Definable t-regularity theorem

Tomohiro Kawakami
Department of Mathematics, Faculty of Education, Wakayama University,
Sakaedani Wakayama 640-8510, Japan
kawa@center.wakayama-u.ac.jp

Abstract

We consider locally definable C^∞ manifolds, locally definable C^∞ maps and study t-regularity of locally definable C^∞ maps

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1 . Introduction.

Let $\mathcal{M} = (\mathbb{R}, +, \cdot, <, e^x, \dots)$ be an exponential o-minimal expansion of the standard structure $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ of the field $\mathbb{R}$ of real numbers. General references on o-minimal structures are [1], [2], see also [7]. For example, the Nash category is a special case of the definable C^∞ category and it coincides with the definable C^∞ category based on $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$ ([8]). Equivariant definable category is studied in [3], [4], [5].

In this paper "definable" means "definable with parameters in $\mathcal{M}$", everything is considered in $\mathcal{M}$, "countable" means finite or countably infinite and each locally definable map is continuous unless otherwise stated.

A subset X of $\mathbb{R}^n$ is called *locally definable* if for every $x \in X$ there exists a definable open neighborhood U of x in $\mathbb{R}^n$ such that $X \cap U$ is a definable subset of X. Clearly every definable set is locally definable, every compact locally definable set is definable and any open subset of $\mathbb{R}^n$ is locally definable.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be locally definable sets. We say that a continuous map $f : U \to V$ is a *locally definable map* if for any $x \in U$ there exists a definable open neighborhood W_x of x in $\mathbb{R}^n$ such that $f|U \cap W_x$ is definable.

Two locally definable maps $f, h : X \to Y$ between locally definable sets are *locally definably homotopic* if there exists a locally definable map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ for all $x \in X$ and $H(x, 1) = h(x)$ for all $x \in X$.

Let M^n, N^p be locally definable C^∞ manifolds of dimension n, p, respectively, $f : M^n \to N^p$ a locally definable C^∞ map, N_1^{p-q} a $(p-q)$-dimensional locally definable C^∞ submanifold of N^p. We say that f is *t-regular* on N_1^{p-q} if for any $x \in f^{-1}(N_1^{p-q})$, $(df)_x(T_x M^n) + T_{f(x)} N_1^{p-q} = T_{f(x)} N^p$.

Theorem 1.1. *Let M^n, N^p be locally definable C^∞ manifolds of dimension n, p, respectively, $f : M^n \to N^p$ a locally definable C^∞ map, N_1^{p-q} a $(p-q)$-dimensional locally definable C^∞ submanifold of N^p. Let*

A be a locally definable closed subset of M^n such that there exists a locally definable open neighborhood U of A such that $f|U$ is t-regular on N_1^{p-q}. For every positive locally definable continuous function $\delta : M^n \to \mathbb{R}$, there exists a locally definable C^∞ map $h : M^n \to N^p$ satisfies the following conditions.

(1) h is locally definable homotopic to f.

(2) g is a δ-approximation of f.

(3) g is t-regular on N_1^{p-q}.

(4) $h|A = f|A$.

Theorem 1.2. *Every n-dimensional locally definable C^∞ manifold X is locally definably C^∞ imbeddable into $\mathbb{R}^{2n+1}$.*

Theorem 1.2 is proved in [6] the case where r is a positive integer.

2 Proof of results

Remark that for any locally definable map f between locally definable sets X and Y, if X is compact, then $f(X)$ is a definable set and $f : X \to f(X)$ $(\subset Y)$ is a definable map.

Note that the maps $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = \sin x, f_2(x) = \cos x$, respectively, are analytic but not locally definable in $\mathcal{R} = (\mathbb{R}, +, \cdot, <)$, and that the field $\mathbb{Q}$ $(\subset \mathbb{R})$ of rational numbers is not a locally definable subset of $\mathbb{R}$. For example, if $\mathcal{M} = \mathbf{R}_{an,exp}$, then $f : (-1,1) \to \mathbb{R}, f(x) = \sin \frac{1}{1-x^2}$ is locally definable but not definable.

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open sets. A C^r map $f : U \to V$ is called a *locally definable C^r map* if f is locally definable. A locally definable C^r map $f : U \to V$ is called a *locally definable C^r diffeomorphism* if there exists a locally definable C^r map $h : V \to U$ such that $f \circ h = id$ and $h \circ f = id$.

Definition 2.1 ([6])**.** Let $1 \le r \le \omega$.
(1) A locally definable subset X of $\mathbb{R}^n$ is called a *d-dimensional locally definable C^r submanifold of* $\mathbb{R}^n$ if for any $x \in X$ there exists a definable C^r diffeomorphism ϕ from some definable open neighborhood U of the origin in $\mathbb{R}^n$ onto some definable open neighborhood V of x in $\mathbb{R}^n$ such that $\phi(0) = x, \phi(\mathbb{R}^d \cap U) = X \cap V$. Here $\mathbb{R}^d = \{x \in \mathbb{R}^n |$ last $(n-d)$ components of x are zero.$\}$
(2) A *locally definable C^r manifold X of dimension d* is a C^r manifold with a countable system of charts $\{\phi_i : U_i \to \mathbb{R}^d\}$ such that for each i and j $\phi_i(U_i \cap U_j)$ is a definable open subset of $\mathbb{R}^d$ and the map $\phi_j \circ \phi_i^{-1}|\phi_i(U_i \cap U_j) : \phi_i(U_i \cap U_j) \to \phi_j(U_i \cap U_j)$ is a definable C^r diffeomorphism. We call these atlas *locally definable C^r*. Locally definable C^r manifolds with compatible atlases are identified. Clearly every definable C^r manifold is a locally definable C^r manifold. A subset Y of a locally definable C^r manifold X is called a *k-dimensional locally definable C^r submanifold of X* if each point $x \in Y$ there exists a locally definable C^r chart $\phi_i : U_i \to \mathbb{R}^d$ of X such that $x \in U_i$ and $U_i \cap Y = \phi_i^{-1}(\mathbb{R}^k)$, where $\mathbb{R}^k \subset \mathbb{R}^d$ is the vectors whose last $(d-k)$ components are zero.
(3) A locally definable C^r manifold is *affine* if it can be imbedded into some $\mathbb{R}^n$ in a locally definable C^r way.

Since a locally definable set X is paracompact, for any countable definable open cover $\{U_\alpha\}$ of X, there exists a partition of unity $\{f_\alpha\}$ subordinate to $\{U_\alpha\}$ such that each f_α is locally definable. Thus we have the following theorem.

Theorem 2.2. *Let X be a locally definable C^∞ manifold. Every locally definable open cover of X has a subordinate locally definable C^∞ partition of unity.*

Definition 2.3. Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be locally definable sets, $f, h : X \to Y$ locally definable maps and $\delta : X \to \mathbb{R}$ a positive locally definable function. We say that g is a *δ-approximation* of f if $d_m(f(x), g(x)) < \delta(x)$ for any $x \in X$, where d_m means the standard metric of $\mathbb{R}^m$.

Proposition 2.4. *Let X be a locally definable C^∞ manifold. Then every C^∞ map $f : X \to \mathbb{R}^n$ is approximated in the C^∞ Whitney topology by a locally definable C^∞ map $h : X \to \mathbb{R}^n$.*

Proof. By Theorem 2.2, we have a locally definable C^∞ partition of unity $\{\phi_j\}_{j=1}^\infty$ subordinates to some locally finite open definable cover $\{X_j\}_{j=1}^\infty$ of X such that $X = \cup_{j=1}^\infty \text{supp } \phi_j$ and $\overline{X_j}$ is compact. For any j, take an open neighborhood U_j of supp ϕ_j in X such that $\overline{U_j}$ is compact. Applying the polynomial approximation theorem, we have a locally definable C^∞ map $h_j : U_j \to \mathbb{R}^n$ which approximates $f|U_j$. If our approximation is sufficiently close, then $\sum_{j=1}^\infty \phi_j h_j$ is a locally definable C^r approximation of f. □

Proof of Theorem 1.2. By Whitney's imbedding Theorem, there exists a C^r imbedding $f : X \to \mathbb{R}^{2n+1}$. Since imbeddings from X to $\mathbb{R}^{2n+1}$ are open in $C^r(X, \mathbb{R}^{2n+1})$, we have the required locally definable C^r imbedding $h : X \to \mathbb{R}^{2n+1}$. □

For a positive number k, $C^n(k)$ means the open ball of $\mathbb{R}^n$ with center 0 and radius k and $\overline{C^n(k)}$ denotes the closure of $C^n(k)$.

Proof of Theorem 1.1. Since N_1^{p-q} is a locally definable C^∞ submanifold, it is covered by a system of chart of N^q such that:

(1) $N_1^{p-q} \subset \cup_{i=1}^\infty Y_i$

(2) (Y_i, k_i) is a chart of N^p.

(3) $k_i : Y_i \cap N_1^{p-q} : Y_i \cap N_1^{p-q} \to \mathbb{R}^{p-q}$.

Let $Y_0 = N^p - N_1^{p-q}$. Then $\{Y_i | i \in \mathbb{N} \cup \{0\}\}$ is a locally definable open cover and $\{f^{-1}(Y_i) | i \in \mathbb{N} \cup \{0\}\}$ is a locally definable open cover of M^n. On the other hand, $M^n = U \cup (M^n - A)$ is a locally definable open cover. Thus there exists a locally definable C^∞ atlas $\{(V_j, h_j) | j \in \mathbb{Z}\}$ such that:

(1) $\{V_j\}$ is a locally finite refinement of $\{f^{-1}(Y_i)\}$ and $\{U, M - A\}$.

(2) $h_j(V_j) = C^n(3)$.

(3) Let $W_j = h_j^{-1}(C^n(1))$. Then $\{W_j\}$ is a locally definable open cover of M^n.

Renumbering V_j, if necessary, $j \leqq 0$ if $V_j \subset U$.

We can take a locally definable C^∞ function $\phi : \mathbb{R}^n \to \mathbb{R}$ such that:

(1) $\phi(\overline{C^n(1)}) = 1$.

(2) $0 < \phi(C^n(2) - \overline{C^n(1)}) < 1$.

(3) $\phi(\mathbb{R}^n - C^n(2)) = 0$.

We define $\phi_i : M^n \to \mathbb{R}$ to be $\phi_i(x) = \begin{cases} \phi \circ h_i(x), & x \in V_i \\ 0, & x \notin V_i \end{cases}$. Then ϕ_i is a locally definable C^∞ function and for each $f(V_j)$, there exists an $i(j)$ such that $f(V_j) \subset Y_{i(j)}$.

By induction, we construct the required map g. Let $f_0 = f$. Then $f_0|U$ is t-regular on N_1^{p-q}. Assume that a locally definable C^∞ map $f_{k-1} : M^n \to N^p$ is constructed such that:

(1) $f_{k-1}| \cup_{j<k} W_j$ is t-regular on N_1^{p-q}.

(2) $f_{k-1}(\overline{U_j}) \subset Y_{i(j)}$.

We now construct a locally definable C^∞ map $f_k : M^n \to N^p$ such that:

(1) $f_k| \cup_{j\leqq k} W_j$ is t-regular on N_1^{p-q}.

(2) $f_k(\overline{U_j}) \subset Y_{i(j)}$.

(3) f_k is a $\frac{\delta}{2^k}$ approximation of f_{k-1}.

Put $i = i(k)$ and $\lambda_k := p_2 \circ k_i \circ f_{k-1} \circ (h_k)^{-1} : C^n(2) \to \mathbb{R}^q$, where $p_2 : \mathbb{R}^{p-q} \times \mathbb{R}^q$ denotes the projection onto the second factor. Then λ_k is a locally definable C^∞ map. For any $\epsilon > 0$, there exist (q, n) matrix A and $(q, 1)$ matrix B such that:

(1) The absolute value of any element of A and B is less that ϵ.

(2) Put $L(x) := Ax + B$. Then 0 is a regular value of $\lambda_k + L$.

Define $f_k(x) =$

$$\begin{cases} k_i^{-1}(k_i \circ f_{k-1}(x) + L(h_k(x))\phi_k(x)), & x \in V_k \\ f_{k-1}(x), & x \in M - U_k \end{cases}.$$

Then f_k is a locally definable C^∞ map.

Since we take sufficiently small A, B, f_k is a $\frac{\delta}{2^k}$ approximation of f_{k-1} and $f_k(\overline{U_j}) \subset Y_{i(j)}$.

Thus $f_k| \cup_{j\leqq k} W_j$ is t-regular on N_1^{p-q}. Let $g(x) = \lim_k f_k(x)$. Then g is a locally definable C^∞ map with required properties. □

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