th residue rings are isomorphic for large enough n.

Fact 2.25. [14] *Let K_1 and K_2 be complete discrete valued fields of mixed characteristic with perfect residue fields. Let $R_{1,n}$ and $R_{2,n}$ be the n-th residue rings of K_1 and K_2 respectively. For $n > e_{\nu_2}(p)(1+e^2_{\nu_1}(p))$, if $R_{1,n}$ and $R_{2,n}$ are isomorphic, then K_1 and K_2 are isomorphic. More generally, for positive integers n_1 and n_2 with $n_2 > e_{\nu_2}(p)(1+e^2_{\nu_1}(p))$, any homomorphism from R_{1,n_1} to R_{2,n_2} is lifted to a homomorphism from K_1 to K_2.*

For model theory of valued fields, we take the following languages for valued fields and their related structures. Let $\mathcal{L}_K = \{+,-,\cdot;0,1;|\}$ be a ring language with a binary relation $|$ for valued fields, where we interpret the binary relation $|$ as $a|b$ if $\nu(a) \le \nu(b)$ for $a, b \in K$. Let $\mathcal{L}_k = \{+', -', \cdot'; 0', 1'\}$ be the ring language for residue fields, and $\mathcal{L}_\Gamma = \{+^*; 0^*; <\}$ be the ordered group language for value groups. For each $n \le 1$, let $\mathcal{L}_{R_n} = \{+_n, -_n, \cdot_n; 0_n, 1_n\}$ be the ring language for the n-th residue ring. For $n = 1$, we identify $\mathcal{L}_{R_1} = \mathcal{L}_k$. We use a language $\mathcal{L}_{vhf} := \{0, 1, \cdot, +, |\}$ for valued hyperfields, where $0, 1$ are constant symbols, $\cdot$ is a binary function symbol, $+$ is a ternary predicate, and $|$ is a binary relation. For a valued hyperfield $(H, \cdot, +, \nu)$, 1^H is interpreted as the identity of the multiplication, 0^H as the identity of the addition, $\cdot^H$ as the multiplication function on H. For $\alpha, \beta, \gamma \in H$, $(\alpha, \beta, \gamma) \in +^H$ if and only if $\gamma \in \alpha + \beta$, and $(\alpha, \beta) \in |^H$ if and only if $\nu(\alpha) \le \nu(\beta)$. For the convention, $(\alpha, 0^H) \in |^H$ for all α in H.

J. Ax and S. Kochen in [1], and Y. Ershov in [8] independently proved that the first order theories of unramifieid valued fields of characteristic 0 are determined by the first order theories of their residue fields and valued groups.

Fact 2.26. [1, 8]*(Ax-Kochen-Ershov principle) Let (K_1, k_1, Γ_1) and (K_2, k_2, Γ_2) be unramified henselian valued fields of characteristic zero. Then we have that*

$$K_1 \equiv K_2 \text{ if and only if } k_1 \equiv k_2 \text{ and } \Gamma_1 \equiv \Gamma_2.$$

In [3], S. A. Basarab generalized Fact 2.26 to the case of finitely ramified valued fields and in [14], W. Lee and the author improved the result of Basarab in the case of perfect residue fields.

Fact 2.27. [3, 14] *Let $(K_1, \nu_1, k_1, \Gamma_1)$ and $(K_2, \nu_2, k_2, \Gamma_1)$ be henselian valued fields of mixed characteristic $(0, p)$ with finite ramification indices. Suppose k_1 and k_2 are perfect fields. Let $n > e_{\nu_2}(p)(1+e^2_{\nu_1}(p))$. The following are equivalent:*

(1) $K_1 \equiv K_2$;
(2) $\Gamma_1 \equiv \Gamma_2$ *and* $R_n(K_1) \equiv R_n(K_2)$.

3. Lifting

In this section, we aim to show that for large enough n, any homomorphism over p between the n-th valued hyperfields of discrete complete valued fields can be lifted to a homomorphism of given valued fields.

Lemma 3.1. *Let $a, b \in K$ and $a_0, \ldots, a_k \in K$. Fix $\gamma \in \Gamma_{\ge 0}$.*

(1) $\bigcup [a]_\gamma = a(1+\mathfrak{m}^\gamma) = \{x| \ \nu(x-a) \ge \gamma + \nu(a)\}$.
(2) $\bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma = \{x \in K| \ \nu(x-(a+b)) > \gamma + \min\{\nu(a), \nu(b)\}\}$.
(3) $0 \in \bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma$ *if and only if* $\bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma = \mathfrak{m}^{\gamma+\min\{\nu(a),\nu(b)\}}$.
(4) $(a_0 + \ldots + a_k) \in \sum^{\mathcal{H}} [a_i]_\gamma$.
(5) Suppose $b \in \bigcup \sum^{\mathcal{H}} [a_i]_\gamma$ *and* $a_0, \ldots, a_k \in R$. *Then* $b = (a_0 + \ldots + a_k) + d$ *for some* $d \in \mathfrak{m}^\gamma$.

Proof. Let $a, b \in K$ and $a_0, \dots, a_k \in K$.

(1) For $x \in K$, we have that

$$\begin{aligned} x \in a(1+\mathfrak{m}^\gamma) &\Leftrightarrow x = a + ad,\ d \in \mathfrak{m}^\gamma \\ &\Leftrightarrow \nu(x-a) = \nu(ad) = \nu(a) + \nu(d), d \in \mathfrak{m}^\gamma \\ &\Leftrightarrow \nu(x-a) > \nu(a) + \gamma. \end{aligned}$$

(2) ($\subseteq$) Let $x \in \bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma$ so that thee are $c, d \in \mathfrak{m}^\gamma$ such that $x = a + b + ac + bd$. Then we have that

$$\nu(x-(a+b)) = \nu(ac+bd) \geq \min\{\nu(ac), \nu(bd)\} > \gamma + \min\{\nu(a), \nu(b)\}.$$

($\supseteq$) Take $x \in K$ such that $\nu(x-(a+b)) > \gamma + \min\{\nu(a), \nu(b)\}$. WLOG we may assume that $\nu(a) \geq \nu(b)$. Then there is $c \in \mathfrak{m}^\gamma$ such that $x = (a+b)+bc = a+b(1+c)$, and $[x]_\gamma \in [a]_\gamma +_{\mathcal{H}} [b]$. So, we conclude that $x \in \bigcup [a]_\gamma +_G [b]_\gamma$.

(3) By (2), $\bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma = (a+b) + \mathfrak{m}^{\gamma + \min\{\nu(a), \nu(b)\}}$. Thus we have that

$$\begin{aligned} 0 \in \bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma &\Leftrightarrow \nu(a+b) > \gamma + \min\{\nu(a), \nu(b)\} \\ &\Leftrightarrow (a+b) \in \mathfrak{m}^{\gamma + \min\{\nu(a), \nu(b)\}} \\ &\Leftrightarrow \bigcup [a]_\gamma +_{\mathcal{H}} [b]_\gamma = \mathfrak{m}^{\gamma + \min\{\nu(a), \nu(b)\}}. \end{aligned}$$

(4) and (5) come from (2). □

Proposition 3.2. *Let K_1 and K_2 be valued fields whose residue fields are of characteristic $p > 0$. Let $\mathcal{H}_\gamma(K_1)$ and $\mathcal{H}_\lambda(K_2)$ be valued hyper fields of K_1 and K_2 respectively. Let $f \in \mathrm{Hom}(\mathcal{H}_\gamma(K_1), \mathcal{H}_\lambda(K_2))$ be over p. Then for all $n \in \mathbb{Z}$, $f([n]) = [n]$. We denote $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_\gamma(K_1), \mathcal{H}_\lambda(K_2))$ for the set of all homomorphisms over p.*

Proof. Let $f \in \mathrm{Hom}(\mathcal{H}_\gamma(K_1), \mathcal{H}_\lambda(K_2))$ be over p. First, we have $f([1]) = [1]$ because $[1]$ is the multiplicative identity. Since $[-1]$ is the additive inverse of $[1]$, $f([-1]) = [-1]$. By Lemma 3.1(2), we have $f([k]) = [k]$ for $1 \leq k \leq p-1$ inductively. Now choose an integer n arbitrary. Since $f([-1]) = [-1]$ and $f([0]) = [0]$, we may assume that $n > 0$. Suppose n and p are relatively prime. We write $n = a_0 + a_1 p + a_2 + \ldots + a_m p^m$ with $0 \leq a_i \leq p-1$ and $a_0 \neq 0$ for some $m \geq 0$. Suppose $f([n]) = [n']$ for some $n' \in K_2$. Then we have that

$$\begin{aligned} [n'] &= f([n]) = f([\sum_i a_i p^i]) \\ &\in \sum_i^{\mathcal{H}} f([a_i p^i]) = \sum_i^{\mathcal{H}} f([a_i]) f([p])^i = \sum_i^{\mathcal{H}} [a_i][p]^i = \sum_i^{\mathcal{H}} [a_i p^i]. \end{aligned}$$

By Lemma 3.1(5), we have that $n' = \sum_i a_i p^i + d = n + d$ for some $d \in \mathfrak{m}_2^\lambda$. Since $n \notin \mathfrak{m}_2$, we have that $n'/n = 1 + d/n \in 1 + \mathfrak{m}_2^\lambda$ and $n'(1+\mathfrak{m}_2^\lambda) = n(1+\mathfrak{m}_2^\lambda)$. Now suppose p divides n. Write $n = n_0 p^l$ for some $l > 0$ and for some n_0 coprime to p. Then we have that $f([n]) = f([n_0 p^l]) = f([n_0]) f([p^l]) = [n_0][p]^l = [n_0 p^l] = [n]$. □

Remark 3.3. *Let (K_1, ν_1, k_1) and (K_2, ν_2, k_2) be finitely ramified valued fields having the same ramification index e. Let $p = char(k_1) = char(k_2) > 0$. Suppose ν_1 and ν_2 are normalized, that is, $\nu_1(p) = \nu_2(p) = 1$. Then for any $n, m \geq 1$ and $f \in$*

$\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2))$, we have that $\nu_1(\alpha) = \nu_2(f(\alpha))$ for every $\alpha \in \mathcal{H}_n(K_1)$. And any homomorphism from K_1 to K_2 induces an isometric homomorphism from $\mathcal{H}_n(K_1)$ to $\mathcal{H}_n(K_2)$.

In the remaining part of this section, **we assume that a complete discrete complete valued field of mixed characteristic $(0,p)$ has the normalized valuation so that $\nu(K^\times) \subset \mathbb{R}$ and $\nu(p) = 1$.** For a henelian valued field (K, ν) of characteristic 0, there is a unique valuation on K^{alg} extending ν and we use the same notion ν for this valuation on K^{alg}.

Remark 3.4. *Let S be the set of Teichmüller representatives of a complete discrete valued field K of mixed characteristic having a perfect residue field. Then, $\mathcal{H}_1(S)$ is a field which is isomorphic to the residue field of K.*

Proof. Let $H = (\mathcal{H}_1(K), +_1, \times_1)$ be the first valued hyperfield of K. For each $a, b \in S$, there is a unique $c \in S$ such that $[c] \in [a] +_1 [b]$. So, $(\mathcal{H}_1(S), +, \times_1)$ forms a field where for $a, b \in S$, $[a] + [b] = [c]$ if $[c] \in [a] +_1 [b]$. Consider a map sending $[a]$ to $a + \mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal of the valuation ring of K and this induces an isomorphism from $\mathcal{H}_1(S)$ to the residue field of K. □

Lemma 3.5. *Let K_1 and K_2 be complete discrete valued fields of mixed characteristic having perfect residue fields. Let S_1 and S_2 be the set of Teichmüller representatives of K_1 and K_2 respectively, and let $f \in \mathrm{Hom}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2))$.*

(1) We have $f(\mathcal{H}_n(S_1)) \subset \mathcal{H}_m(S_2)$ and $f \restriction_{\mathcal{H}_n(S_1)}$ is injective.
(2) If f is over p, then $f \restriction_{\mathcal{H}_n(\mathrm{W}(k_1))}$ is induced by a unique homomorphism from $\mathrm{W}(k_1)$ to $\mathrm{W}(k_2)$, where $\mathrm{W}(k_1)$ and $\mathrm{W}(k_2)$ are Witt subrings of K_1 and K_2 respectively.

Proof. (1) The proof is similar to the proof of [14, Lemma 3.3]. Let R_1 and R_2 be the valuation rings of K_1 and K_2 respectively. By Fact 2.3(3), S_i is contained in $\mathrm{W}(k_i)^\times$ where k_i is the residue field of K_i for $i = 1, 2$. For each $\lambda \in S_1$, let η_s be any representative of $f(\lambda^{1/p^s}(1 + \mathfrak{m}_1^n))$ so that $(\eta_s)^{p^s}(1 + \mathfrak{m}_2^m) = f(\lambda(1 + \mathfrak{m}_1^n))$. Since η_s is in $\mathrm{W}(k_2)^\times$, we have that $\eta_s(1 + \mathfrak{m}_2^m) = \eta_s + \mathfrak{m}_2^m$ and $(\eta_s)^{p^s}(1 + \mathfrak{m}_2^m) = (\eta_s)^{p^s} + \mathfrak{m}_2^m$ as a set. For any other representative θ_s of $f(\lambda^{1/p^s}(1 + \mathfrak{m}_1^n))$, we have that $\eta_s + \mathfrak{m}_2^m = \theta_s + \mathfrak{m}_2^m$. If we write $\eta_s = \theta_s + \pi_2^{n_2} a$ for some a in R_2, the following binomial expansion

$$\begin{aligned}\eta_s^{p^s} &= (\theta_s + \pi_2^{n_2} a)^{p^s} \\ &= \theta_s^{p^s} + p^s \theta_s^{p^s - 1} \pi_2^n a + \dots + (\pi_2^n a)^{p^s}\end{aligned}$$

shows $\eta_s^{p^s} - \theta_s^{p^s} \in \mathfrak{m}_2^s$. Since η_{s+1}^p is a representative of $f(\lambda^{1/p^s}(1 + \mathfrak{m}_1^n))$, the calculation above shows that $(\eta_s^{p^s})$ is a Cauchy sequence and $\lim_{s\to\infty} \eta_s^{p^s}$ is well-defined in R_2. Since $\eta_s^{p^s}(1+\mathfrak{m}_2^m) = f(\lambda(1+\mathfrak{m}_1^n))$ and $1+\mathfrak{m}_2^m$ is topologically closed in R_2,

$$f\left(\lambda(1 + \mathfrak{m}_1^n)\right) = \left(\lim_{s\to\infty} \eta_s^{p^s}\right)(1 + \mathfrak{m}_2^m).$$

Similarly, we have

$$f\left((\lambda)^{1/p}(1 + \mathfrak{m}_1^n)\right) = \left(\lim_{s\to\infty} \eta_s^{p^{s-1}}\right)(1 + \mathfrak{m}_2^m).$$

Since

$$\lim_{s\to\infty} \eta_s^{p^s} = \left(\lim_{s\to\infty} \eta_s^{p^{s-1}}\right)^p,$$

we obtain
$$\lim_{s\to\infty}\eta_s^{p^s}\in S_2$$
by 2.3(3). Therefore, we have that $f(\mathcal{H}_n(S_1))\subset\mathcal{H}_m(S_2)$.

By Remark 3.4, the restriction map $f\restriction\mathcal{H}_1(S_1)$ induces an homomorphism between residue fields of K_1 and K_2. So, we have that $f\restriction\mathcal{H}_1(S_1)$ is trivial or injective. Since $1\in S_1$, we have that $f([1]_n)=[1]_m$ and $f\restriction\mathcal{H}_1(S_1)$ is not trivial. Therefore, $f\restriction\mathcal{H}_1(S_1)$ must be injective.

(2) Suppose f is over p, that is, $f([p]_n)=[p]_m$. Note that each $a\in \mathrm{W}(k_i)$ is uniquely written as $\sum\limits_{k\geq 0}a_kp^k$ for $a_k\in S$ for $i=1,2$. By Fact 2.3(1), Remark 3.4, and (1), we have a homomorphism $\bar f:\mathrm{W}(k_1)\to\mathrm{W}(k_2)$ such that $f([a]_n)=[\bar f(a)]_m$ for $a\in S_1$. Take $a\in\mathrm{W}(k_1)$ and write $a=p^l\sum\limits_{k\geq 0}a_kp^k$ for $l\geq 0$ and $a_k\in S_1$ with $a_0\neq 0$. We have that
$$\begin{aligned}f([a]_n)&=[p]_m^l f([\sum_{k\geq 0}a_kp^k])_m\\&=[p]_m^l f([a_0+a_1p+a_2p^2+\cdots+a_sp^s])_m,\end{aligned}$$
where $s=\max\{n,m\}$. It is enough to show that
$$f([\sum_{0\leq k\leq s}a_kp^k]_n)=[\sum_{0\leq k\leq s}\bar f(a_k)p^k]_m$$
for $a_0\neq 0$ and $s\geq\max\{n,m\}$. Take $a=\sum\limits_{0\leq k\leq s}a_kp^k\in\mathrm{W}(k_1)$ with $a_0\neq 0$ and $s\geq\max\{n,m\}$. Then,
$$\begin{aligned}f([a])&\in\sum_{0\leq k\leq s}^{\mathcal{H}}f([a_k]_n)[p]_m^k\\&=\sum_{0\leq k\leq s}^{\mathcal{H}}[\bar f(a_k)]_m[p]_m^k\\&=\sum_{0\leq k\leq s}^{\mathcal{H}}[\bar f(a_k)p^k]_m.\end{aligned}$$
So, we have that
$$\begin{aligned}f([a]_n)&=[\sum_{0\leq k\leq s}\bar f(a_k)p^k+d]_m\\&=[\bar f(a)+d]\end{aligned}$$
for some $d\in\mathfrak{m}_2^m$ by Lemma 3.1(5). Since $(\bar f(a)+d)/\bar f(a)=1+d/\bar f(a)\in 1+\mathfrak{m}_2^m$, we have that $[\bar f(a)+d]_m=[\bar f(a)]_m$. Therefore, $f([a]_n)=[\bar f(a)]_m$ for each $a\in\mathrm{W}(k_1)$. □

3.1. **Tamely ramified case.** We first recall some embedding lemma in [12] for tame algebraic extensions of valued fields. We say that an algebraic extension (L,ν_L) of a henselian valued field (K,ν_K) is *tame* if the following conditions hold: For every finite subextension (F,ν_F) of (L,ν_L) over (K,ν),

- the residue field extension k_L over k_K is separable;

- if the characteristic of k_K is $p > 0$, then the ramification index $(\Gamma_F : \Gamma_K)$ is prime to p; and
- $[F : K] = [k_F : k_K](\Gamma_F : \Gamma_K)$.

If K is a complete discrete tamely ramified valued field with a perfect residue field k, then K is the tame extension of the fraction field of $\mathrm{W}(k)$.

Fact 3.6. [12, Lemma 3.1] *Let K be an arbitrary valued field. Let L be an algebraic tame extension of some henselization of K and F be an arbitrary henselian extension of K. Any K_1-embedding from L_1 to F_1 is induced from a K-embedding from L to F. Furthermore if $k_K = k_L$, then any (group) G^1_K-embedding from G^1_L to G^1_F is induced from a K-embedding from L to F.*

By adapting the ideas of the proofs of [12, Lemma 3.1], we prove the following theorems.

Theorem 3.7. *Let K_1 and K_2 be complete discrete valued fields of mixed characteristic $(0, p)$ with perfect residue fields. Suppose K_1 is tamely ramified. Any homomorphism over p from $\mathcal{H}_n(K_1)$ to $\mathcal{H}_m(K_2)$ is induced from a unique homomorphism from K_1 to K_2. From this, we conclude that there is one-to-one correspondence between $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2))$ and $\mathrm{Hom}(K_1, K_2)$.*

Proof. Take $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2))$. Let F_i be the fraction field of $\mathrm{W}(k_i)$ where k_i is the residue field of K_i for $i = 1, 2$. By Lemma 3.5(2), $f \restriction_{\mathcal{H}_1(F_1)}$ is induced from a homomorphism $\bar{f} : F_1 \to F_2$. Since K_1 is a totally tamely ramified extension of F_1, $K_1 = F_1(\sqrt[e]{pa})$ for some a in $\mathrm{W}(k_1)^\times$ where e is the ramification index of K_1 (c.f. Chapter 2 of [13]). Let $\pi_1 = \sqrt[e]{pa}$ and $f([\pi_1]_n) = [\pi_2']_m$ for some $\pi_2' \in K_2$ so that $[\pi_2']^e_m = [p\bar{f}(a)]_m$. Consider a polynomial $P(X) = X^e - (p\bar{f}(u))/\pi_2'^e \in K_2[X]$. Since $P(1) \in \mathfrak{m}_2^m$ and $P'(1) \notin \mathfrak{m}_2$, by Hensel's lemma, there is unique $b \in R_2^\times$ such that $b^e = (p\bar{f}(a))/\pi_2'^e$ and $(b - 1) \in \mathfrak{m}_2^m$. Let $\pi_2 := b\pi_2'$ so that $\pi_2^e = p\bar{f}(a)$. Note that $[\pi_2]_m = [\pi_2']_m$ and π_2 is such a unique zero of the polynomial $X^e - p\bar{f}(a)$. So we have a homomorphism $\tilde{f} : K_1 \to K_2$ extending $\bar{f} \cup \{(\pi_1, \pi_2)\}$ and it induces f. □

We generalize Theorem 3.7 to the case of infinitely tamely ramified valued fields. To do this, we first recall the Ax-Sen-Tate theorem.

Definition 3.8. [2] *Let (K, ν) be a henselian valued field and (K^{alg}, ν) be the algebraic closure of K. For $a \in K^{alg}$, define*

$$\Delta_K(a) := \min\{\nu(\sigma(a) - a)|\ \sigma \in G_K\},$$

where G_K is the Galois group of K^{alg} over K.

Fact 3.9. [2, Proposition 1, Proposition 2'] *Let (K, ν) be a complete valued field and (K^{alg}, ν) be the algebraic closure of K. Suppose $\Gamma_{K^{alg}}$ is archimedean. Let $K \subset F \subset K^{alg}$.*

1. *Suppose K is of mixed characteristic $(0, p)$. Then for all $a \in K^{alg}$, there exists $b \in F$ such that*
$$\nu(a - b) \geq \Delta_F(a) - (p/(p - 1)^2)\nu(p).$$
2. *Suppose K is of equal characteristic $p \geq 0$. Then for all $a \in K^{alg}$ and for all $\gamma \in (\Gamma_{K^{alg}})_{>0}$, there exists b in the perfect closure of F such that*
$$\nu(a - b) \geq \Delta_F(a) - \gamma.$$

Fact 3.10 (Ax-Sen-Tate Theorem)**.** [9, Proposition 3.8] *Let (K, ν) be a complete valued field and (K^{alg}, ν) be the algebraic closure of K with $\Gamma_{K^{alg}}$ archimedean. Let C be the completion of K^{alg} which is algebraically closed and let L be a perfect complete subfield of C containing K. Then L is the completion of $L \cap K^{alg}$.*

Proof. See [9, Proposition 3.8] with Fact 3.9. □

Corollary 3.11. *Let p be a prime number. Let K be a tamely ramified valued field of mixed characteristic $(0, p)$ with a perfect residue field k, and F be the fraction field of $\mathrm{W}(k)$. Suppose K is a subfield of the completion of F^{alg}, and either*

- *K is an algebraic extension of F, or*
- *complete.*

For a complete valued field L of mixed characteristic $(0, p)$ with a perfect residue field, any homomorphism over p from $\mathcal{H}_\gamma(K)$ to $\mathcal{H}_\lambda(L)$ is induced from a unique homomorphism K to L for any $\gamma \in (\Gamma_K)_{\geq 0}$ and $\lambda \in (\Gamma_L)_{\geq 0}$.

Proof. Let L be a complete valued field of mixed characteristic $(0, p)$ with a perfect residue field. Let F be the fraction field of $\mathrm{W}(k)$. Fix $\gamma \in (\Gamma_K)_{\geq 0}$ and $\lambda \in (\Gamma_L)_{\geq 0}$.

Suppose K is an algebraic extension of F. Since K is tamely ramified, $K = \bigcup K_i$ where K_i is a tamely totally ramified finite extension of F. Take $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_\gamma(K), \mathcal{H}_\lambda(L))$. Set $f_i := f \restriction_{\mathcal{H}_\gamma(K_i)}$ and we have $f := \varprojlim_{K_i} f_i = \bigcup f_i$. By Theorem 3.7, f_i is induced from a unique homomorphism $\sigma_i : K_i \to L$. Take $\sigma := \varprojlim_{K_i} \sigma_i = \bigcup \sigma_i : K \to L$ and it is a unique homomorphism inducing f.

Suppose K is complete. By Fact 3.10, K is the completion of $K' := K \cap F^{alg}$. Let $f' = f \restriction_{\mathcal{H}_\gamma(K')}$. By above result, f' is induced from a unique homomorphism $\tilde{f}' : K' \to L$. Since K is a completion of K', $\tilde{f}'$ induces a unique homomorphism $\tilde{f} \to L$, which induces f. □

We can not drop the condition of being over p in Theorem 3.7.

Example 3.12. *Consider $K_1 = \mathbb{Q}_3(\sqrt{3})$ and $K_2 = \mathbb{Q}_3(\sqrt{-3})$, which are not isomorphic by Kummer Theory. Note that their residue fields are isomorphic to $\mathbb{F}_3$ and so their Teichmüller representatives are $\{-1, 0, 1\}$. Let $\pi_1 = \sqrt{3}$ and $\pi_2 = \sqrt{-3}$. Every elements of K_1 is of the form $\sum_{i \geq n} a_i \pi_1^n$ for some integer n and $a_i \in \{-1, 0, 1\}$ with $a_n \neq 0$. So each elements in $\mathcal{H}_1(K_1)$ is of the form $\pi_1^n a_n(1+\mathfrak{m}_1)$ for some integer n and $a_n \in \{-1, 1\}$(†). The same formulas hold for K_2. We will show that $\mathrm{Hom}(\mathcal{H}_1(K_1), \mathcal{H}_1(K_2)) = \{f_0, f_1\}$, where f_i sends $[a]$ to $[a]$ for $a \in \{-1, 1\}$, $[\pi_1]$ to $[(-1)^i \pi_2]$, and $[3]$ to $[-3]$ for $i = 1, 2$, so that $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_1(K_1), \mathcal{H}_1(K_2)) = \emptyset$. Note that $[1] \neq [-1]$. Let $f \in \mathrm{Hom}(\mathcal{H}_1(K_1), \mathcal{H}_1(K_2))$. Then $f([-1]) = [-1]$ because $f([1]) = [1]$ and $0 \in f([1] + [-1])$. Since π_1 is an uniformizer, $f([\pi_1]) = [\pi_2]$ or $= [-\pi_2]$. In both cases, $f([3]) = f([\pi_1]^2) = [\pi_2]^2 = [\pi_2^2] = [-3]$. By (†), such f induces an isomorphism between $\mathcal{H}_1(K_1)$ and $\mathcal{H}_1(K_2)$.*

Without base fields, even the residue fields of K_1 and K_2 are primes fields so that the residue fields are equal, we can not lift a group homomorphism from $\mathcal{H}_1^\times(K_1)$ to $\mathcal{H}_1^\times(K_2)$ to a homomorphism from K_1 to K_2(c.f. Fact 3.6).

Example 3.13. *Let $K_1 = K_2 = \mathbb{Q}_5$. The set of Teichmüller representatives of $\mathbb{Q}_5$ is $\{0, i, i^2, i^3, i^4\}$ where $i = \sqrt{-1}$. Consider a group isomorphism $f : \mathcal{H}_1^\times(\mathbb{Q}_5) \to$*

$\mathcal{H}_1^\times(\mathbb{Q}_5)$ by mapping $[i] \mapsto [-i]$ and $[5] \mapsto [5]$. Then f is never induced from an automorphism of $\mathbb{Q}_5$ since any automorphism of $\mathbb{Q}_5$ sends i to i.

3.2. **Generally ramified case.** We first introduce of a notion of lifting map of homomorphisms of the n-th valued hyperfields, which is an analogy to a lifting map of homomorphisms of the n-th residue rings in [14, Definition 3.1].

Definition 3.14. *Let K_1 and K_2 be complete discrete valuation rings of characteristic 0 with perfect residue fields k_1 and k_2 of characteristic p respectively. Let π_i be a uniformizer of K_i and ν_i be a corresponding valuation of K_i for $i = 1, 2$. For any homomorphism $\phi : \mathcal{H}_n(K_1) \to \mathcal{H}_m(K_2)$, we say that a homomorphism $g : K_1 \to K_2$ is a (n, m)*-lifting *of ϕ at π_1 if g satisfies the following:*

- *There exists a representaive b of $\phi([\pi_1])$ which satisfies*
$$\nu_2\big(g(\pi_1) - b\big) > M(K_1).$$
- *$\phi_{red,1} \circ \mathcal{H}_1 = \mathcal{H}_1 \circ g$ where $\phi_{red,1} : \mathcal{H}_1(K_1) \to \mathcal{H}_1(K_2)$ denotes the natural reduction map of ϕ.*

When such g is unique, we denote g by $\mathrm{L}^{\mathcal{H}}_{\pi_1,n,m}(\phi)$. When $\mathrm{L}^{\mathcal{H}}_{\pi_1,n,m}(\phi)$ exists for all $\phi : \mathcal{H}_n(K_1) \to \mathcal{H}_m(K_2)$, we write $\mathrm{L}^{\mathcal{H}}_{\pi_1,n,m} : \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2)) \to \mathrm{Hom}(K_1, K_2)$. When $n = m$, we briefly write $\mathrm{L}^{\mathcal{H}}_{\pi_1,n,m} = \mathrm{L}^{\mathcal{H}}_{\pi_1,n}$ and say that $\mathrm{L}^{\mathcal{H}}_{\pi_1,n}$ is an n-lifting at *π_1.*

The following result is analogous to Proposition 2.9(2) for n-th residue rings in [14].

Proposition 3.15. *Let K_1 and K_2 be complete discrete valued fields of characteristic 0 with perfect residue fields k_1 and k_2 of characteristic p respectively. Let π_i be a uniformizer of K_i and ν_i be a corresponding valuation of K_i for $i = 1, 2$. Let R_i be the valuation ring of K_i for $i = 1, 2$. The definition of liftings is independent of the choice of uniformizer of K_1. More precisely, saying that $g : K_1 \to K_2$ is a (n, m)-lifting of $\phi : \mathcal{H}_n(K_1) \to \mathcal{H}_m(K_2)$ at π_1 is equivalent to the following:*

1. *For any x in R_1, there exists a representative b_x of $\phi(x(1 + \mathfrak{m}_1^n))$ which satisfies*
$$\nu_2\big(g(x) - b_x\big) > M(K_1).$$
2. *$\phi_{red,1} \circ \mathcal{H}_1 = \mathcal{H}_1 \circ g$*

We write $\mathrm{L}^{\mathcal{H}}_{\pi_1,n,m} = \mathrm{L}^{\mathcal{H}}_{n,m}$ and say that $\mathrm{L}^{\mathcal{H}}_{n,m}$ is a (n, m)-lifting. *Moreover, there is at most one (n, m)-lifting for $m > M(K_1)e_2$, where e_2 is the ramification index of K_2.*

Proof. The proof is similar to the proof of [14, Proposition 3.5 (2)]. Let S_i be the Teichmüller representatives of K_i for $i = 1, 2$. Fix a uniformizer π of K_1. Let $g : K_1 \to K_2$ be a (n, m)-lifting of $\phi : \mathcal{H}_n(K_1) \to \mathcal{H}_m(K_2)$ at a uniformizer π. Let $b \in K_2$ be a representative of $\phi([\pi]))$ such that $\nu_2(g(x) - b) > M(K_1)$. Note that $\nu_2(b) > 0$. Take $x \in R_1$. Then $x = \sum_{i\geq 0} \lambda_i\pi^i$. Take $l > 0$ such that $\nu_1(\sum_{i>l} \lambda_i\pi^i) > M(K_1)$. Denote $x^{\leq l} := \sum_{i\leq l} \lambda_i\pi^i$ and $x^{>l} \sum_{i>l} \lambda_i\pi^i$ so that

$x = x^{\leq l} + x^{>l}$. Then we have that

$$\begin{aligned}\phi([x]) &= [x^{\leq l} + x^{>l}])\\ &\in \big(\phi([x^{\leq l}] +_{\mathcal{H}} \phi([x^{>l}])\big)\\ &\subset \Big(\sum_{i\leq l}^{\mathcal{H}} \phi([\lambda_i])\phi([\pi]^i) + \phi([x^{>l}])\Big)\\ &\subset \Big(\sum_{i\leq l}^{\mathcal{H}} [g(\lambda_i)][b]^i + \phi([x^{>l}])\Big).\end{aligned}$$

There is a representative b_x of $\phi([x])$ of the form:

$$b_x = \sum_{i\leq l} g(\lambda_i) b^i + d$$

for some $\nu_2(d) > M(K_1)$. Compute

$$\begin{aligned}\nu_2(g(x) - b_x) &= \nu_2\big(\sum_i (g(\lambda_i)g(\pi)^i) - b_x\big)\\ &= \nu_2\big(\sum_{0<i\leq l} g(\lambda_i)(g(\pi)^i - b^i) + (\sum_{i>l} g(\lambda_i)g(\pi)^i - d)\big)\\ &\geq \min\{\nu_2(\sum_{0<i\leq l} g(\lambda)(g(\pi)^i - b^i)), \nu_2(\sum_{i>l} g(\lambda_i)g(\pi)^i - d)\}\\ &> M(K_1)\end{aligned}$$

because $\nu_2(g(\pi)^i - \beta^i) = \nu_2(g(\pi) - \beta) + \nu_2(g(\pi)^{i-1} + \cdots + \beta^{i-1}) > M(K_1)$, and $\nu_2(\sum_{i>l} g(\lambda_i)g(\pi)^i), \nu_2(d) > M(K_1)$.

Now we show moreover part. Assume $m > M(K_1)e_2$ and there are two (n,m)-liftings $\mathrm{L}, \mathrm{L}' : \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2)) \to \mathrm{Hom}(K_1, K_2)$. Fix $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2))$. By Lemma 3.5 and (2), we have that $\mathrm{L}(f) \restriction \mathrm{W}(k_1) = \mathrm{L}'(f) \restriction \mathrm{W}(k_1)$ (†). It remains to show that $\mathrm{L}(f)(\pi_1) = \mathrm{L}'(f)(\pi_1)$. Set $\pi := \mathrm{L}(f)(\pi_1)$ and $\pi' := \mathrm{L}'(f)(\pi_1)$. By (†), π and π' are conjugates over the fraction field of $\mathrm{W}(k_2)$ (‡). By (1), there are two representatives b and b' of $f([\pi_1])$ such that

$$\nu_2(\pi - \beta), \nu_2(\pi' - \beta') > M(K_1).$$

Since $\nu_2(b - b') > m/e_2 > M(K_1)$, we have that

$$\begin{aligned}\nu_2(\pi - \pi') &> \max\{\nu_2(\pi - b), \nu_2(b - b'), \nu_2(\pi' - b')\}\\ &> M(K_1).\end{aligned}$$

By (‡), we conclude that $\nu_2(\pi - \pi') = \infty$ and $\pi = \pi'$. □

Remark 3.16. *Let K_1 and K_2 be complete discrete valued field of mixed characteristic $(0,p)$ with perfect residue fields. Suppose K_1 is tamely ramified. By Theorem 3.7, for every $n \geq 1$ there is a unique bijective n-lifting map*

$$\mathrm{L}_n^{\mathcal{H}} : \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_n(K_2)) \to \mathrm{Hom}(K_1, K_2)$$

such that $f([a]_n) = [\mathrm{L}_n^{\mathcal{H}}(f)(a)]$ for $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_n(K_2))$ and $a \in K_1$.

Fact 3.17 (Krasner's lemma)**.** *Let (K,ν) be henseilan valued field whose value group is contained in $\mathbb{R}$ and let $a,b \in K^{alg}$. Suppose a is separable over $K(b)$. Suppose that for all embeddings $\sigma (\neq id)$ of $K(a)$ over K, we have*

$$\widetilde{\nu}(b-a) > \widetilde{\nu}\big(\sigma(a)-a\big).$$

Then $K(a) \subset K(b)$.

We show that a lifting map of homomorphisms of n-th valued hyperfields for large enough n

Theorem 3.18. *Let K_1 and K_2 be complete discrete valued field of mixed characteristic $(0,p)$ with perfect residue fields. Let e_1 and e_2 be ramification indices of K_1 and K_2 respectively. Let $\mathcal{H}_n(K_1)$ and $\mathcal{H}_m(K_2)$ be valued hyper fields of K_1 and K_2 respectively. Suppose $m > M(K_1)e_1e_2$. There is a unique lifting map $\mathrm{L}^{\mathcal{H}}_{n,m} : \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2)) \to \mathrm{Hom}(K_1, K_2)$.*

Proof. Fix a homomorphism f in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1), \mathcal{H}_m(K_2))$. By Lemma 3.5, f induces a map $f \restriction_{\mathcal{H}_n(S_1)}$ from $\mathcal{H}_n(S_1)$ to $\mathcal{H}_m(S_2)$. This map $f \restriction_{\mathcal{H}_n(S_1)}$ induces a homomorphism from k_1 to k_2 and by the functoriality of Witt ring, we have a homomorphism $\bar{f} : \mathrm{W}(k_1) \to \mathrm{W}(k_2)$. Note that $\bar{f}$ induces a homomorphism from $\mathrm{W}(k_1)[X]$ to $\mathrm{W}(k_2)[X]$ by acting on coefficients and we denote this homomorphism by $\bar{f}$ also. By Fact 2.3(2), there is a uniformizer π_1 of K_1 such that $R_1 = \mathrm{W}(k_1)[\pi_1]$. Let $q(X)$ be the irreducible polynomial of π_1 over the fraction field of $\mathrm{W}(k_1)$, which is in $\mathrm{W}(k_1)[X]$. Write $q(X) = X^e + a_{e-1}X^{e-1} + \ldots + a_0$ where $e = \nu_1(p)$ is the ramification index of K_1. Let $\pi_2' \in R_2$ such that $[\pi_2']_m = f([\pi_1]_n)$. For $a \in \mathrm{W}(k_1)$, we can uniquely write $a = a_0 + a_1 p + a_2 p^2 + \ldots$ with $a_i \in S_1$. For $l \geq 0$, define $a^{\leq l} := a_0 + a_1 p + \ldots + a_l p^l$ and $a^{>l} := a - a^{\leq l}$. And define $q^{\leq l}(X) := X^e + a_{e-1}^{\leq l} X^{e-1} + \ldots + a_0^{\leq l}$ and $q^{>l}(X) = q(X) - q^{\leq l}(X)$, which are in $\mathrm{W}(k_1)[X]$. Then we have that

$$\begin{aligned}
0 &= f([q(\pi_1)]_n) \\
&= f([q^{\leq m}(\pi_1) + q^{>m}(\pi_1)]_n) \\
&\in f([q^{\leq m}(\pi_1)]_n) +_{\mathcal{H}} f([q^{>m}(\pi_1)]_n) \\
&\subset [\pi_2'^{e}]_m +_{\mathcal{H}} f[a_{e-1}^{\leq m}\pi_2'^{e-1}]_m +_{\mathcal{H}} \ldots +_{\mathcal{H}} f[a_0^{\leq m}]_m +_{\mathcal{H}} f([q^{>m}(\pi_1)]_n) \\
&\subset [\pi_2'^{e}]_m +_{\mathcal{H}} \sum_{0\leq i<e}^{\mathcal{H}} \sum_{0\leq j\leq m}^{\mathcal{H}} [\pi_2'^{i} \bar{f}(a_i^j) p^j]_m +_{\mathcal{H}} f([q^{>m}(\pi_1)]_n).
\end{aligned}$$

Note that $\nu_2(q^{>m}(\pi_1)) \geq m/e_2$. By Lemma 3.1(2), we have that

$$0 = \bar{f}(q^{\leq m})(\pi_2') + d \quad (*)$$

for some $d \in \mathfrak{m}_2^m$. Since $q = q^{\leq m} + q^{>m}$, we have that $\bar{f}(q)(\pi_2') = \bar{f}(q^{\leq m})(\pi_2') + \bar{f}(q^{>m})(\pi_2')$ and $\nu_2(\bar{f}(q^{>m})(\pi_2')) \geq m/e_2$. From $(*)$ we have that

$$0 = \bar{f}(q)(\pi_2') + d'$$

for some $d' \in \mathfrak{m}_2^m$, and $\nu_2(\bar{f}(q)(\pi_2')) \geq m/e_2 > M(K_1)e_1$. By Krasner's lemma, there is $\pi_2 \in K_2$ such that $\bar{f}(q)(\pi_2) = 0$ with $\nu_2(\pi_2 - \pi_2') > M(K_1)$. Define $\tilde{f} : K_1 \to K_2, \pi_1 \mapsto \pi_2$ extending $\bar{f}$, and set $\mathrm{L}^{\mathcal{H}}_{n,m}(f) = \tilde{f}$. $\square$

Corollary 3.19. *Let (K_1,ν_1,k_1,Γ_1) and (K_2,ν_2,k_2,Γ_1) be finitely ramified complete valued fields of mixed characteristic. Let $n > e_{\nu_2}(p)(1+e_{\nu_1}^2(p))$. The following are equivalent:*

1. *$K_1 \cong K_2$;*
2. *$R_n(K_1) \cong R_n(K_2)$; and*
3. *$\mathcal{H}_n(K_1) \cong_{\mathcal{H}_n(\{p\})} \mathcal{H}_n(K)$.*

The following is an analogy of [14, Proposition 4.4] for a lifting map of homomorphisms of n-th valued hyperfields, which gives a funtoriality of lifting map in Section 4.

Proposition 3.20. *Let (K_1,ν_1), (K_2,ν_2), and (K_3,ν_3) be complete discrete valued fields of mixed characteristic $(0,p)$ with perfect residue fields. Suppose K_1 and K_2 have the same ramification index. Suppose $m,k > \max\{e_{\nu_2}(p)(1+e_{\nu_1}^2(p), e_{\nu_3}(p)(1+e_{\nu_2}^2(p))\}$. Let $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1),\mathcal{H}_m(K_2))$ and $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_m(K_2),\mathcal{H}_k(K_3))$. Then $\mathrm{L}_{n,k}^{\mathcal{H}}(g\circ f) = \mathrm{L}_{m,k}^{\mathcal{H}}(g)\circ \mathrm{L}_{n,m}^{\mathcal{H}}(f)$.*

Proof. Take $f \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_n(K_1),\mathcal{H}_m(K_2))$ and $g \in \mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_m(K_2),\mathcal{H}_k(K_3))$. First, we have that

$$\begin{aligned}
(g\circ f)\circ \mathcal{H}_1 &= g\circ (f\circ \mathcal{H}_1)\\
&= g\circ (\mathcal{H}_1\circ \mathrm{L}_{n,m}^{\mathcal{H}}(f))\\
&= (g\circ \mathcal{H}_1)\circ \mathrm{L}_{n,m}^{\mathcal{H}}(f)\\
&= (\mathcal{H}_1\circ \mathrm{L}_{m,k}^{\mathcal{H}}(g))\circ \mathrm{L}_{n,m}^{\mathcal{H}}(f)\\
&= \mathcal{H}_1\circ (\mathrm{L}_{m,k}^{\mathcal{H}}(g)\circ \mathrm{L}_{n,m}^{\mathcal{H}}(f)).
\end{aligned}$$

It remains to show that for a uniformizer π_1 of K_1, there is a representative b of $g\circ f([\pi_1])$ such that

$$\nu_3(\mathrm{L}_{m,k}^{\mathcal{H}}(g)\circ \mathrm{L}_{n,m}^{\mathcal{H}}(f)(\pi_1) - b) > M(K_1).$$

Take a representative b_1 of $f([\pi_1])$ such that $\nu_2(\mathrm{L}_{n,m}^{\mathcal{H}}(f)(\pi_1)-b_1) > M(K_1)$, and we have that $\nu_3(\mathrm{L}_{m,k}^{\mathcal{H}}(g)\circ\mathrm{L}_{n,m}^{\mathcal{H}}(f)(\pi_1)-\mathrm{L}_{m,k}^{\mathcal{H}}(g)(b_1)) > M(K_1)$. Next consider a representative b_2 of $g([b_1])(= g(f([\pi_1])))$ such that $\nu_3(\mathrm{L}_{m,k}^{\mathcal{H}}(g)(b_1)-b_2) > M(K_2)$. Since K_1 and K_2 have the same ramification indices, we have that $M(K_1) = M(K_2)$. Thus we conclude that $\nu_3(\mathrm{L}_{m,k}^{\mathcal{H}}(g)\circ\mathrm{L}_{n,m}^{\mathcal{H}}(f)(\pi_1)-b_2) > M(K_1)$, and set $b := b_2$. □

4. Valued hyperfields, truncated DVRs, and valued fields

In this section, we figure out relationships between valued hyperfields, truncated DVRs, and complete discrete valued fields of mixed characteristic. We first define a notion of finitely ramified valued hyperfields.

Definition 4.1. *Let (H,ν) be a discrete valued hyperfield of length l. Let $R_l(H) = R$. Suppose the residue field $k(H)$ has the characteristic $p \geq 0$.*

1. *We say H is of* equal characteristic (p,p) *if $p = 0$ in R. Otherwise, we say H is of* mixed characteristic $(0,p)$.
2. *Suppose H is* of mixed characteristic $(0,p)$. *We say H is* finitely ramified *if $\nu_R(p) = e < \infty$, and e is called the* ramification index *of H.*
3. *Suppose H is finitely ramified. We say H is normalized if $\theta_H = 1/e$ where e is the ramification index of H.*

Note that any discrete valued hyperfield of mixed characteristic need not be finitely ramified.

Proposition 4.2. *Let K be a discrete valued field. Let K_1, K_2, and K_3 be discrete valued fields of mixed characteristic $(0,p)$ having the same ramification index e.*

(1) $T_n(K) = (R_n(K), M_n(K), \epsilon_n)$ is a triple, where $M_n(K) := \mathfrak{m}_K/\mathfrak{m}_K^{n+1}$ and the map ϵ_n is induced by the inclusion $\mathfrak{m}_K \subset R(K)$.
(2) For every triple $T = (R_n(K), M, \epsilon)$, we have $T \cong T_n(K)$.
(3) Let $n > e$. Each morphism $f \in \mathrm{Hom}(R_n(K_1), R_n(K_2))$ induces a morphism $T_n(f) \in \mathrm{Hom}(T_n(K_1), T_n(K_2))$. For $f \in \mathrm{Hom}(R_n(K_1), R_n(K_2))$ and $g \in \mathrm{Hom}(R_n(K_2), R_n(K_3))$, we have $T_n(g \circ f) = T_n(g) \circ T_n(f)$.

Proof. (1) It is clear.

(2) Let $R := R_n(K)$. Given a triple $T = (R, M, \epsilon)$, let Π be a generator of M. Since $\epsilon(M) = \mathfrak{m}_R$, $\epsilon(\Pi) = \pi + \mathfrak{m}_K$ for a uniformizer π of K. Define a map $\eta : M \to \mathfrak{m}_K/\mathfrak{m}_K^{n+1}$, $\Pi \mapsto \pi/\mathfrak{m}_K^{n+1}$. Then the triple $(1, \mathrm{Id}_R, \eta)$ gives an isomorphism from T to $T_n(K)$.

(3) Let $f \in \mathrm{Hom}(R_n(K_1), R_n(K_2))$. Let π_1 and π_2 be uniformizers of K_1 and K_2 respectively. By the choice of n, $f(\pi_1/\mathfrak{m}_{K_1}^n)$ generates the maximal ideal of $R_n(K_2)$ so that $f(\pi_1/\mathfrak{m}_{K_1}^n) = (a_f\pi_2)/\mathfrak{m}_{K_2}^{n+1}$ for a unit $a_f \in R(K_2)^\times$. Define a map $\eta_f : M_n(K_1) \to M_n(K_2)$ sending $\pi_1/\mathfrak{m}_{K_1}^{n+1}$ to $(a_f\pi_2)/\mathfrak{m}_{K_2}^{n+1}$. Then the triple $T_n(f) := (1, f, \eta_f)$ gives a morphism from $T_n(K_1)$ to $T_n(K_2)$. Note that $T_n(f)$ does not depend on the choices of π_1 and π_2 because $M_n(K_1)$ and $M_n(K_2)$ are free $R_n(K_1)$ and $R_n(K_2)$-modules of rank 1 respectively.

Now we show T_n is commute with the composition. Let π_3 be a uniformizer of K_3. Fix $f \in \mathrm{Hom}(R_n(K_1), R_n(K_2))$ and $g \in \mathrm{Hom}(R_n(K_2), R_n(K_3))$. There are $a_f \in R(K_2)^\times$, $b_g, b_{g\circ f} \in R(K_3)^\times$ such that $f(\pi_1/\mathfrak{m}_{K_1}^n) = (a_f\pi_2)/\mathfrak{m}_{K_2}^n$, $g(\pi_2/\mathfrak{m}_{K_1}^n) = (b_g\pi_3)/\mathfrak{m}_{K_3}^n$, and $g \circ f(\pi_1/\mathfrak{m}_{K_1}^n) = (b_{g\circ f}\pi_3)/\mathfrak{m}_{K_3}^n$. Let $b_f \in R(K_3)^\times$ such that $g(a_f/\mathfrak{m}_{K_2}^n) = b_f/\mathfrak{m}_{K_3}^n$. Then we have that $b_{g\circ f}/\mathfrak{m}_{K_3}^n = (b_g b_f)/\mathfrak{m}_{K_3}^n$. By choosing b_g and b_f properly, we may assume that $b_g b_f - b_{g\circ f} \in \mathfrak{m}_{K_3}^{n+1}$ (†). Define $\eta_{g\circ f} : M_n(K_1) \to M_n(K_3), \pi_1/\mathfrak{m}_{K_1}^{n+1} \mapsto (b_{g\circ f}\pi_3)/\mathfrak{m}_{K_3}^{n+1}$. By (†), we have that $\eta_{g\circ f} = \eta_g \circ \eta_f$. Therefore,

$$\begin{aligned} T_n(g \circ f) &= (1, g \circ f, \eta_{g\circ f}) \\ &= (1, g \circ f, \eta_g \circ \eta_f) \\ &= (1, g, \eta_g) \circ (1, f, \eta_f) \\ &= T_n(g) \circ T_n(f). \end{aligned}$$

□

Corollary 4.3. *Let H be a finitely ramified discrete valued hyperfield of mixed characteristic with the perfect residue field and let e be the ramification index. Suppose $l(:= l(H)) > e(1 + \nu_R(e))$ so that e is not zero in $R(:= R_l(H))$. Then there is a unique complete discrete valued field K (up to isomorphic) such that $\mathcal{H}_l(K) \cong H$.*

Remark/Definition 4.4. *Let H_1 and H_2 be finitely ramified discrete valued hyperfield of mixed characteristic $(0,p)$ with perfect residue fields. Let $\mathrm{Tr}(H_1) = (R_1, M_1, \epsilon_1)$ and $\mathrm{Tr}(H_2) = (R_2, M_2, \epsilon_2)$, and we identify $H_1 = \mathrm{U}(\mathrm{Tr}(H_1))$ and*

$H_2 = \mathrm{U}(\mathrm{Tr}(H_2))$. Suppose they have the same length l and the same ramification index e so that $n = \nu_1(e) = \nu_2(e) \in \{0, 1, \ldots, l-1\} \cup \{\infty\}$, where $\nu_1 = \nu_{R_1}$ and $\nu_2 = \nu_{R_2}$. Suppose $l > e$. Since $l > e$, p is not zero in R_l and R_2 and $\nu_1(p) = \nu_2(p) = e$. So $p = a_1\pi_1^e$ in R_1 and $p = a_2\pi_2^e$ in R_2 for some units a_1, a_2 and some uniformizers π_1, π_2 in R_1 and R_2 respectively. Take $\Pi_i \in M_i$ such that $\epsilon_i(\Pi_i) = \pi_i$ for $i = 1, 2$. Then $\nu_{\mathrm{Tr}(H_i)}(a_i\Pi_i^{\otimes e}) = e$ for $i = 1, 2$. We say a homomorphism $f : H_1 \to H_2$ is over p *if $f(a_1\Pi_1^{\otimes e}) = a_2\Pi_2^{\otimes e}$. Denote $\mathrm{Hom}_{\mathbb{Z}}(H_1, H_2)$ for the set of all homomorphisms from H_1 to H_2, which are over p.*

Example 4.5. *Let $K_1 = K_2 = \mathbb{Q}_3(\sqrt{3})$ and $\mathfrak{m}$ be the maximal ideal of the valuation ring $\mathbb{Z}_3[\sqrt{3}]$. Note that $R := R_4(K_1) = R_4(K_2) \cong (\mathbb{Z}_3/9\mathbb{Z}_3)[x]/(x^2-3)$. Then there is an isomorphism $f : a + bx \mapsto a + 4bx$ in $\mathrm{Hom}(R_4(K_1), R_4(K_2))$. Then f induces an isomorphism $T_4(f) : (R, \mathfrak{m}/\mathfrak{m}^4, \epsilon_4) \to (R, \mathfrak{m}/\mathfrak{m}^4, \epsilon_4)$ and it induces an isometric isomorphism $U(T_4(f)) : \mathcal{H}_4(K_1) \to \mathcal{H}_4(K_2), \sqrt{3}(1+\mathfrak{m}^4) \mapsto 4\sqrt{3}(1+\mathfrak{m}^4)$, which is not in $\mathrm{Hom}_{\mathbb{Z}}(\mathcal{H}_4(K_1), \mathcal{H}_4(K_2))$. Suppose $U(T_4(f))$ is over p. Then $3(1+\mathfrak{m}^4) = U(T_4(f))(3(1+\mathfrak{m}^4)) = U(T_4(f))((\sqrt{3})^2(1+\mathfrak{m}^4)) = (4\sqrt{3})^2(1+\mathfrak{m}^4) = 4^2 3(1+\mathfrak{m}^4)$. So, we have $(1+\mathfrak{m}^4) = 4^2(1+\mathfrak{m}^4)$, which is impossible, because $15 \notin \mathfrak{m}^4 = 9\mathbb{Z}_3$.*

Now we introduce some categories of valued hyperfields, truncated DVRs, and valuation rings and we study relationships between them. We recall two categories of truncated DVRs and valuation rings, which were used to generalize the functoriality of unramified valuation rings in [14, Section 4]. For a prime number p and a positive integer e, let $\mathcal{C}_{p,e}$ be a category consisting of the following data :

- $\mathrm{Ob}(\mathcal{C}_{p,e})$ is the family of complete discrete valuation rings of mixed characteristic having perfect residue fields of characteristic p and the ramification index e; and
- $\mathrm{Mor}_{\mathcal{C}_{p,e}}(R_1, R_2) := \mathrm{Hom}(R_1, R_2)$ for R_1 and R_2 in $\mathrm{Ob}(\mathcal{C}_{p,e})$.

Let $\mathcal{R}_{p,e}^n$ be a category consisting of the following data :

- For $n \le e$, $\mathrm{Ob}(\mathcal{R}_{p,e}^n)$ is the family of truncated DVRs $\overline{R}$ of length n with perfect residue fields of characteristic p, and for $n > e$, $\mathrm{Ob}(\mathcal{R}_{p,e}^n)$ is the family of truncated DVRs $\overline{R}$ of length n with perfect residue fields of characteristic p such that $p \in \overline{\mathfrak{m}}^e \setminus \overline{\mathfrak{m}}^{e+1}$ where $\overline{\mathfrak{m}}$ is the maximal ideal of $\overline{R}$; and
- $\mathrm{Mor}_{\mathcal{R}_{p,e}^n}(\overline{R_1}, \overline{R_2}) := \mathrm{Hom}(\overline{R_1}, \overline{R_2})$ for $\overline{R_1}$ and $\overline{R_2}$ in $\mathrm{Ob}(\mathcal{R}_{p,e,}^n)$,

Note that for $e_1, e_2 \ge 1$ and for $n \le e_1, e_2$, two categories $\mathcal{R}_{p,e_1}^n, \mathcal{R}_{p,e_2}^n$ are the same. For each $m > n$, let $\mathrm{Pr}_n : \mathcal{C}_{p,e} \to \mathcal{R}_{p,e}^n$ and $\mathrm{Pr}_n^m : \mathcal{R}_{p,e}^m \to \mathcal{R}_{p,e}^n$ be the canonical projection functors respectively. Given a prime number p and a positive integer e, let $l_{p,e} := e(1+\nu(e))$ for some(every) $R \in \mathrm{Ob}(\mathcal{C}_{p,e})$. Note that for $p \not| e$, we have that $l_{p,e} = e$.

Fact 4.6. [14, Definition 4.2, Theorem 4.7] *Fix a prime number p and a positive integer e. For every $n > l_{p,e}$, there is a functor $\mathrm{L} : \mathcal{R}_{p,e}^n \longrightarrow \mathcal{C}_{p,e}$, called an n-th* lifting functor*, which satisfies the following:*

1. *$(\mathrm{Pr}_n \circ \mathrm{L})(\overline{R}) \cong \overline{R}$ for each $\overline{R}$ in $\mathrm{Ob}(\mathcal{R}_{p,e}^n)$.*
2. *$\mathrm{Pr}_1 \circ \mathrm{L}$ is equivalent to Pr_1^n.*
3. *$\mathrm{L} \circ \mathrm{Pr}_n$ is equivalent to $\mathrm{Id}_{\mathcal{C}_{p,e}}$.*

Moreover, there is a unique n-th lifting functor L satisfying

(4) *For each* $g \in \mathrm{Mor}_{\mathcal{R}^n_{p,e}}(\overline{R}_1, \overline{R}_2)$ *and for any* $x \in \mathrm{L}(\overline{R}_1)$, *there is a representative* b_x *of* $(\mathrm{Pr}_n \circ \mathrm{L})(g)(x + \mathfrak{m}_1^n)$ *such that*
$$\nu_2(\mathrm{L}(g)(x) - b_x) > M(\mathrm{L}(\overline{R}_1)).$$

Next we introduce two categories of valued hyperfields. Let $\mathcal{H}^n_{p,e}$ be a category consisting of the following data :

- $\mathrm{Ob}(\mathcal{H}^n_{p,e})$ is the family of discrete normalized valued hyperfield of length n and mixed characteristic $(0,p)$ having perfect residue fields, and for $n > e$ in addition, having ramification indices e; and
- For $n \le e$, $\mathrm{Mor}_{\mathcal{H}^n_{p,e}}(H_1, H_2) := \mathrm{Iso}(H_1, H_2)$ and for $n > e$, $\mathrm{Mor}_{\mathcal{H}^n_{p,e}}(H_1, H_2) := \mathrm{Iso}_{\mathbb{Z}}(H_1, H_2)$, where $\mathrm{Iso}_{\mathbb{Z}}(H_1, H_2) = \mathrm{Hom}_{\mathbb{Z}}(H_1, H_2) \cap \mathrm{Iso}(H_1, H_2)$ for H_1 and H_2 in $\mathrm{Ob}(\mathcal{H}^n_{p,e})$.

Let $\widehat{\mathcal{H}}^n_{p,e}$ be a category consisting of the following data :

- $\mathrm{Ob}(\widehat{\mathcal{H}}^n_{p,e}) = \mathrm{Ob}(\mathcal{H}^n_{p,e})$; and
- $\mathrm{Mor}_{\widehat{\mathcal{H}}^n_{p,e}}(H_1, H_2) := \mathrm{Iso}(H_1, H_2)$ for H_1 and H_2 in $\mathrm{Ob}(\widehat{\mathcal{H}}^n_{p,e})$.

By Remark/Definition 2.20, we have functors $\mathrm{Res}^n_{p,e} : \mathcal{H}^n_{p,e} \to \mathcal{R}^1_{p,e}$ and $\widehat{\mathrm{Res}} : \widehat{\mathcal{H}}^n_{p,e} \to \mathcal{R}^1_{p,e}$. Next we introduce lifting functors for two categories $\mathcal{H}^n_{p,e}$ and $\widehat{\mathcal{H}}^n_{p,e}$, which are analogous to the lifting functors of $\mathcal{R}^n_{p,e}$ in [14, Definition 4.2].

Definition 4.7. *Fix a prime number p and a positive integer e. We say that the category $\mathcal{C}_{p,e}$ is* n-$\mathcal{H}$-liftable *if there is a functor* $\mathrm{L}^{\mathcal{H}} : \mathcal{H}^n_{p,e} \to \mathcal{C}_{p,e}$ *which satisfies the following:*

- $(\mathcal{H}_n \circ \mathrm{L}^{\mathcal{H}})(\overline{R}) \cong \overline{R}$ *for each* $\overline{R}$ *in* $\mathrm{Ob}(\mathcal{R}^n_{p,e})$.
- $\mathrm{Pr}_1 \circ \mathrm{L}^{\mathcal{H}}$ *is equivalent to* $\mathrm{Res}^n_{p,e}$.
- $\mathrm{L}^{\mathcal{H}} \circ \mathcal{H}_n$ *is equivalent to* $\mathrm{Id}_{\mathcal{C}_{p,e}}$.

We say that $\mathrm{L}^{\mathcal{H}}$ *is an* n-th $\mathcal{H}$-lifting functor *of* $\mathcal{C}_{p,e}$.

Now we see relationships between the categories $\mathcal{C}_{p,e}$, $\mathcal{R}^n_{p,e}$, $\mathcal{H}^n_{p,e}$, and $\widehat{\mathcal{H}}^n_{p,e}$. Define a functor $\widehat{\mathrm{U}} : \mathcal{R}^n_{p,e} \to \widehat{\mathcal{H}}^n_{p,e}$ as follows:

- For $\overline{R} \in \mathrm{Ob}(\mathcal{R}^n_{p,e})$, $\widehat{\mathrm{U}}(\overline{R}) := (\mathrm{U} \circ T_n \circ \mathrm{L})(\overline{R})$ after rescaling $\theta_{(\mathrm{U} \circ T_n \circ \mathrm{L})(\overline{R})} = 1/e$, where L is the n-th lifting in Fact 4.6; and
- For $f \in \mathrm{Mor}_{\mathcal{R}^n_{p,e}}(\overline{R}_1, \overline{R}_2)$, $\widehat{\mathrm{U}}(f) := (\mathrm{U} \circ \mathrm{T}_n)(f)$,

and define a functor $\widehat{\mathrm{To}} : \widehat{\mathcal{H}}^n_{p,e} \to \mathcal{R}^n_{p,e}$ as follows:

- For $H \in \mathrm{Ob}(\widehat{\mathcal{H}}^n_{p,e})$, $\widehat{\mathrm{To}}(H) := \overline{R}$ where $\overline{R}$ is a truncated DVR in $\mathrm{Tr}(H)$.
- For $g \in \mathrm{Mor}_{\widehat{\mathcal{H}}^n_{p,e}}(H_1, H_2)$, $\widehat{\mathrm{To}}(g)$ is a morphism in $\mathrm{Mor}_{\mathcal{R}^n_{p,e}}(\widehat{\mathrm{To}}(H_1), \widehat{\mathrm{To}}(H_1))$ such that $\mathrm{Tr}(f) = (1, \widehat{\mathrm{To}}(g), \eta)$.

By Remark 2.21 and Proposition 4.2, we have the following result.

Theorem 4.8. *Let p be a prime number and e be a positive integer. Fix $n > e$.*

(1) $\widehat{\mathrm{U}} \circ \widehat{\mathrm{To}}$ *is equivalent to* $\mathrm{Id}_{\widehat{\mathcal{H}}^n_{p,e}}$.

(2) $\widehat{\mathrm{To}} \circ \widehat{\mathrm{U}}$ *is equivalent to* $\mathrm{Id}_{\mathcal{R}^n_{p,e}}$

Therefore, two categories $\widehat{\mathcal{H}}^n_{p,e}$ and $\mathcal{R}^n_{p,e}$ are equivalent.

Remark 4.9. [14, Proposition 4.9] *Let $R_1/\mathrm{W}(k)$ and $R_2/\mathrm{W}(k)$ be totally ramified extensions of degree e. Then $R_{1,e}$ is isomorphic to $R_{2,e}$ as $\mathrm{W}(k)$-algebras. So, the assumption that $n > e$ is natural in Theorem 4.8.*

Each $R \in \mathrm{Ob}(\mathcal{C}_{p,e})$ gives a discrete valued hyperfield $\mathcal{H}(R) := \mathcal{H}_n(K)$ in $\mathrm{Ob}(\mathcal{H}^n_{p,e})$ where K is the fraction field of R after rescaling $\theta_{\mathcal{H}_n(R)} = 1/e$. Also each $f \in \mathrm{Mor}_{\mathcal{C}_{p,e}}(R_1, R_2)$ induces a morphism $\mathcal{H}(f) : \mathcal{H}(R_1) \to \mathcal{H}(R_2)$. So we have a functor $\mathcal{H} : \mathcal{C}_{p,e} \to \mathcal{H}^n_{p,e}$. For $n > l_{p,e}$, we define a functor $\mathrm{L}^{\mathcal{H}} : \mathcal{H}^n_{p,e} \to \mathcal{C}_{p,e}$. Since $\mathcal{H}^n_{p,e}$ is a subcategory of $\widehat{\mathcal{H}}^n_{p,e}$, we have a functor $\mathrm{To} : \mathcal{H}^n_{p,e} \to \mathcal{R}^n_{p,e}$ by restricting $\widehat{\mathrm{To}}$ to $\mathcal{H}^n_{p,e}$. For each $H \in \mathrm{Ob}(\mathcal{H}^n_{p,e})$, there is a unique(up to isomorphic) complete valued field K such that $H \cong \mathcal{H}_n(K)$, where K is the fraction field of $(\mathrm{L} \circ \mathrm{To})(H)$ by Corollary 3.19. For $H_1, H_2 \in \mathrm{Ob}(\mathcal{H}^n_{p,e})$ and $f \in \mathrm{Mor}_{\mathcal{H}^n_{p,e}}(H_1, H_2)$, we have a morphism $\mathrm{L}^{\mathcal{H}}(f) \in \mathrm{Mor}_{\mathcal{C}_{p,e}}(R_1, R_2)$ where $R_1 = (\mathrm{L} \circ \mathrm{To})(H_1)$ and $R_2 = (\mathrm{L} \circ \mathrm{To})(H_2)$ by Theorem 3.18. By Proposition 3.20, $\mathrm{L}^{\mathcal{H}}$ forms a functor. By Remark 3.3, Theorem 3.18, and Proposition 3.20, we have the following result which is analogous to Fact 4.6 in the case of valued hyperfields.

Theorem 4.10. *Let p be a prime number and e be a positive integer. Fix $n > l_{p,e}$. Then $\mathcal{H}^n_{p,e}$ is n-$\mathcal{H}$-liftable and there is an n-th $\mathcal{H}$-lifting functor $\mathrm{L}^{\mathcal{H}}$ of $\mathcal{H}^n_{p,e}$ satisfying the following:*

(1) There is an isometric isomorphism, which is over p, between $(\mathcal{H} \circ \mathrm{L}^{\mathcal{H}})(H)$ and H for each H in $\mathrm{Ob}(\mathcal{H}^n_{p,e})$.
(2) $\mathrm{Pr}_1 \circ \mathrm{L}^{\mathcal{H}}$ is equivalent to $\mathrm{Res}^n_{p,e}$.
(3) $\mathrm{L}^{\mathcal{H}} \circ \mathcal{H}$ is equivalent to $\mathrm{Id}_{\mathcal{C}_{p,e}}$.

Moreover, there is a unique n-th lifting functor $\mathrm{L}^{\mathcal{H}}$ satisfying

(4) For each $g \in \mathrm{Mor}_{\mathcal{H}^n_{p,e}}(H_1, H_2)$ and for any $x \in \mathrm{L}^{\mathcal{H}}(H_1)$, there is a representative b_x of $(\mathcal{H}_n \circ \mathrm{L})(g)([x])$ such that
$$\nu_2(\mathrm{L}^{\mathcal{H}}(g)(x) - b_x) > M(\mathrm{L}^{\mathcal{H}}(H_1)).$$

If we take L satisfying (1)-(4) and $\mathrm{L}^{\mathcal{H}}$ satisfying (1)-(4), then $\mathrm{L}^{\mathcal{H}} = \mathrm{L} \circ \mathrm{To}$.

Moreover, by Theorem 3.7, we have the following theorem.

Theorem 4.11. *Let p be a prime number and e be a positive integer. Suppose p does not divide e so that $l_{p,e} = e$. For any $n > e$, two categories $\mathcal{H}^n_{p,e}$ and $\mathcal{C}_{p,e}$ are equivalent*

In summary, for $n > l_{p,e}$, we have the following diagram between categories of $\mathcal{C}_{p,e}$, $\mathcal{R}^n_{p,e}$, $\mathcal{H}^n_{p,e}$, and $\widehat{\mathcal{H}}^n_{p,e}$:

$$\begin{tikzcd}
\mathcal{H}^n_{p,e} \arrow[r, shift left, "\mathrm{L}^{\mathcal{H}}"] \arrow[d, hook] \arrow[dr, "\mathrm{To}"] & \mathcal{C}_{p,e} \arrow[l, shift left, "\mathcal{H}"] \arrow[d, shift left, "\mathrm{Pr}_n"] \\
\widehat{\mathcal{H}}^n_{p,e} \arrow[r, shift right, "\widehat{\mathrm{To}}"'] & \mathcal{R}^n_{p,e} \arrow[l, shift right, "\widehat{\mathrm{U}}"'] \arrow[u, shift left, "\mathrm{L}"]
\end{tikzcd}$$

, and we can take $\mathrm{L}^{\mathcal{H}} = \mathrm{L} \circ \mathrm{To}$.

Question 4.12. *Let p be a prime number and e be a positive integer. Fix $n > l_{p,e}$. We know that two categories $\widehat{\mathcal{H}}^n_{p,e}$ and $\mathcal{R}^n_{p,e}$ are equivalent, and $\mathrm{L}^{\mathcal{H}} \circ \mathcal{H}$ is equivalent* *to* $\mathrm{Id}_{\mathcal{C}_{p,e}}$. *And for the tame case, that is,* $p \not| e$, $\mathcal{H}^n_{p,e}$ *and* $\mathcal{C}_{p,e}$ *are equivalent. Is* $\mathcal{H}_n \circ \mathrm{L}^{\mathcal{H}}$ *equivalent to* $\mathrm{Id}_{\mathcal{H}^n_{p,e}}$ *if* $p|e$*?*

5. Relative completeness via hyper fields

In this section, we aim to prove an AKE-type relative completeness theorem in terms of valued hyperfields for finitely ramified valued fields. We first recall basic facts on coarsenings of valuations.

Remark/Definition 5.1. [15] *Let* (K,ν,k,Γ) *be valued field. Let* Γ° *be a convex subgroup of* Γ *and* $\dot\nu : K\setminus\{0\} \longrightarrow \Gamma/\Gamma^\circ$ *be a map sending* $x(\neq 0)\in K$ *to* $\nu(x)+\Gamma^\circ \in \Gamma/\Gamma^\circ$. *The map* $\dot\nu$ *is a valuation, called* a coarse valuation *of* ν *with respect to* Γ°. *The residue field* K°, *called* the core field *of* (K,ν) *with respect to* Γ°, *of* $(K,\dot\nu)$ *forms a valued field equipped with a valuation* ν° *induced from* ν *and the value groups* Γ°. *More precisely, the valuation* ν° *is defined as follows: Let* $\mathrm{pr}_{\dot\nu} : R_{\dot\nu} \longrightarrow K^\circ$ *be the canonical projection map and let* $x\in R_{\dot\nu}$. *If* $x^\circ := \mathrm{pr}_{\dot\nu}(x) \in K^\circ\setminus\{0\}$, *then* $\nu^\circ(x^\circ) := \nu(x)$. *And* $x^\circ = 0 \in K^\circ$ *if and only if* $\nu(x) > \Gamma^\circ$, *that is,* $\nu(x) > \gamma$ *for all* $\gamma\in\Gamma^\circ$. *If* K *is of characteristic* 0 *and* Γ° *is non-trivial, then* $(K,\dot\nu)$ *is always of equal characteristic* $(0,0)$.

Fact 5.2. *Let* (K,ν,Γ) *be valued field. Let* $\dot\nu$ *be the coarse valuation and* K° *be the core field with respect to* Γ° *for a non-trivial convex subgroup* Γ° *of* Γ.

1. *Let* R_ν, $R_{\dot\nu}$, *and* R_{ν° *be the valuation rings of* (K,ν), $(K,\dot\nu)$, *and* (K°,ν°) *respectively. Then* $(\mathrm{pr}_{\dot\nu})^{-1}(R_{\nu^\circ}) = R_\nu$.
2. *If* (K,ν) *is finitely ramified, then* (K°,ν°) *is finitely ramified, and* K *and* K° *have the same ramification index.*
3. *If* (K,ν) *is finitely ramified and* $\aleph_1$*-saturated, and* Γ° *is the smallest non-trivial convex subgroup, then* (K°,ν°) *is complete.*

Definition 5.3. *Let* (K,ν,Γ) *be valued field. Let* $\Gamma^\circ\subset\Gamma$ *be a non-trivial convex subgroup and* $\gamma\in(\Gamma^\circ)_{>0}$. *Let* $0^\gamma_{\Gamma^\circ} := \{\alpha\in\mathcal{H}_\gamma(K) |\ \nu_{\mathcal{H}}(\alpha) > \Gamma^\circ\}$. *If* γ *is obvious, we write* $0^\gamma_{\Gamma^\circ} = 0_{\Gamma^\circ}$. *Define* $\mathcal{H}^\times_\gamma(K,\Gamma^\circ) := \{\alpha\in\mathcal{H}_\gamma(K) : \nu_{\mathcal{H}}(\alpha)\in\Gamma^\circ\}$ *and* $\mathcal{H}_\gamma(K,\Gamma^\circ) := \mathcal{H}^\times_\gamma(K,\Gamma^\circ)\cup\{0^\gamma_{\Gamma^\circ}\}$.

Remark 5.4. *Let* Γ° *be a non-trivial convex subgroup of* Γ. *Then* $\mathcal{H}_\gamma(K,\Gamma^\circ)$ *forms a valued hyperfield.*

Proof. Let $(\mathcal{H}_\gamma(K), +_{\mathcal{H}}, \cdot_{\mathcal{H}}, \nu_{\mathcal{H}})$ be the valued γ-hyperfield of K. First, $\mathcal{H}^\times_\gamma(K,\Gamma^\circ)$ is a multiplicative subgroup of $(\mathcal{H}^\times_\gamma(K),\cdot_{\mathcal{H}})$ because Γ° is a subgroup of Γ. Note that for $\alpha,\beta\in\mathcal{H}^\times_\gamma(K)$, if there is $x\in\alpha+_{\mathcal{H}}\beta$ such that $\nu(x)>\Gamma^\circ$, then $0_{\Gamma^\circ}\subset\alpha+_{\mathcal{H}}\beta(\dagger)$. For $\alpha=[a]$ and $\beta=[b]$, $\bigcup\alpha+_{\mathcal{H}}\beta = (a+b)+\mathfrak{m}^{\gamma+\min\{\nu(a),\nu(b)\}}$ by Lemma 3.1(2). Suppose $x\in\bigcup\alpha+_{\mathcal{H}}\beta$ such that $\nu(x)>\Gamma^\circ$. If $\nu(a+b)<\gamma+\min\{\nu(a),\nu(b)\}$, then $\nu(x)=\nu(a+b)<\gamma+\min\{\nu(a),\nu(b)\}\in\Gamma^\circ$ because $\gamma,\nu(a),\nu(b)\in\Gamma^\circ$. So we have that $\nu(a+b)\geq\gamma+\min\{\nu(a),\nu(b)\}$ and $\bigcup\alpha+_{\mathcal{H}}\beta=\mathfrak{m}^{\gamma+\min\{\nu(a),\nu(b)}\supset 0_{\Gamma^\circ}$. We define a multivalued operation $+^{\Gamma^\circ}_{\mathcal{H}}$ on $\mathcal{H}_\gamma(K,\Gamma^\circ)$ as follows: For $\alpha,\beta\in\mathcal{H}^\times_\gamma(K)$,

- $\alpha+^{\Gamma^\circ}_{\mathcal{H}} 0_{\Gamma^\circ} = \alpha = 0_{\Gamma^\circ} +^{\Gamma^\circ}_{\mathcal{H}}\alpha$.
- $\alpha+^{\Gamma^\circ}_{\mathcal{H}}\beta = \begin{cases}\alpha+_{\mathcal{H}}\beta & \text{if } \forall x\in\bigcup\alpha+_{\mathcal{H}}\beta,\ \nu(x)\in\Gamma^\circ\\ (\alpha+_{\mathcal{H}}\beta)\cap(\mathcal{H}_n)^\times(K)\cup\{0_{\Gamma^\circ}\} & \text{if } \exists x\in\bigcup\alpha+_{\mathcal{H}}\beta,\ \nu(x)>\Gamma^\circ\end{cases}.$

By (†) and covexity of Γ°, $+_{\mathcal{H}}^{\Gamma^\circ}$ is well-defined. Define a map $\nu_{\mathcal{H}}^{\Gamma^\circ}$ on $\mathcal{H}_\gamma(K,\Gamma^\circ)$ as follows: For $\alpha\in\mathcal{H}_\gamma(K,\Gamma^\circ)$, $\nu_{\mathcal{H}}^{\Gamma^\circ}(\alpha)=\nu_{\mathcal{H}}(\alpha)$ if $\alpha\neq 0_{\Gamma^\circ}$ and $\nu_{\mathcal{H}}^{\Gamma^\circ}(\alpha)=\infty$ if $\alpha=0_{\Gamma^\circ}$. Then $(\mathcal{H}_\gamma(K,\Gamma^\circ),+_{\mathcal{H}}^{\Gamma^\circ},\cdot_{\mathcal{H}},\nu_{\mathcal{H}}^{\Gamma^\circ})$ forms a valued hyperfield. □

Lemma 5.5. *Let (K,ν,Γ) be valued field. Let $\dot\nu$ be the coarse valuation and K° be the core field with respect to Γ° for a non-trivial convex subgroup Γ° of Γ. For each $\gamma\in\Gamma^\circ_{>0}$, $\mathcal{H}_\gamma(K,\Gamma^\circ)$ and $\mathcal{H}_\gamma(K^\circ)$ are isomorphic.*

Proof. Consider a map $f:\mathcal{H}_\gamma(K,\Gamma^\circ)\to\mathcal{H}_\gamma(K^\circ)$ defined as follows: For $\alpha\in\mathcal{H}_\gamma(K,\Gamma^\circ)$, $f(\alpha)=[a^\circ]$ if $\alpha=[a]$, and $f(\alpha)=0$ if $\alpha=0_{\Gamma^\circ}$. The map f is well-defined. Suppose $[a]=[b]\in\mathcal{H}_\gamma(K,\Gamma^\circ)$. Then $a/b\in(1+\mathfrak{m}^\gamma)$ and $\nu(a)=\nu(b)\in\Gamma^\circ$. So, we have that $(a/b)^\circ=a^\circ/b^\circ\in(1+(\mathfrak{m}^\circ)^\gamma)$, and it implies $[a^\circ]=[b^\circ]$. And it is clear that for all $x\in K$ with $\nu(x)>\Gamma^\circ$, $x^\circ=0$. It is routinely to check that f is a homomorphism. □

We recall the following facts before proving a relative completeness theorem.

Fact 5.6. *(Keisler-Shelah Isomorphism Theorem) Let $\mathcal{M}$ and $\mathcal{N}$ be two first order structures. If $\mathcal{M}\equiv\mathcal{N}$, then there is a ultrafilter $\mathcal{U}$ on an infinite set I such that*

$$\mathcal{M}^{\mathcal{U}}\cong\mathcal{N}^{\mathcal{U}},$$

where $\mathcal{M}^{\mathcal{U}}$ and $\mathcal{N}^{\mathcal{U}}$ are the ultrapowers of $\mathcal{M}$ and $\mathcal{N}$ with respect to $\mathcal{U}$.

Fact 5.7. [5, Lemma 1.5] *Let (K,ν) be a finitely ramified henselian valued field of mixed characteristic $(0,p)$. Then the valuation ring $R(K)$ of (K,ν) is definable by the formula*

$$\phi_q(x)\equiv\exists y\ y^q=1+px^q$$

for some $q>0$ such that $p\not| q$ and $q>e_\nu(p)$.

Theorem 5.8. *Let (K_1,ν_1,k_1,Γ_1) and (K_2,ν_2,k_2,Γ_1) be finitely ramified henselian valued fields of mixed characteristic $(0,p)$. Suppose k_1 and k_2 are perfect fields. Let $n>e_{\nu_2}(p)(1+e^2_{\nu_1}(p))$. The following are equivalent:*

1. *$K_1\equiv K_2$.*
2. *$R_n(K_1)\equiv R_n(K_2)$ and $\Gamma_1\equiv\Gamma_2$.*
3. *$\mathcal{H}_n(K_1)\equiv_{\mathcal{H}_n(\{p\})}\mathcal{H}_n(K_2)$.*

Proof. For (1) ⇔ (2), see Fact 2.27. It is clear that (1) implies (3). We show (3) implies (1). Suppose (3) holds. By Fact 5.6, we may assume that $\mathcal{H}_n(K_1)\cong_{\mathcal{H}_n(\mathbb{Z})}\mathcal{H}_n(K_2)$. By Remark 2.8, $\Gamma_1\cong\Gamma_2$ and we may assume that $\Gamma_1=\Gamma_2=\Gamma$. By taking ultrapowers of K_1 and K_2 with respect to a nonprincipal ultrafilter on $\mathbb{N}$, we may assume that K_1 and K_2 are $\aleph_1$-saturated. Let $\dot\nu_1$ and $\dot\nu_2$ be the coarse valuation of ν_1 and ν_2 with respect to the smallest non-trivial convex subgroup Γ°. We have two valued fields $(K_1,\dot\nu_1)$ and $(K_2,\dot\nu_2)$ of equal characteristic $(0,0)$ with residue fields K_1° and K_2° respectively. By Fact 5.2(2) and (3). (K_1°,ν_1°) and (K_2°,ν_2°) are complete discrete valued fields of mixed characteristic $(0,p)$. Note that the value groups of $(K_1,\dot\nu_1)$ and $(K_2,\dot\nu_2)$ are Γ/Γ°. Since $\mathcal{H}_n(K_1)\cong_{\mathcal{H}_n(\mathbb{Z})}\mathcal{H}_n(K_2)$, we have that $\mathcal{H}_n(K_1^\circ)\cong_{\mathcal{H}_n(\mathbb{Z})}\mathcal{H}_n(K_1^\circ)$ by Lemma 5.5. Since $e_{\nu_i^\circ}(p)=e_{\nu_i}(p)$ for $i=1,2$, we have the $K_1^\circ\cong K_2^\circ$ by Corollary 3.19. By Fact 2.26, we have that $(K_1,\dot\nu_1)\equiv(K_2,\dot\nu_2)$. To show that $(K_1,\nu_1)\equiv(K_2,\nu_2)$, it is enough to show that the valuation rings $R_\nu(K_1)$ of (K_1,ν_1) and $R_\nu(K_2)$ of (K_2,ν_2) are definable in $(K_1,\dot\nu_1)$ and $(K_2,\dot\nu_2)$ by the same formula. Recall the following result on a definability of a residue ring. Take $l>0$ large enough so that $q:=p^l+1>\max\{e_{\nu_1}(p),e_{\nu_2}(p)\}$.

By Fact 5.7, $\phi_q(x)$ defines the residue rings $R_{\nu^\circ}(K_1)$ and $R_{\nu^\circ}(K_2)$ of (K_1°, ν_1°) and (K_2°, ν_2°). By Fact 5.2(2), the valuation rings $R_\nu(K_1)$ and $R_\nu(K_2)$ are definable by the same formula in $(K_1, \dot{\nu}_1)$ and $(K_2, \dot{\nu}_1)$ so that $(K_1, \nu_1) \equiv (K_2, \nu_2)$. □

By Theorem 3.7 and the proof of (3) ⇒ (1) of Theorem 5.8, we have the following result.

Corollary 5.9. *Let $(K_1, \nu_1, k_1, \Gamma_1)$ and $(K_2, \nu_2, k_2, \Gamma_1)$ be finitely tamely ramified henselian valued fields of mixed characteristic $(0, p)$. Suppose k_1 and k_2 are perfect fields. The following are equivalent:*

(1) $K_1 \equiv K_2$.
(2) $\mathcal{H}_1(K_1) \equiv_{\mathcal{H}_1(\{p\})} \mathcal{H}_1(K_2)$.

For local fields of mixed characteristic, they are elementary equivalent if and only if they are isomorphic. So we have the following corollary.

Corollary 5.10. *Let K_1 and K_2 be local fields of mixed characteristic. Let $n > e_{\nu_2}(p)(1 + e_{\nu_1}^2(p))$. The followings are equivalent:*

(1) $\mathcal{H}_n(K_1) \equiv_{\mathcal{H}_n(\{p\})} \mathcal{H}_n(K_2)$.
(2) $\mathcal{H}_n(K_1) \cong_{\mathcal{H}_n(\{p\})} \mathcal{H}_n(K_2)$.

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