FORKING AND DIVIDING IN NTP$_2$ THEORIES

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Abstract. We prove that in theories without the tree property of the second kind (which include dependent and simple theories) forking and dividing over models are the same, and in fact over any extension base. As an application we show that dependence is equivalent to bounded forking assuming NTP$_2$.

1. Introduction

Background.
The study of forking in the dependent setting was initiated by Shelah in full generality [Sheb] and by Dolich in the case of nice *o*-minimal theories [Dol04]. A lot of further results appear in [Adlb], [HP], [OUb] and [Sta]. The main trouble is that apparently non-forking independence outside of the simple context no longer corresponds to a notion of dimension in any possible way. Moreover it is neither symmetric nor transitive (at least in the classical sense). However in dependent theories it corresponds to invariance of types, which is undoubtedly a very important concept, and it is a meaningful combinatorial tool.

Main results.
The crucial property of forking in simple theories is that it equals dividing (thus the useful concept - forking - becomes somewhat more understandable in real-life situations). It is known that there are dependent theories in which forking does not equal dividing in general (for example in circular order over the empty set, see section 5). However there is a natural restatement of the question due to Anand Pillay: whether forking and dividing are equal over models? After failing to find a counter-example we decided to prove it instead. And so the main theorem of the paper is:

Theorem 1.1. *Let T be an NTP_2 theory (a class which includes dependent and simple theories). Then forking and dividing over models is the same.*

In fact, a more general result is attained. Namely that:

Theorem 1.2. *Let T be NTP_2. Then for a set A, the following are equivalent:*

(1) *A is an extension base for $\mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}^f$ (non-forking) (see definition 2.11).*
(2) *$\mathop{\smile\hskip-0.9em\raise0.2ex\hbox{$|$}}^f$ has left extension over A (see definition 2.9).*
(3) *Forking equals dividing over A.*

So theorem 1.1 is a corollary of 3.9 (as type over models are finitely satisfiable, so (1) is true), and of course,

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Corollary 1.3. *If T is NTP$_2$ and all sets are extension bases for non-forking, then forking equals dividing. (This class contains simple theories, o-minimal and c-minimal theories).*

A short overview of what follows.
In section 2 we recall briefly the definitions and notions needed.
In section 3, after proving the easy direction of theorem 1.1, we prove the so-called broom lemma, which is the technical key to the rest of the paper. Essentially it says that if a formula is covered by finitely many formulas arranged in a "nice position", then we can throw away the dividing ones, by passing to an intersection of finitely many conjugates.
We then use the broom lemma to show that in NTP$_2$ theories there is still some symmetry going on over sets which satisfy the conditions of theorem 3.9 (in particular - models). More precisely, every type has a global non-forking (even invariant) non-coforking extension (we called it a strictly invariant extension - see definition 3.12). This gives us a right analogue of Kim's lemma in the NTP$_2$ context over such sets and allows to deduce that in NTP$_2$ theories forking equals dividing over such sets.
We also give some corollaries, among them that in dependent theories forking is type definable, has left extension over models (answering a question of Itai Ben Yaacov), and that if p is a global φ type which is invariant over a model, then it can be extended to a global type invariant over the same model (strengthening a result that appeared in [HP]).
In section 4 we show that assuming NTP$_2$, dependence of a theory is equivalent to boundedness of non-forking, which is a generalization of a well-known analogous result describing the subclass of stable theories inside the class of simple theories and gives a partial answer to a question of Hans Adler from [Adlb]). Finally in section 5 we present 2 examples that show why we assume NTP$_2$ and work over models. One of them is a variant of an example due to Martin Ziegler of a theory in which forking does not equal dividing over models (and more). In the end we ask some questions and propose further research directions.

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2. Preliminaries

Here we give here all the definitions and claims needed for the rest of the paper.

Notation 2.1.

(1) Our big saturated monster will be denoted by $\mathfrak{C}$, and no distinction is made between tuples and single elements unless stated explicitly.

(2) If $\langle a_i \,|i<\lambda\rangle$ is a sequence then $a_{<i}$ is the set $\{a_j \,|j<i\}$.
(3) $a \equiv_A b$ will mean $tp\,(a/A) = tp\,(b/A)$.

We start by recalling some standard definitions.

Definition/Claim 2.2.

(1) $AutfL\,(\mathfrak{C}/A)$ is the subgroup of all automorphisms of $\mathfrak{C}$ generated by the set $\{f \in Aut\,(\mathfrak{C}/M)\,|A \subseteq M \text{ some small model}\}$.
(2) We say that a and b have the same Lascar strong type over A ($lstp\,(a/A) = lstp\,(b/A)$) if there is some automorphism $f \in AutfL\,(\mathfrak{C}/A)$ taking a to b.
(3) Having the same Lascar strong type over A is the same as being in the transitive closure of the relation $E\,(a,b)$ = there exists some indiscernible sequence over A, I, such that aI, bI are both indiscernible sequences over A.

Definition 2.3.

(1) A formula $\varphi\,(x,a)$ divides over A iff it k-divides for some k iff there is some sequence $\langle a_i \,|i<\omega\rangle$ with $a_i \equiv_A a$ for all i, such that $\{\phi(x,a_i)\,|i<\omega\}$ is k-inconsistent (see definition 2.7).
(2) A formula $\varphi\,(x,a)$ forks over A if for some n there are $\phi_0(x,a_0)$, ..., $\phi_n(x,a_n)$ such that $\varphi(x,a) \vdash \bigvee_{i<n} \phi_i(x,a_i)$, and each $\phi_i\,(x,a_i)$ divides over A.
(3) A (partial) type divides/forks over A if some formula in it divides/forks over A.
(4) A formula $\varphi\,(x,a)$ quasi (Lascar) divides over A if there are $\langle a_i \,|i<m\rangle$ for some $m<\omega$ such that $a_i \equiv_A a$ ($lstp\,(a_i/A) = lstp\,(a/A)$) and $\{\varphi\,(x,a_i)\,|i<n\}$ is inconsistent.

Fact 2.4. [Cas07, Lemma 3.1] *$tp\,(a/Ab)$ does not divide over A ($a \mathop{\smile\!\!\!\!\!\!\perp}^{d}_{A} b$) iff for every indiscernible sequence over A, I, with $b \in I$, there is $J \equiv_{Ab} I$ such that J is Aa indiscernible.*

Definition 2.5. A theory T has the independence property if there is a formula $\phi(x,y)$ and tuples $\{a_i \,|i<\omega\}$, $\{b_u \,|u \subseteq \omega\}$ such that $\phi(a_i,b_u)$ if and only if $i \in u$. T is dependent iff it does not have the independence property (also known as NIP).

Definition/Claim 2.6. We recall

(1) The alternation number of a formula: $alt\,(\varphi\,(x,y)) =$

$$= max\,\{n<\omega\,|\exists \langle a_i \,|i<\omega\rangle \text{ indiscernible}, \exists b : \varphi\,(a_i,b) \leftrightarrow \neg\varphi\,(a_{i+1},b) \text{ for } i<n-1\}$$

(2) T is dependent iff every formula has finite alternation rank.
(3) If $I = \langle a_i \,|i<\omega\rangle$ is an indiscernible sequence, and C is any set, then we define the average type $Av\,(I,C) \in S\,(C)$ as

$$\{\varphi\,(x,c)\,|c \in C, \models \varphi\,(a_i,c) \text{ for all } i \text{ big enough}\}$$

It is well defined when T is dependent.

Definition 2.7. A theory T has TP$_2$ (the tree property of the second kind) if there exists a formula $\phi(x,y)$, a number $k<\omega$ and an array of elements $\left\langle a_i^j \,|i,j<\omega\right\rangle$ such that:

- Every row (j) is k-inconsistent:
  For all $j < \omega$ and $\forall i_0 < i_1 < ... < i_k < \omega$,
  $$\phi(x, a^j_{i_0}) \wedge \phi(x, a^j_{i_1}) \wedge ... \wedge \phi(x, a^j_{i_k}) = \emptyset$$
- Every vertical path is consistent:
  $\forall f : \omega \to \omega \bigwedge_{j<\omega} \phi(x, a^j_{f(j)}) \neq \emptyset$

We say that T is NTP$_2$ when it does not have TP$_2$.

Fact 2.8. *Every dependent theory as well as every simple one is NTP$_2$.*

Proof. Exercise. □

Since some of our proofs and theorems require certain abstract properties of pre-independence relations we define them here. By a pre-Independence relation we shall mean a ternary relation $\downfree$ between subsets of the monster model, which satisfies one or more of the properties below. For a more general definition of a pre-independence relation see e.g. [Adlb, Section 5]. Note that since normally our relation is not symmetric many properties can be postulated both on the left side and on the right side.

Definition 2.9. The following are the properties we consider for a pre-independence relation for this paper (below A, B, C, D are sets or tuples)

1. Invariance: If $A \downfree_C B$ and $A'B'C' \equiv ABC$ then $A' \downfree_{C'} B'$.
2. Monotonicity: If $A \downfree_C B$ and $A' \subseteq A, B' \subseteq B$ then $A' \downfree_C B'$.
3. Base monotonicity: If $A \downfree_C BD$ then $A \downfree_{CB} D$.
4. (Right) extension: if $A \downfree_C B$ and $D \supseteq B$ is some set then there is $D' \equiv_{BC} D$ such that $A \downfree_C D'$.
5. Transitivity on the left: $A_1 \downfree_C B$ and $A_2 \downfree_{CA_1} BA_1$ implies $A_1A_2 \downfree_C B$.
6. Stronger than invariance: if $A \downfree_C B$ then $A \downfree^i_C B$ which means: there is a global extension of $tp(A/BC)$, $p \in S(\mathfrak{C})$ such that p is strongly Lascar invariant over C (every automorphism from $Autf L(\mathfrak{C}/C)$ fixes p).
7. Preservation of indiscernibles: if I is a C-indiscernible sequence and $A \downfree_C I$ then I is AC-indiscernible.
8. Left extension: if $A \downfree_C B$ and $D \supseteq A$ then there is $D' \equiv_{AC} D$ such that $D' \downfree_C B$.

Note 2.10. If $\downfree$ satisfies extension then (6) and (7) are equivalent for it.

Definition 2.11.

1. A set A is an extension base for $\downfree$ (or, if $\downfree$ is clear from the context, we shall omit it), if for all a, $a \downfree_A A$.
2. A set A is a good extension base for $\downfree$, if every $B \supseteq A$ is an extension base for $\downfree$.
3. A set A is a left extension base for $\downfree$, if it is an extension base and $\downfree$ has left extension over A.

Fact 2.12. *T arbitrary.*

1. Co-inheritance: (denoted by $\downfree^u$) - $a \downfree^u_C b$ iff $tp(a/bC)$ is finitely satisfiable in C. It satisfies (1) - (7), and also (8) when C is a model.

(2) If $\downfree$ is any pre-independence relation satisfying (1) - (5), and C is a good extension base for it, then it also satisfies left extension over C. So a good extension base for $\downfree$ is a left extension base for it.
(3) Invariance - $\downfree^i$ (see (6) in 2.9) satisfies (1) - (7).
(4) Non-forking ($\downfree^f$) - $a \downfree^f_C b$ iff $tp(a/bC)$ does not fork over C - satisfies (1) - (5).
(5) T dependent: Non-forking also satisfies (6) (so (7)), in fact $\downfree^f = \downfree^i$.

Proof. □

(1) The fact that ind^u satisfies (1) - (7) can be seen in e.g. [Adlb, section 5]. For left extension over models: Consider inheritance ($\downfree^h$) over a model M: $A \downfree^h_M B$ iff $tp(A/BM)$ is a heir over M, iff $B \downfree^u_M A$. It is well known that $\downfree^h$ satisfies extension and existence over models. So if $A \downfree^u_M B$ and $A \subseteq C$, then $B \downfree^h_M A$, so by extension we can find $C' \equiv_{MA} C$ such that $B \downfree^h_M C'$, so $C' \downfree^u_M B$.
(2) Assume C is a good extension base. Assume that $A \downfree_C B$ (so by extension and invariance $A \downfree_C CB$) and $A \subseteq D$ is some set. As CA is an extension base, $D \downfree_{CA} CA$ and by extension and invariance there is some $D' \equiv_{CA} D$ such that $D' \downfree_{CA} BCA$ so $D' \downfree_{CA} B$ by monotonicity, so $AD' \downfree_C B$ by transitivity on the left.
(3) Can be checked directly, and also appears in [Adlb, section 5] .
(4) Can be checked directly, and also appears in [Adlb, section 5] and [Adla].
(5) Appears in [Sheb, 5.4] (and also in [Adlb]). In fact, p is a global non-forking type over C iff p is strongly Lascar invariant over C, so $\downfree^f = \downfree^i$.

Definition 2.13. For the sake of this paper, we shall call theories where every set is an extension base for $\downfree$, $\downfree$-*extensible* theories. Note that for dependent theories, being $\downfree^i$-extensible is the same as being $\downfree^f$-extensible.

Example 2.14. [HP, 2.14] c-minimal and o-minimal theories are $\downfree^f$-extensible dependent theories.

Definition 2.15.
(1) A global $\downfree$-free type over A is a global type p such that for any $B \supseteq A$, and $c \models p|_B$, $c \downfree_A B$.
(2) A Morley sequence $\langle a_i \mid i < \omega \rangle$ for $\downfree$, with base A over $B \supseteq A$ is an indiscernible sequence over B, such that for all i, $a_i \downfree_A Ba_{<i}$.

Remark 2.16.
(1) If N is $|A|^+$ saturated and $p \in S(N)$ has a global A invariant extension (every automorphism from $Aut(\mathfrak{C}/C)$ fixes it), then it is unique. So if $\downfree$ is stronger than invariance ($\downfree^i$), and $c \downfree_A N$ where N is $|M|^+$ saturated for some model $M \supseteq A$, then $tp(c/N)$ has a unique global $\downfree$ free extension over A (see e.g. [Usvb, Lemma 2.23]).
(2) If p is a global $\downfree$ free type over A which is also invariant over $B \supseteq A$ (so this is true if $\downfree$ satisfies (6), and B contains a model containing A), then p generates a Morley sequence with base A over B as follows: let $a_0 \models p|_B$

and inductively $a_{i+1} \models p|_{Ba_{<i+1}}$. In addition, there is a global $\downfree$-free over A type $P^{(\omega)}$, such that $I \models p^{(\omega)}|_B$ iff if I is a Morley sequence generated by p over B. For more on that see [HP, Lemma 2.3] .

3. Main results

3.1. Consequences of forking equals dividing.

Here we prove the easy direction of theorem 1.2 ((3) implies (2) and (1)) i.e. we prove that:

Theorem 3.1. *Let T be any theory. Then for a set A, if forking equals dividing over A, then A is an extension base for $\downfree^f$ (non-forking) and $\downfree^f$ has left extension over A.*

In fact we have some more consequences on the behavior of forking over A if forking equals dividing over A. So for this section, assume that forking equals dividing over A.

Claim 3.2. A is an extension base for non-forking.

Proof. No type divides over its domain. □

Claim 3.3. We have left extension for non-forking over A.

Proof. Suppose $a \downfree_A b$ and we have some c. We want to find some $c' \equiv_{Aa} c$ such that $c'a \downfree_A b$. Let $p = tp(c/Aa)$. If no such c' exists, then it means that

$$p(x) \cup \{\neg\varphi(x, a, b) \,|\varphi \text{ is over } A \text{ and } \varphi(x, y, b) \text{ divides over } A\}$$

is inconsistent. If not, then $p(x) \vdash \bigvee_{i<m} \varphi_i(x, a, b)$ for some $m < \omega$ and $\varphi_i(x, y, z)$ such that $\varphi_i(x, y, b)$ divides over A. Let $\varphi(x, y, z) = \bigvee_{i<m} \varphi_i(x, y, z)$. So $\varphi(x, y, b)$ forks over A, hence divides over A, and $p(x) \vdash \varphi(x, a, b)$. $\varphi(x, y, b)$ divides over A, but $a \downfree_A b$ so by 2.4 $\varphi(x, y, b)$ divides over Aa, so there is some indiscernible sequence over Aa, $\langle b_i \,|i < \omega\rangle$ such that $b_i \equiv_{Aa} b$ that witnesses dividing. As $p \in S(Aa)$, $p \vdash \varphi(x, a, b_i)$ for each i, but this is a contradiction, as p is itself consistent of course. □

Note 3.4. This last claim answers (modulo theorem 1.1) a question of Itai Ben-Yaacov which appeared in a preprint of his [BY].

Claim 3.5. Non-forking is non-degenerate over A, i.e. $a \downfree_A^f a$ iff $a \in acl(A)$.

Proof. One always have that $a \downfree_A a$ implies $a \in acl(A)$ (if $a \notin acl(A)$, then it has unbounded many conjugates, so we can find an infinite indiscernible sequence over A, $\langle a_i \,|i < \omega\rangle$, and it would witness the dividing of $x = a$ over A, i.e. $tp(a/aA)$ forks over A).
On the other hand, assume that $tp(a/Aa)$ forks over A. Hence it divides over A, so there is some formula over A, $\varphi(x, a)$, such that $\models \varphi(a, a)$ and $\varphi(x, a)$ divides over A. So it follows that $\varphi(x, a) \wedge x = a$ has these properties as well. So this means that there is an indiscernible sequence over A, $\langle a_i \,|i < \omega\rangle$ which witnesses dividing of $\varphi(x, a) \wedge x = a$. If $a \in acl(A)$, then for infinitely many i s a_i is constant, say c. So it follows that $\varphi(x, c) \wedge x = c$ is inconsistent, so $\neg\varphi(c, c)$. But $c \equiv_A a$, so this is a contradiction. □

Claim 3.6. Non-forking is rigid, i.e $a \mathop{\smile\!\!\!\!\!|}^{f}_{A} b$ iff $a \mathop{\smile\!\!\!\!\!|}^{f}_{acl(A)} b$ iff $a \mathop{\smile\!\!\!\!\!|}^{f}_{A} acl(Ab)$ iff $acl(Aa) \mathop{\smile\!\!\!\!\!|}^{f}_{A} b$ iff $acl(Aa) \mathop{\smile\!\!\!\!\!|}_{acl(A)} acl(Ab)$.

Proof. First note that $\varphi(x,b)$ divides over A iff it divides over $acl(A)$ (why? the "if" direction is clear. The "only if" one follows from the fact that if I is an indiscernible sequence over A then it is also indiscernible over $acl(A)$ - any two increasing sequences from I have the same Lascar strong type, hence the same strong type over A). In particular, it follows that forking equals dividing over $acl(A)$.
Assume now that $a \mathop{\smile\!\!\!\!\!|}^{f}_{A} b$. By extension, there is some $c \equiv_{Ab} acl(Ab)$ such that $a \mathop{\smile\!\!\!\!\!|}^{f}_{A} c$, but then $c = acl(Ab)$ as a set, so by base monotonicity, $a \mathop{\smile\!\!\!\!\!|}^{f}_{acl(A)} b$. In the same way, by left extension, we have $acl(Aa) \mathop{\smile\!\!\!\!\!|}^{f}_{A} b$.
From this it is easy to conclude. □

3.2. **The Broom lemma.**

We start the proof of the 2nd direction of theorem 1.2 by eliminating the main technical obstacle.

Lemma 3.7. *Suppose that $\mathop{\smile\!\!\!\!\!|}$ satisfies (1) - (7) from 2.9 and (8) over A, and that* $\alpha(x,e) \vdash \psi(x,c) \vee \bigvee_{i<n} \varphi_i(x,a_i)$, *where*

(1) For $i<n$, $\varphi_i(x,a_i)$ is k dividing, as witnessed by the indiscernible sequence $I_i = \langle a_{i,l} \,|l<\omega\rangle$ where $a_{i,0} = a_i$.
(2) For each $i<n$ and $1 \le l$, $a_{i,l} \mathop{\smile\!\!\!\!\!|}_{A} a_{i,<l} I_{<i}$.
(3) $c \mathop{\smile\!\!\!\!\!|}_{A} I_{<n}$.

then for some $m<\omega$ and $\{e_i \,|i<m\}$ with $e_i \equiv_A e$, $\bigwedge_{i<m} \alpha(x,e_i) \vdash \psi(x,c)$. In particular, if $\psi = \bot$, then $\alpha(x,e)$ quasi divides over A.

Proof. By induction on n. For $n=0$ there is nothing to prove. Assume that the claim is true for n and we prove it for $n+1$. Let $b_0 = a_{n,0} \ldots a_{n,k-2}, b_1 = a_{n,1} \ldots a_{n,k-1}$. By preservation of indiscernibles, as $c \mathop{\smile\!\!\!\!\!|}_{A} I_{n-1}$, we have

$$cb_1 \equiv_A cb_0$$

We build by induction on $0 \le j < k$ sequences $\left\langle I^{l,j}_{<n} \,|1 \le l \le j \right\rangle$ (so $I^{l,j}_{<n} = I^{l,j}_0 \ldots I^{l,j}_{n-1}$) such that:

(1) $I^{0,j}_{<n} = I_{<n}$.
(2) $I^{l,j}_{<n} c a_{n,l} \equiv_A I^{0,j}_{<n} c a_{n,0}$ for all $0 \le l \le j$ and
(3) For all $0 \le l < j$, $cI^{j,j}_{<n} I^{j-1,j}_{<n} \ldots I^{l+1,j}_{<n} \mathop{\smile\!\!\!\!\!|}_{A} I^{l,j}_{<n}$ and $c \mathop{\smile\!\!\!\!\!|}_{A} I^{j,j}_{<n}$ (which already follows from (2)).

For $j=0$, use (1).
So suppose we have this sequence for j and build it for $j+1<k$.
By (1), let $I^{0,j+1}_{<n} = I_{<n}$.
As $cb_1 \equiv_A cb_0$ we can find some:

$$J^{j+1,j+1}_{<n} J^{j,j+1}_{<n} \ldots J^{1,j+1}_{<n} cb_1 \equiv_A I^{j,j}_{<n} I^{j-1,j}_{<n} \ldots I^{0,j}_{<n} cb_0$$

As $cb_1 \mathop{\smile\!\!\!\!\!|}_{A} a_{n,0} I_{i<n}$ (by transitivity on the left), by left extension, we can find $\left\langle I^{l,j+1}_{<n} \,|1 \le l \le j+1 \right\rangle$

$$I^{j+1,j+1}_{<n} I^{j,j+1}_{<n} \ldots I^{1,j+1}_{<n} cb_1 \equiv_A J^{j+1,j+1}_{<n} J^{j,j+1}_{<n} \ldots J^{1,j+1}_{<n} cb_1$$

and $\left\langle I_{<n}^{l,j+1} \middle| 1 \leq l \leq j+1 \right\rangle cb_1 \mathop{\smile\!\!\!\!\!\!|}_A a_{n,0} I_{i<n}$.
Now to check that we have our conditions satisfied:
For (2), first of all, $I_{<n} ca_{n,0} \equiv_A I_{<n}^{1,j+1} ca_{n,1}$ by the equations above. For $1 \leq l \leq j$

$$I_{<n} ca_{n,0} \equiv_A I_{<n}^{l,j} ca_{n,l}$$

by the hypothesis regarding j. By the equation above,

$$I_{<n}^{l,j} ca_{n,l} \equiv_A I_{<n}^{l+1,j+1} ca_{n,l+1}$$

and so we have (2) for $1 \leq l \leq j+1$. (3) follows from the construction and invariance of $\mathop{\smile\!\!\!\!\!\!|}$ and the induction hypothesis about j. This completes the construction, and so for $j = k-1$ we have $\left\langle I_{<n}^{l,k-1} \middle| 0 \leq l \leq k-1 \right\rangle$. We shall now use only this last sequence.
There are some $\langle e_l \,|l < k \rangle$ such that $e_0 = e$ and for $0 < l$, $e_l I_{<n}^{l,k-1} ca_{n,l} \equiv_A eI_{<n} ca_{n,0}$, so applying some automorphism fixing Ac, we replace $a_{n,0}$ by $a_{n,l}$, e by e_l and $I_{<n}$ by $I_{<n}^{l,k-1}$. So

$$\alpha(x, e_l) \vdash \psi(x, c) \vee \bigvee_{i<n} \varphi_i \left(x, a_i^{l,k-1} \right) \vee \varphi_n (x, a_{n,l})$$

where $a_i^{l,k-1}$ starts $I_i^{l,k-1}$. Hence $\alpha^0 = \bigwedge_{l<k} \alpha(x, e_l)$ implies the conjunction of these formulas. But as I_n witnesses that $\varphi_n(x, a_n)$ is k dividing, we have the following:

$$\alpha^0 \vdash \psi(x, c) \vee \bigvee_{i<n, l<k} \varphi_i \left(x, a_i^{l,k-1} \right)$$

Define a new formulas $\psi^r(x, c^r) = \psi(x, c) \vee \bigvee_{i<n, r \leq l<k} \varphi_i \left(x, a_i^{l,k-1} \right)$ for $r \leq k$. By induction on $r \leq k$, we find α^r such that α^r is a conjunction of conjugates over A of $\alpha(x, e)$, and $\alpha^r \vdash \psi^r(x, c^r)$. It will follow of course, that $\alpha^k \vdash \psi(x, c)$ as desired.
For $r = 0$, we already found α^0. Assume we found α^r, so we have

$$\alpha^r \vdash \psi^{r+1} \left(x, c^{r+1} \right) \vee \bigvee_{i<n} \varphi_i \left(x, a_i^{r,k-1} \right)$$

One can easily see that the hypothesis of the lemma is true for this implication (using $c = c^{r+1}$, and $I_i = I_i^{r,k-1}$) so by the induction hypothesis (on n), there is some α^{r+1}(which is a conjunction of conjugates of α^r over A) such that $\alpha^{r+1} \vdash \psi^{r+1} \left(x, c^{r+1} \right)$. □

Remark 3.8. The name of this lemma is due to its method of proof, which reminded the authors (and also Itai Ben Yaacov who thought of the name) of a sweeping operation.

3.3. **Working Abstractly.**
In this section we shall prove the following theorem:

Theorem 3.9. *Let T be NTP$_2$. Then for A, (1) implies (2) where:*
1. *There exists a pre-independence relation $\mathop{\smile\!\!\!\!\!\!|}$ that satisfies (1) - (7) from 2.9 and (8) over A, and A is an extension base for it.*
2. *forking equals dividing over A.*

If T is dependent then they are equivalent.

So assume T is NTP$_2$, and that $\downfree$ is a pre-independence relation satisfying (1) - (7) from 2.9 and that A is an extension base for $\downfree$. We do not assume left extension until later.

Claim 3.10. Assume $\varphi(x,a)$ divides over A. Then there is a model $A \subseteq M$ and a global $\downfree$-free type over A, $p \in S(\mathfrak{C})$, extending $tp(a/M)$, such that any (some) Morley sequence generated by p over M witnesses that $\varphi(x,a)$ divides. (i.e. $\{\varphi(x,a_i) \,| i < \omega\}$ is inconsistent whenever $\langle a_i \,| i < \omega \rangle$ is a Morley sequence generated by p over M).

Proof. Let $I = \langle b_i \,| i < \omega \rangle$ be an A-indiscernible sequence that witnesses k dividing of $\varphi(x,a)$ (so $tp(b_i/A) = tp(a/A)$). Let M be some small model containing A, and let N be an $|M|^+$ saturated model containing M. Let $\lambda = \left(2^{|N|+|T|}\right)^+$, and let $I' = \langle b_i \,| i < \lambda \rangle$ be an indiscernible sequence over A with the same EM type as I. As A is an extension base, $I' \downfree_A A$, so by invariance and extension , we may assume that $I' \downfree_A N$. As λ is longer than the number of types on N, there are infinitely many indexes such that $tp(b_i/N)$ is the same, wlog the first ω. Call this type p'. As N was saturated enough, and $\downfree$ is stronger than invariance ($\downfree^i$), there is a unique extension of p' to a global type p which is $\downfree$ free over A. Let $q' = tp(\langle b_i \,| i < \omega \rangle / N)$, and let q be it's unique global extension to an $\downfree$ free type over A. So $q|_{x_i} = p$ for all $i < \omega$.
Now let $\left\langle \bar{d}^n \,| n < \omega \right\rangle$ be a Morley sequence generated by q over M, i.e. $\bar{d}^n \models q|_{M\bar{d}_{<n}}$. In the array $\langle d_i^n \,| n, i < \omega \rangle$, every row $\langle d_i^n \,| i < \omega \rangle$ is indiscernible with the same type over A as $\langle b_i \,| i < \omega \rangle$ (the original sequence witnessing diving), and for every $\eta : \omega \to \omega$, $\left\langle d^n_{\eta(n)} \,| n < \omega \right\rangle \models p^{(\omega)} \restriction M$.
So we have an array $\langle \varphi(x, d_i^n) \,| i, n < \omega \rangle$ such that every row is k inconsistent, so by the definition of an NTP$_2$ theory, it must be that for all/any $\eta : \omega \to \omega$, $\left\{\varphi\left(x, d_i^{\eta(i)}\right) | i < \omega \right\}$ is inconsistent, so it follows that any Morley sequence generated by p over M witnesses dividing over A of $\varphi(x,a)$.
Let $a' \models p|_M$, so $a' \equiv_A a$, so there is some automorphism σ fixing A taking a' to a. $\sigma(p)$ is also $\downfree$ free over A, and any Morley sequence generated by $\sigma(p)$ over $\sigma(M)$ witnesses dividing over A of $\varphi(x,a)$, and $a \models \sigma(p)|_{\sigma(M)}$. So wlog σ is id, and we are done. $\square$

Remark. The above claim and proof, with some modifications and generalizations is due to Usvyatsov and Onshuus in [OUa]. It should be noted that H.Adler and A.Pillay were the first to realize that NTP$_2$ is all the assumption one needs.

From here on assume that $\downfree$ has left extension over A.

Corollary 3.11. *Forking over A implies quasi dividing over A.*

Proof. Suppose $\alpha(x,e)$ forks over A, then $\alpha(x,e) \vdash \bigvee_{i<n} \varphi_i(x,a_i)$ where for all $i < n$, $\varphi_i(x,a_i)$ divides over A. By claim 3.10, for each i, there is some p_i which is a global $\downfree$ free extension of $tp(a_i/A)$, and a model $M_i \supseteq A$ as above. So let I_0 be some indiscernible sequence witnessing dividing of $\varphi_0(x,a_0)$. For $0 < i$, let I_i be a Morley sequence generated by p_i as follows: $a_0^i = a_i \models p_i|_{M_i}$, and for all $j > 0$, $a_j^i \models p_i|_{M_i I_{<i} a^i_{<j}}$. This will set us in the situation of the broom lemma 3.7 hence α quasi divides. $\square$

Definition 3.12. We say that $tp(a/Bb)$ is strictly invariant over B (denoted by $a \downfree^{ist}_B b$) if there is a global extension p, which is strongly Lascar invariant over B (so $a \downfree^{i}_B b$) and for any $C \supseteq B$, if $c \models p|_C$ then $C \downfree^{f}_B c$.

Remark 3.13. $\downfree^{ist}$ satisfies extension, invariance and monotonicity.

Claim 3.14. If $\varphi(x,a)$ divides over B, and p is a global $\downfree^{ist}$ free type extending $tp(a/B)$, and $M \supseteq B$ is some model, then any (some) Morley sequence generated by p over M witnesses dividing of $\varphi(x,a)$.

Proof. Suppose $\varphi(x,a)$ divides over B. Suppose $M \supseteq B$ is a model and N is $|M|^+$saturated. Let $b \models p|_N$. So $\varphi(x,b)$ divides over B and suppose I is an indiscernible sequence starting with b which witnesses k dividing. So $b \downfree^{ist}_B N$, hence $N \downfree^{f}_B b$ and by 2.4, there is $I' \equiv_{Bb} I$ that is indiscernible over N. Denote $I' = \langle b_i \,|i < \omega\rangle$. For each $i < \omega$, $b_i \models p|_N$. Consider $q = tp(I'/N)$ and generate a sequence $\langle I_i \,|i < \omega\rangle$ in N as follows: $I_i \models q|_{MI_{<i}}$. Denote $I_i = \left\langle a^i_j \,|j < \omega \right\rangle$.$\langle I_i \,|i < \omega\rangle$ is not necessarily a Morley sequence as we do not know that $I' \downfree^{ist}_B N$. Nevertheless for each $\eta : \omega \to \omega$, $\left\langle a^i_{\eta(i)} \,|i < \omega \right\rangle$ is a Morley sequence of type $p^{(\omega)}|_M$ (because p is M invariant). If $\left\{\varphi\left(x, a^i_0\right) |i < \omega\right\}$ was consistent, than every vertical path in the array $\left\langle \varphi\left(x, a^i_j\right) |i,j < \omega \right\rangle$ is consistent, but we know that every row is k inconsistent so this is a contradiction to NTP$_2$. $\square$

Claim 3.15. If B is an extension base for $\downfree^{ist}$, then forking equals dividing over B.

Proof. Suppose $\varphi(x,a) \vdash \bigvee_{j<n} \varphi_j(x,a_j)$, each $\varphi_j(x,a_j)$ divides over B. Let $\bar{a} = aa_0 \dots a_{n-1}$ Let $M \supseteq B$ be a model, and $N \supseteq M$ an $|M|^+$ saturated model. By existence, $\bar{a} \downfree^{ist}_B B$, so by extension and invariance, we may assume that $\bar{a} \downfree^{ist}_B N$. Let q be the global unique type extending $tp(\bar{a}/N)$, $\downfree$ free over B. Let $\left\langle \bar{a}^i \,|i < \omega \right\rangle$ be a Morley sequence generated by q over M. So for each $i < \omega$, $\varphi(x,a_i) \vdash \bigvee_{j<n} \varphi_j\left(x, a^i_j\right)$. By 3.14 each indiscernible sequence $\left\langle a^i_j \,|j < \omega \right\rangle$ witnesses that $\varphi_j(x,a_j)$ divides over B (notice that it does not necessarily starts with a_j). If $\{\varphi(x,a_i) \,|i < \omega\}$ was consistent, than we would have a contradiction (if $\varphi(c,a_i)$ for all i, then there is some $j < n$ such that $\varphi_j\left(c, a^i_j\right)$ for infinitely many i s). Hence $\varphi(x,a)$ divides over B. $\square$

Claim 3.16. A is an extension base for $\downfree^{ist}$.

Proof. We want to show that for any a, $a \downfree^{ist}_A A$. So let $q = tp(a/A)$. We shall show that the following set is consistent

$$\begin{aligned} q \quad &\cup \quad \{\neg\psi(x,d) \,|\psi \text{ is over } A \text{ and } \psi(a,y) \text{ forks over } A\} \\ &\cup \quad \{\varphi(x,e) \leftrightarrow \varphi(x,f) \,|\varphi \text{ is over } A \text{ and } lstp(e/A) = lstp(f/A)\} \end{aligned}$$

Because if it is consistent, then we let p be a global type containing it, and it will satisfy the requirements.

So suppose not, then $q \vdash \bigvee_i \neg(\varphi_i(x,e_i) \leftrightarrow \varphi_i(x,f_i)) \vee \psi(x,d)$ where $lstp(e_i/A) = lstp(f_i/A)$ and $\psi(a,y)$ forks over A. Because $\psi(a,y)$ forks over A, it quasi divides over A (by 3.11). So there are $a_1, \dots, a_n$ such that $a_i \equiv_A a$ and $\{\psi(a_i, y) \,|i < n\}$ is inconsistent. Let $r = tp(a_1, \dots, a_n/A)$. So

$$r|_{x_j} = q \vdash \bigvee_i \neg(\varphi_i(x_j, e_i) \leftrightarrow \varphi_i(x_j, f_i)) \vee \psi(x_j, d)$$

for $j < n$. So

$$r \vdash \bigwedge_j \left[\bigvee_i \neg(\varphi_i(x_j, e_i) \leftrightarrow \varphi_i(x_j, f_i)) \vee \psi(x_j, d)\right]$$

But $\neg\exists y \left(\bigwedge_j \psi(x_j, y)\right) \in r$, so $r \vdash \bigvee_{i,j} \neg(\varphi_i(x_j, a_i) \leftrightarrow \varphi_i(x_j, b_i))$. But this is a contradiction, as r is over A, and as A is an extension base for $\downarrow$, there is a global invariant type extending r. $\square$

Remark 3.17. Alex Usvyatsov noticed that one can use the broom lemma to prove that types over models can be extended to global non-forking heirs (see [Usva]). A very similar proof as the above proof can show that they can also be extended to global non-coforking coheirs - p is a global non-coforking coheir over M, if for any $C \supseteq M$, $c \models p|_C$, $c \downarrow^u_M C$ and $C \downarrow^f_M c$ (so they are also strictly invariant types).

Corollary 3.18. *Forking equals dividing over A.*

Proof. Just combine the last two claims. $\square$

By this we have proved one direction of theorem 3.9. As for the other one:

Claim 3.19. (T dependent) (2) implies (1) in 3.9.

Proof. $\downarrow^f$ satisfies all of the demands: as forking equals dividing over A and types never divide over their domain, A is an extension base for $\downarrow^f$. $\downarrow^f$ is the same as $\downarrow^i$ in dependent theories and by 3.3, we have left extension over A. $\square$

Remark 3.20.

(1) Strictly invariant types are a special case of strictly non-forking types. We say that $tp(a/Bb)$ strictly does not fork over B (denoted by $a \downarrow^{st}_B b$) if there is a global extension p, which does not fork over B, and for any $C \supseteq B$, if $c \models p|_C$ then $C \downarrow^f_B c$. They coincide in dependent theories, and in stable theories they are the same as non-forking. The notion originated in [Sheb, 5.6], and the proof of the next lemma is based on ideas from section 5 there. More on strict non-forking can be found in [Usva].

(2) Lemma 3.14 above is an analog of what is known as Kim's lemma in simple theories, that states in the simple context, every Morley sequence witnesses dividing. It was noticed independently by Alex Usvyatsov (see [Usva]). In fact, a more proper analog (and a generalization of 3.14) would be

Lemma 3.21. *If $\varphi(x, a)$ divides over B, and $\langle a_i \,|i < \omega\rangle$ is a $\downarrow^{ist}$ sequence over B (i.e. $a_i \downarrow^{ist}_B a_{<i}$) such that $a_i \equiv_B a$, then $\{\varphi(x, a_i) \,|i < \omega\}$ is inconsistent.*

Proof. Let I be an indiscernible sequence witnessing k dividing of $\varphi(x, a)$ starting with a. We build by induction on n a sequence of indiscernible sequences $I_i = \left\langle a^i_j \,|j < \omega\right\rangle$ for $i < n$, such that I_i is indiscernible over $BI_{<i}a_0^{>i}$, $a_0^i = a_i$ and $I_i \equiv_B I$. By compactness we can find such a sequence of length ω. This will give us

an array $\left\{\varphi\left(x,a^i_j\right)|i<\omega\right\}$ in which each row is k inconsistent and for $\eta:\omega\to\omega$, $\left\langle a^i_{\eta(i)}|i<\omega\right\rangle\equiv_B\langle a_i|i<\omega\rangle$, so by the definition of NTP$_2$, we are done.
For $n=1$, there is such an I_1 because $a\equiv_B a_1$. Assume we have such a sequence for n. As $a_{n+1}\equiv_B a$, there is some indiscernible sequence $I'\equiv_B I$ starting with a_{n+1}. As $a_{n+1}\mathop{\smile\!\!\!\!\!\!|}^{ist}_B a_{<n+1}$, we may assume by extension and invariance of $\mathop{\smile\!\!\!\!\!\!|}^{ist}$ that $a_{n+1}\mathop{\smile\!\!\!\!\!\!|}^{ist}_B I_{<n+1}$ (here we change the sequence we already built, but we retain all its properties). So $I_{<n+1}\mathop{\smile\!\!\!\!\!\!|}^{f}_B a_{n+1}$. By 2.4, there is some $I_{n+1}\equiv_{Ba_{n+1}}I'$, which is indiscernible over $BI_{<n+1}$. For $i<n+1$, I_i is indiscernible over $C=BI_{<i}a_0^{n>,>i}$, so by preservation of indiscernibles (see 2.9), as $a_{n+1}\mathop{\smile\!\!\!\!\!\!|}^{i}_C I_i$ (by base monotonicity), it follows that I_i is indiscernible over Ca_{n+1}, which is exactly what we needed to prove. $\square$

3.4. **Applying the previous section.**

Here we assume T is NTP$_2$ unless stated otherwise.

Theorem 3.22. *Forking equals dividing over models.*

Proof. Let $\mathop{\smile\!\!\!\!\!\!|}$ be $\mathop{\smile\!\!\!\!\!\!|}^u$ (co-inheritance) and M a model. Then $\mathop{\smile\!\!\!\!\!\!|}^u$ has all properties listed in 2.9 except left extension, which it has over M (see 2.12), and M is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^u$. So it has all the assumption of the previous section, hence we can apply corollary 3.18. $\square$

Theorem 3.23. *If A is a left extension base for $\mathop{\smile\!\!\!\!\!\!|}^i$, then forking equals dividing over A (so in the dependent context, replace $\mathop{\smile\!\!\!\!\!\!|}^i$ by $\mathop{\smile\!\!\!\!\!\!|}^f$), moreover, A is an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{ist}$.*

Proof. Let $\mathop{\smile\!\!\!\!\!\!|}$ be $\mathop{\smile\!\!\!\!\!\!|}^i$ - invariance. Then $\mathop{\smile\!\!\!\!\!\!|}$ and A have all the properties required by the previous section. $\square$

By 2.12,

Corollary 3.24. *In $\mathop{\smile\!\!\!\!\!\!|}^i$-extensible NTP$_2$ theories, forking equals dividing over any set, and it is $\mathop{\smile\!\!\!\!\!\!|}^{ist}$-extensible as well.*

So, for left extension bases for $\mathop{\smile\!\!\!\!\!\!|}^i$, not only is forking the same as dividing, but also they are $\mathop{\smile\!\!\!\!\!\!|}^{ist}$ extension base, and so if $\varphi(x,b)$ forks over A, every global $\mathop{\smile\!\!\!\!\!\!|}^{ist}$ free type extending $tp(b/A)$ witnesses it: every Morley sequence generated by it witnesses it.
If A is just an extension base for $\mathop{\smile\!\!\!\!\!\!|}^f$, or if $\mathop{\smile\!\!\!\!\!\!|}^f$ has left extension over it, then we can also conclude that forking equals dividing, but may no longer be an extension base for $\mathop{\smile\!\!\!\!\!\!|}^{ist}$. But first we shall need a small lemma, which is always true.

Lemma 3.25. *(T any theory) If $a\mathop{\smile\!\!\!\!\!\!|}^f_A b$ and $\varphi(x,b)$ forks over A then it forks over Aa.*

Proof. Recall that if $a\mathop{\smile\!\!\!\!\!\!|}^f_A c$ and $\phi(x,c)$ divides over A then it divides over Aa (see 2.4). $\varphi(x,b)$ forks over A, so there are $n<\omega$, $\phi_i(x,y_i)$ and b_i for $i<n$ such that $\phi_i(x,b_i)$ divides over A and $\varphi(x,b)\vdash\bigvee_{i<n}\phi_i(x,b_i)$. By extension, we may assume $a\mathop{\smile\!\!\!\!\!\!|}^f_A b\langle b_i|i<n\rangle$. Hence $\phi_i(x,b_i)$ divides over Aa. Hence $\varphi(x,b)$ forks over Aa. $\square$

So now we can finish the proof of theorem 1.2.

Theorem 3.26. *Assume that*

1. *A is an extension base for non-forking; or*
2. *Non-forking satisfies left extension over A*

Then forking equals dividing over A.

Proof. So suppose $\varphi(x,b)$ forks over A. Let $M \supseteq A$ be some model.
If (1) is true, $M \mathop{\smile\!\!\!\!\!\!\perp}^f_A A$, and by extension and invariance we may assume (maybe replacing M by some $M' \equiv_A M$) that $M \mathop{\smile\!\!\!\!\!\!\perp}^f_A b$.
If (2) is true, then, as $A \mathop{\smile\!\!\!\!\!\!\perp}^f_A b$ (even $A \mathop{\smile\!\!\!\!\!\!\perp}^u_A b$), by left extension, we may assume $M \mathop{\smile\!\!\!\!\!\!\perp}^f_A b$.
By the previous lemma, $\varphi(x,b)$ forks over M, so by theorem 3.22, it divides over M. As $M \supseteq A$, it divides over A, as we wanted. □

Corollary 3.27. *Theorem 1.2, i.e. for a set A, the following are equivalent:*

1. *A is an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^f$ (non-forking) (see definition 2.11).*
2. *$\mathop{\smile\!\!\!\!\!\!\perp}^f$ has left extension over A (see definition 2.9).*
3. *forking equals dividing over A.*

Proof. Combine 3.26 and 3.1. □

And we can deduce a stronger version of 3.23:

Corollary 3.28. *A is an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^i$ iff A is an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^{ist}$ and in that case forking equals dividing over A.*

Proof. The "only if" direction (the "if" direction is clear): $\mathop{\smile\!\!\!\!\!\!\perp}^i$ is stronger than $\mathop{\smile\!\!\!\!\!\!\perp}^f$, so by 1.2, forking equals dividing over A. The proof of 3.16 only uses the fact that forking implies quasi dividing over A and that A is an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^i$. So in our case the proof works. □

3.5. Some corollaries for dependent theories.

From here on, assume A is a set over which forking equals dividing, for example - a model, and T is dependent. We start with an obvious improvement of theorem 1.2:

Theorem 3.29. *The following are equivalent for A:*

1. *A is an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^f$.*
2. *$\mathop{\smile\!\!\!\!\!\!\perp}^f$ has left extension over A.*
3. *Forking equals dividing over A.*
4. *A is an $\mathop{\smile\!\!\!\!\!\!\perp}^{ist}$ extension base.*

Proof. If A is an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^{ist}$, then obviously it's an extension base for $\mathop{\smile\!\!\!\!\!\!\perp}^f$, so the theorem follows from 1.2.
If one of (1) - (3) is true, then all are true by 1.2. As T is dependent, $\mathop{\smile\!\!\!\!\!\!\perp}^f = \mathop{\smile\!\!\!\!\!\!\perp}^i$, by 3.28 we are done. □

Corollary 3.30. *The following are equivalent for a formula $\varphi(x,a)$:*

- *φ forks over A.*
- *φ quasi Lascar divides over A.*

- *φ divides over A.*

Proof. We only need to show the equivalence of the first 2. If φ forks over A, then it quasi Lascar divides because forking equals dividing over A. If φ does not fork over A, then extend it to a global non forking type over A, p. By dependence, p is strongly Lascar invariant over A. This means that it contains all Lascar conjugates of φ over A, and in particular it is impossible for φ to quasi Lascar divide. □

Remark 3.31. Dividing in type definable, so in dependent theories all these notions are type-definable over A (i.e. dependent theories are low, see [Bue99])

Proof. (Due to Itai Ben Yaacov) First we shall see that for any set B, if $\varphi(x,a)$ divides over B then it $k := alt(\varphi)$ divides over B. if $\langle a_i \,|i < \omega\rangle$ is an indiscernible sequence witnessing $m > k$ dividing but not k dividing, it means that $\exists x \bigwedge_{i<k} \varphi(x,a_i)$, and by indiscerniblity, $\exists x \bigwedge_{i<k} \varphi(x,a_{mi})$. So assume $\varphi(c,a_{mi})$ for $i < k$. But for each i, there must be some $m(i-1) < j_i \leq mi - 1$ such that $\neg\varphi(c,a_{j_i})$. This is a contradiction to the definition of *alt* (see definition 2.6).
So it follows that dividing is type definable - there is a (partial) type $\pi(x,Y)$ (Y the same length as B) such that $\pi(a,B)$ iff $\varphi(x,a)$ divides over B. The type would say that there exists a sequence $\langle x_i \,|i < \omega\rangle$ of elements having the same type as x over Y, and that every k subset of formulas of the form $\varphi(y,x_i)$ is inconsistent. □

The following is a strengthening of [HP, Lemma 9.10]

Corollary 3.32. *Let p be a partial type which is Lascar invariant over A. Then there exists some global Lascar invariant over A extension of p in $S(\mathfrak{C})$.*

Proof. If $\varphi_1,\ldots,\varphi_n \in p$, then $\bigwedge_i \varphi_i$ does not Lascar quasi divide over A (because all the conjugates of φ_i are in p for all i). Hence p does not fork over A, hence there is a global non-forking (hence Lascar invariant) extension. □

4. Bounded Forking + NTP$_2$ = Dependent

It is well-known that stable theories can be characterized as those simple theories in which every type over model has boundedly many non-forking extensions (see e.g. [Adlb, theorem 45]). Our aim in this section is to prove a generalization of this fact: if non-forking is bounded, and the theory is NTP$_2$, then the theory is actually dependent. By doing this we give a partial answer to a question of Hans Adler.

We quote from [Adlb, Section 6, Corollary 38]:

Fact 4.1. *The following are equivalent for a theory T:*

1. Every type over model has boundedly many global non-forking extensions.
2. For every global type p and every model M, p does not fork over M if and only if it is invariant over M.

So to conclude it is enough to find a global non-forking type over a model which is not invariant over it.

Theorem 4.2. *Assume T is NTP$_2$, but has the independence property, then there is a global non-forking type over a model which is not invariant.*

So assume $\varphi(x,y)$ has the independence property. This means that there is an infinite set A (may be a set of tuples), such that for any subset $B \subseteq A$, there is some b such that for all $a \in A$, $\varphi(b,a)$ iff $a \in B$. Let $r(x) = \{x \neq a \,|a \in A\}$ be a partial type over A. Since it is finitely satisfiable in A there is a global type p containing r which is finitely satisfied in A. p is A invariant, so $p^{(\omega)}$ is well defined. It is even finitely satisfiable in A. Let $\psi(x,y,z)=\varphi(x,y)\wedge\neg\varphi(x,z)$.

Claim 4.3. There is some $|C|\leq\aleph_1$ such that if $I=\langle a_i\,|i<\omega\rangle$ is an indiscernible sequence such that for every $i<\omega$, $a_i\models p^{(2)}|_{AC}$, then $\Gamma_I=\{\psi(x,a_i)\,|i<\omega\}$ is consistent.

Proof. Assume not. Define by induction $\left\langle I^j\,|j<\omega_1\right\rangle$ such that

- For all $i<\omega$, $I^j=\left\langle a_i^j\,|i<\omega\right\rangle$ is indiscernible (over $\emptyset$).
- For all $i,j<\omega$, $a_i^j\models p^{(2)}|_{AI^{<j}}$.
- $\Gamma_j=\Gamma_{I^j}$ is inconsistent.

How? for $C=\emptyset$ the conclusion is false, so we can find I_0 as above, such that Γ_0 is inconsistent. For $0<j$, by our assumption $C=I^{<j}$ is not the desired C, so we can continue. So for each $j<\omega_1$, Γ_j is inconsistent, so, by I^j being indiscernible, it follows that it is k_j inconsistent for some $k_j<\omega$. So for infinitely many js, Γ_j is k inconsistent for some k, so we may assume that this is the case for $j<\omega$. Consider the array $\left\{\psi\left(x,a_i^j\right)|i,j<\omega\right\}$. Each row is k-inconsistent, and to get a contradiction to NTP$_2$, it's enough to show that each vertical path is consistent. For all $\eta:\omega\to\omega$, $tp\left(\left\langle a_{\eta(j)}^j\right\rangle_{j<\omega}/A\right)=p^{(\omega)}|_A$, so it's enough to show that some vertical path is consistent. So take the first column, and show that $\left\{\psi\left(x,a_0^j\right)|j<\omega\right\}$ is consistent, so we need to show that $\left\{\psi\left(x,a_0^j\right)|j<n\right\}$ is consistent for all $n<\omega$. Assume not, so $\neg\exists x\bigwedge_{j<n}\psi\left(x,a_0^j\right)$, but $\left\langle a_0^j\,|j<n\right\rangle\models p^{(2n)}|_A$, so it is finitely satisfiable in A, so there are distinct $a_j,b_j\in A$ (i.e. with no repetitions), such that $\neg\exists x\bigwedge_{j<n}\psi(x,a_j,b_j)$. But that is a contradiction to the choice of A and φ.

Let M be a model containing AC. $ab\models p^{(2)}|_M$. So the formula $\psi(x,a,b)$does not divide over M (by the claim above), thus does not fork by 3.22. Then there is some global type containing it and non-forking over M, which is certainly not invariant over M (because of the choice of ψ and the fact that $a\equiv_M b$) and we are done. □

5. Optimality of results

In general, forking is not the same as dividing, and Shelah already gave an example in [She90, III,2], and Kim gave another example in his thesis ([Kim96, Example 2.11]) - circular ordering. Both examples were over the empty set, and the theory was dependent.

Here we give 2 examples. The first shows that outside the realm of NTP$_2$, our results are not necessarily true, and the second shows that even in dependent theories, forking is not the same as dividing even over sets containing models.

In both examples, we use the notion of a (directed) circular order, so here is the definition:

Definition 5.1. A circular order on a finite set is the ternary relation obtained by placing the points on a circle and taking all triples in anticlockwise order. For an infinite set, a circular order is a ternary relation such that the restriction to any finite set is a circular order.

5.1. **Example 1.** Here we present a variant of an example found by Martin Ziegler, showing that

(1) forking and dividing over models are different in general.
(2) Strictly non-forking types need not exist over models (see 3.20), so in particular, strictly invariant types and non-forking heirs need not necessarily exist over models.

Let L be a 2 sorted language: one sort for "points", which will use the variables t and another for "sets", which we denote with s. L consists of 1 binary relation $E(t,s)$ to denote "membership", and 2 4-ary relations: $C(t_1,t_2,t_3,s)$ and $D(s_1,s_2,s_3,t)$. Consider the class K of all finite structures of this language satisfying:

(1) For all s, $C(-,-,-,s)$ is a circular order on the set of all t such that $E(t,s)$, and if $C(t_1,t_2,t_3,s)$ then $E(t_i,s)$ for $i=1,2,3$, and
(2) For all t, $D(-,-,-,t)$ is a circular order on the set of all s such that $\neg E(t,s)$, and if $D(s_1,s_2,s_3,t)$ then $\neg(E(t,s_i))$ for $i=1,2,3$.

This class has the Hereditary property, the Joint embedding property and the Amalgamation property as can easily be verified by the reader. Hence, as the language has no function symbols, there is a Fraïssé limit - a theory T which is complete, ω-categorical and eliminates quantifiers. See [Hod93, Theorem 7.4.1] for the details.

Let M be a model of T and M_1 be an $|M|^+$ saturated extension. We choose $t_0,s_0\in M_1\backslash M$, such that for all $t\in M$, $\neg E(t,s_0)$ and for all $s\in M$, $E(t_0,s)$. Now, $E(x,s_0)$ forks over M, and $\neg E(t_0,y)$ forks over M, but non of them (quasi) divides.

Why? Non quasi dividing is straight forward from the construction of T.

We show that $\neg E(t_0,y)$ forks (for $E(x,s_0)$ use the same argument): choose some circular order on $Points^M$, and choose s_i' for $i<\omega$ such that:

- $\neg E(t_0,s_i')$ for $i<\omega$.
- $D(s_i',s_j',s_k',t_0)$ whenever $i<j<k$.
- For all $i<\omega$, for all $t\in M$, $E(t,s_i')$, and $C(-,-,-,s_i')$ orders $Points^M$ using the prechosen circular order.

Now, $\neg E(t_0,y)\vdash D(s_0',y,s_1',t_0)\vee D(s_1',y,s_0',t_0)$ and $D(s_0',y,s_1',t_0)$ divides over Mt_0 as witnessed by $\langle s_i's_{i+1}' \,|i<\omega\rangle$, and so does $D(s_1',y,s_0',t_0)$.

Let $p(t)$ be $tp(t_0/M)$. We show that p is not a strictly non-forking type over M: suppose q is a global strictly non-forking extension, and let $t_0'\models q|_{M_1}$. Then $t_0'\downarrow_M M_1$ and $M_1\downarrow_M t_0'$. But $s_0\in M_1$ and surely $\neg E(t,s_0)\in q$, so $\neg E(t_0',s_0)$. $t_0'\equiv_M t_0$ so $s_0\not\downarrow_M t_0'$ - a contradiction.

Note that T has the tree property of the 2nd kind: Let s_i for $i<\omega$ be such that they are all different, and for each i, let t_j^i for $j<\omega$, be such that for $j<k<l$, $C(t_j^i,t_k^i,t_l^i,s_i)$. The array $\{C(t_j^i,x,t_{j+1}^i,s_i)\,|i,j<\omega\}$ witnesses TP$_2$.

5.2. **Example 2.** We give an example showing that even if T is dependent, and S contains a model, forking is not necessarily the same as dividing over S. Hence models are not good extension bases (see 2.11) in dependent theories in general.

Let L be a 2 sorted language. One sort for "points", which will use the variables t and another for "sets", which we denote with s. L contains a binary ordering relation $<$, a binary "membership" function f from the points sort to the sets sort,

and a 4-ary relation $C(t_1, t_2, t_3, s)$.
Let T have the axioms:

(1) $<$ is a dense linear order without end points on $Sets$.
(2) $C(-,-,-,s)$ is a dense circular order on the infinite set $f^{-1}(s) = \{t \,|\, f(t) = s\}$ (i.e. for all t_1, t_2 from $f^{-1}(s)$, there is t_3 such that $C(t_1, t_3, t_2, s)$), and $C(t_1, t_2, t_3, s)$ implies $f(t_i) = s$ for $i = 1, 2, 3$.

It is easy to see that T is complete and has quantifier elimination (for example because it is the Fraïssé limit of an age of the appropriate class of finite structures). Moreover, T is dependent: To show this, it's enough to show that all formulas $\varphi(x, y)$ where x is one variable have finite alternation number. As T eliminates quantifiers, it's enough to consider atomic formulas (see e.g. [Adlb, Section 1]), and this is left to the reader.
Let M be a model, M_1 an $|M|^+$ saturated model. Let $s \in M_1 \backslash M$. Let $t_1, t_2 \in f^{-1}(s)$, then $C(t_1, x, t_2, s)$ divides over Ms (because one can find a sequence $\langle t_i \,|\, i < \omega \rangle$ in $f^{-1}(s)$ such that $C(t_i, t_j, t_k, s)$ iff $i < j < k$ starting with t_1 and t_2, $t_i t_{i+1} \equiv_{Ms} t_1 t_2$, and so $\langle t_i t_{i+1} \,|\, i < \omega \rangle$ witnesses 2 dividing).
So $E(x, s) \vdash C(t_1, x, t_2, s) \vee C(t_2, x, t_1, s)$ forks but does not divide over $M \cup \{s\}$.

6. Questions and remarks

(1) Are simple theories $\mathop{\smile\!\!\!\!\!\!|}^{i}$-extensible NTP$_2$ theories?
(2) In [Sta] Starchenko gives a quite natural and informative characterization of non-forking in o-minimal theories. It would be very nice to find a proper generalizations to dependent theories (or at least to dp-minimal theories).
(3) What about generalizing the results of this paper to n-dependent theories? (See [Shea]) (Is Ziegler's example n-dependent for some n?) To SOP_n? $NSOP_2$ or even to $NSOP$?
(4) It would be nice to find some purely semantic characterization of theories in which forking equals dividing over models. For example we know that all NTP$_2$ theories are such, however the opposite is not true: there is a theory with TP$_2$ in which forking=dividing (essentially the example from section 5, but with dense linear orders instead of circular ones).

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