

DEFINABLE GROUPS AND IMAGINARIES IN DENSE PAIRS OF GEOMETRIC STRUCTURES

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ABSTRACT. We study definable groups in dense/codense expansions of geometric theories. We show that under some tame assumptions, definably amenable groups definable in the old language remain amenable in the expansion. Under the same assumptions, we show that the connected component G_P^{00} in the expansion agrees with the connected component G^{00} in the original language. We also analyze imaginaries in the expansion when the underlying theory has disintegrated algebraic closure.

1. INTRODUCTION

In this paper we study definable groups in expansions of geometric structures M with dense-codense subsets. Examples include lovely pairs, H-structures and groups with the Mann property. This family of examples shares many common features, key among them a classification of definable subsets of M into large or small sets, or more generally, sets of large dimension n in M^n .

This subject has been actively studied in recent years, for example, Eleftheriou, Günaydin and Hieronymi in [14] study definable groups in pairs when the underlying theory is o-minimal and try to recover the group in terms of the interplay of their large and small pieces (see the introduction of section 3 for more details). There is also the work of Martin-Pizarro and Blossier on groups definable in lovely pairs of algebraically closed fields and well as the study by Baro and Martin-Pizarro of small groups in dense-codense pairs of real closed fields. A description of definable groups in H-structures of stable geometric theories was developed by the authors of this paper in [3]. In the supersimple SU-rank 1 case, Zou [20] has shown that groups definable in H-structures are definably isomorphic to type-definable groups in the base language. While we were finishing this paper, Eleftheriou [15] has shown that any large group definable in a dense-codense expansion of an o-minimal theory is definable in the original theory.

This paper deals mostly with the questions of preservation of definable amenability under dense-codense expansions, and their effect on the connected component G^{00} of the group. We obtain positive results, namely, a left invariant measure extends naturally to the new definable sets and the connected components are invariant under these expansions.

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In Section 3, we focus on a group G definable in real closed fields. We consider expansions by dense-codense predicates and show that there are no new definable subgroups of G of the same “large dimension”.

In Section 4, we consider the general setting of geometric theories. We show how to extend left-invariant definable measures, and whenever generic types exist, how to extend them to the new language in order for them to remain generic. We show that definable subgroups of 1-dimensional groups are either definable in the old language or small. Similarly, in larger dimensions we show that large definable subgroups are also definable in the old language. We also study the connected component of the group G^{00} in both languages and show that it is invariant under the expansion.

In Section 5, we deal with small groups in lovely pairs of geometric structures and G -structures, as developed in [4]. In the case of lovely pairs of geometric structures, we start with a definable group in the old language whose structure is enriched by traces of definable groups from the large model. As before, we show how to extend measures to the new language and how to express generics in the expansion in terms of the generics in the old language.

In Section 6, we move into the setting of a real closed field expanded with a multiplicative subgroup G with the Mann property, and we analyze the definable subgroups of G in the new language and show that they are definable in the pure group language. We prove similar results when G is a subgroup of the circle group \mathbb{S}^1 with the Mann property. As before, the connected components G^{00} agree in the two languages.

In Section 7, we study imaginaries in dense-codense expansions, focusing on the case when T is geometrically trivial, and show how to adapt the criterion for elimination of imaginaries from [17] to reduce the “new” imaginaries to the “old” ones.

2. PRELIMINARIES

Recall that a complete theory T is *geometric* if (1) it eliminates the quantifier \exists^∞ and (2) the algebraic closure satisfies the exchange property in models of T . By *independence* we shall mean algebraic independence and use the symbol \perp to denote this independence relation. Whenever $M \models T$ and $\vec{a} \in M$, we write $\text{acl}(\vec{a})$ for the algebraic closure of \vec{a} inside M and $\text{tp}(\vec{a})$ for the type of \vec{a} inside M .

We start by defining dense/codense expansions:

Definition 2.1. Let T be a complete geometric theory in a language \mathcal{L} and let $M \models T$. Let P be a new unary predicate and let $\mathcal{L}_P := \mathcal{L} \cup \{P\}$ be the corresponding extended language. Let $(M, P(M))$ denote an expansion of M to \mathcal{L}_P , where $P(M) := \{x \in M \mid P(x)\}$.

- (1) $(M, P(M))$ is called a *dense/co-dense* expansion if, for any non-algebraic \mathcal{L} -type $p(x) \in S_1(A)$ where $A \subset M$ has a finite dimension, $p(x)$ has realizations both in $P(M)$ and in $M \setminus \text{acl}_T(A \cup P(M))$.
- (2) A dense/co-dense expansion $(M, P(M))$ is called a *lovely pair* if $P(M)$ is an elementary substructure of M .
- (3) A dense/co-dense expansion $(M, P(M))$ is called an *H-structure* if $P(M)$ is an \mathcal{L} -algebraically independent subset of M .

- (4) Assume M is a real field or a real closed field and $P(M)$ is the multiplicative group generated by $H(M)$ where $(M, H(M))$ is an H -structure. We call such a pair $(M, P(M))$ a G -structure. It is also a dense/codense expansion.
- (5) Assume $T = ACF$ or $T = RCF$ and $P(M)$ is a multiplicative subgroup. We say $P(M)$ has the *Mann property* if...

We will also consider pairs (M, P) , where P is a binary predicate (for example the group of roots of unity in the unit circle [2]), but the tools used for these expansions will be similar to the ones used for unary predicates. Whenever we consider them, we will say how the arguments change in these other settings.

Definition 2.2. Given a complete theory T_P which is dense/dense and $(M, P) \models T_P$ and $A \subset M$, we write $P(A)$ for $P(M) \cap A$. We say such a set A is P -independent if $A \downarrow_{P(A)} P(M)$, that is, if A is algebraically independent from $P(M)$ over $P(A)$.

Whenever $\vec{a} \in M$, we write $\text{acl}_P(\vec{a})$ for the algebraic closure of \vec{a} inside (M, P) and $\text{tp}_P(\vec{a})$ for the type of \vec{a} inside (M, P) .

Definition 2.3. Let T be geometric let $M \models T$ and let (M, P) be a dense/codense expansion. We say the pair satisfies the **Type Equality Assumption (TEA)** if whenever $\vec{a}, \vec{b}, \vec{c} \in M$ are such that \vec{a} is P -independent,

$$\begin{aligned} \vec{b} &\downarrow_{\vec{a}} P(M), \\ \vec{c} &\downarrow_{\vec{a}} P(M), \end{aligned}$$

and

$$\text{tp}(\vec{b}\vec{a}) = \text{tp}(\vec{c}\vec{a}).$$

Then

$$\text{tp}_P(\vec{b}\vec{a}) = \text{tp}_P(\vec{c}\vec{a}).$$

Remark 2.4. *The theories of lovely pairs, H-structures and G-structures satisfy TEA.*

Proof. Using transitivity and the fact that \vec{a} is P -independent we get $\vec{b}\vec{a} \downarrow_{P(\vec{a})} P(M)$ and $\vec{c}\vec{a} \downarrow_{P(\vec{a})} P(M)$. Thus both tuples $\vec{b}\vec{a}, \vec{c}\vec{a}$ are P -independent. Then the statement is clear for the cases of lovely pairs and H-structures. For the case of G-structures, it suffices to note that $G(\vec{b}) = G(\vec{c}) \subset G(\vec{a})$. Therefore, the G -parts of $\vec{b}\vec{a}$ and $\vec{c}\vec{a}$ coincide with $G(\vec{a})$ (and, in particular, have the same group type). \square

This assumption TEA allows us to describe new definable sets in terms of old definable sets up to small sets. A version of the following result for sets of dimension one was central to many arguments on dense/codense pairs [3, 5, 6], in particular for understanding which model-theoretic properties transfer from $Th(M)$ to $Th(M, P)$.

Proposition 2.5. *Let (M, P) be a dense/codense-structure satisfying TEA, let $Z \subset M^n$ be \mathcal{L} -definable and let $Y \subset Z$ be \mathcal{L}_P -definable. Then there is $X \subset Z$ \mathcal{L} -definable such that $\text{ldim}(Y \triangle X) < \dim(Z)$. Moreover, if both Y and Z are definable over a P -independent tuple \vec{a} , then X can also be chosen definable over \vec{a} .*

Proof. Let $k = \dim(Z)$. If $\text{ldim}(Y) < k$ or $\text{ldim}(Z \setminus Y) < k$, the result is clear, so we may assume that both Y and $Z \setminus Y$ have ldim equal to k . We may also assume that (M, P) is sufficiently saturated. Assume that Y and Z are definable over \vec{a} and

that \vec{a} is P -independent. Let $\vec{b} \in Y$ be such that $\text{ldim}(\vec{b}/\vec{a}) = k$ and let $\vec{c} \in Z \setminus Y$ be such that $\text{ldim}(\vec{c}/\vec{a}) = k$. Since $\vec{b} \in Z$ we have that $\text{dim}(\vec{b}/\vec{a}) = k = \text{ldim}(\vec{b}/\vec{a})$, so $\vec{b} \perp_{\vec{a}} P$. Similarly, we get $\vec{c} \perp_{\vec{a}} P$. Then by TEA, $\text{tp}(\vec{b}/\vec{a}) \neq \text{tp}(\vec{c}/\vec{a})$, so there is $X_{\vec{b}\vec{c}}$, a set \mathcal{L} -definable over \vec{a} , such that $\vec{b} \in X_{\vec{b}\vec{c}}$ and $\vec{c} \notin X_{\vec{b}\vec{c}}$. By compactness, we may first assume that $X_{\vec{b}\vec{c}}$ only depends on $\text{tp}(\vec{b}/\vec{a})$ and applying compactness again we may assume that $X_{\vec{b}\vec{c}}$ does not depend on $\text{tp}(\vec{b}/\vec{a})$ and we will call it simply X . Thus for $\vec{b}' \in Y$ and $\vec{c}' \in Z \setminus Y$ such that $\text{ldim}(\vec{b}'/\vec{a}) = k$ and $\text{ldim}(\vec{c}'/\vec{a}) = k$, we have $\vec{b}' \in X$ and $\vec{c}' \in Z \setminus X$. This shows that $\text{ldim}(Y \triangle X) < k$. \square

3. DEFINABLE GROUPS IN EXPANSIONS OF REAL CLOSED FIELDS

We begin this section with the study of definable groups in pairs of the form (R, P) , where R is a real closed field and P is a new predicate where P stands for a small subset of R or of a power of R . Some properties of groups definable in expansions similar to the ones we are considering were studied by Eleftheriou, Günaydin and Hieronymi in [14]. In particular, they conjectured:

Let $(F, *)$ be a L_P -definable group. Then there is a short exact sequence $0 \rightarrow B \rightarrow U \rightarrow K \rightarrow 0$ and a map $\tau : U \rightarrow H$ where

- U is \vee -definable.
- B is \vee -definable in L with $\text{dim}(B) = \text{ldim}(F)$.
- K is definable and small.
- $\tau : U \rightarrow F$ is a surjective group homomorphism and
- all maps involved are \vee -definable.

While the conjecture remains open in our setting, we present some positive results when F is the subgroup of an \mathcal{L} -definable group and the pair satisfies some tameness condition. We start with one dimensional groups, where the arguments are more transparent.

Definition 3.1. We say (R, P) satisfies the **Partition Assumption** if for all $X \subset R^n$ which is \mathcal{L} -definable and 1-dimensional and all $Y \subset X$ which is L_G -definable, there is a finite partition of X into cells $\{C_i : i \leq m\}$ such that each C_i is either a point or a 1-dimensional cell and $C_i \cap Y$ is small or cosmall in $C_i \cap X$ for $i = 1, \dots, m$.

Lemma 3.2. *Assume that TEA holds for (R, P) , then the partition assumption also holds.*

Proof. It follows from Proposition 2.5. \square

Thus there are many expansions that also satisfy the partition assumption, for example dense pairs, H -structures, and expansions with groups satisfying the Mann property of finite index inside $R^{>0}$ (the work of Günaydin and van den Dries [11]) and inside \mathbb{S} (the work of Belegradek and Zil'ber [2]). It also holds for G -structures (see [?]).

We begin with divisible groups:

Lemma 3.3. *Assume $F \leq (R, +)$ is \mathcal{L}_P -definable and the Partition Assumption holds. Then either F is small or $F = (R, +)$.*

Proof. Assume $F \leq (R, +)$ is \mathcal{L}_P -definable. If F is small there is nothing to prove. Assume otherwise. Then by the Partition Assumption there is a partition $-\infty = a_0 < a_1 < \dots < a_m = \infty$ of R and an interval (a_i, a_{i+1}) such that F is cosmall in (a_i, a_{i+1}) . Now let $f \in F \cap (a_i, a_{i+1})$, then $(F \cap (a_i, a_{i+1})) - f$ is cosmall in $(a_i, a_{i+1}) - f$ and contains 0. Thus, after possibly going to a different partition, we may assume that (a_i, a_{i+1}) contains 0.

Claim $F \cap (a_i, a_{i+1}) = (a_i, a_{i+1})$

Assume otherwise and let $c \in (a_i, a_{i+1}) \setminus F$. Then the coset $c + F$ is cosmall in $(c + a_i, c + a_{i+1})$. Since (a_i, a_{i+1}) is an open set containing 0, $(a_i, a_{i+1}) \cap (c + a_i, c + a_{i+1})$ is an open interval and both F and $c + F$ are cosmall in the interval, so they must intersect, a contradiction.

Thus $F \cap (a_i, a_{i+1}) = (a_i, a_{i+1})$, in particular the elements $a_i/2, a_{i+1}/2$ belong to F and thus both elements a_i, a_{i+1} belong to F .

Now consider $(a_i, a_{i+1}) + (a_i, a_{i+1}) = (2a_i, 2a_{i+1})$ which is a subset of F . Since $F \cap (a_{i+1}, a_{i+2}) \supset (a_{i+1}, 2a_{i+1}) \cap (a_{i+1}, a_{i+2})$, the set F must be cosmall in the interval (a_{i+1}, a_{i+2}) . Similarly, it is easy to show that F must be cosmall in the interval and (a_{i-1}, a_i) . It follows as in the claim that $F \cap (a_{i+1}, a_{i+2}) = (a_{i+1}, a_{i+2})$ and $F \cap (a_{i-1}, a_i) = (a_{i-1}, a_i)$. Proceeding inductively we get that $F = R$. \square

A similar argument also works for more general 1-dimensional groups:

Lemma 3.4. *Assume that (T, \cdot) is a 1-dimensional \mathcal{L} -definable group and let $F \leq (T, \cdot)$ be \mathcal{L}_P -definable. Also assume that the Partition Assumption holds. Then either F is small or F is \mathcal{L} -definable.*

Proof. Let $F \leq (T, \cdot)$ be \mathcal{L}_G -definable of dimension 1. If F is small there is nothing to prove. Assume otherwise. Since we need to see (T, \cdot) as topological group, we give (T, \cdot) the t -topology. Then we can write $T = U_1 \cup \dots \cup U_r$ where the sets U_i are definable and there is a definable bijection between U_i and some open subset V_i in R .

Then by the generalized partition assumption, there is a partition of U_j into cells $\{C_i^j : i \leq m_j\}$ such that each C_i^j is either a point or a 1-dimensional cell and $C_i^j \cap F$ is small or cosmall in C_i^j for $i = 1, \dots, m$. Then for some pair of indexes i, j , the cell C_i^j is 1-dimensional and $C_i^j \cap F$ is cosmall.

Let $f \in C_i^j \cap F$ and apply to T the left translation by f^{-1} , which is a continuous bijection in T . Then the set $D_F = f^{-1} \cdot (C_i^j \cap F)$ is 1-dimensional, contains the identity and is cosmall in the t -open set $D = f^{-1} \cdot C_i^j$.

Claim $F \cap D = D$. Assume otherwise and let $d \in D \setminus F$. Then the coset dF is cosmall in $d \cdot D$. Since D is a 1-dimensional t -open set containing e , $d \cdot D$ is also a t -open set and they intersect in a t -open set where both F and $d \cdot F$ are cosmall, a contradiction.

Thus F contains an open set around the identity in the t -topology. Since translation is a homeomorphism, we get that F is t -open.

Assume now that C_l^k is another 1-dimensional t -open set such that $F \cap C_l^k \neq \emptyset$. Then the intersection is t -open non-empty and thus not small. It follows as in the claim that the intersection must agree with C_l^k . We get that F is the union of

the 1-dimensional t -open sets C_i^k that it intersects and maybe some extra points and so it must be \mathcal{L} -definable. \square

The arguments can be generalized to higher dimensional subgroups under stronger assumptions about the expansion.

Proposition 3.5. *Let $M \models RCF$ and let (M, P) be a dense/codense expansion satisfying TEA. Assume that (T, \cdot) is an \mathcal{L} -definable group in M^n and let $F \leq (T, \cdot)$ be \mathcal{L}_P -definable. Then either $\text{ldim}(F) < \dim(T)$ or F is \mathcal{L} -definable.*

Proof. Let $k = \dim(T)$ and let $F \leq (T, \cdot)$ be \mathcal{L}_P -definable. If $\text{ldim}(F) < k$ there is nothing to prove. Assume otherwise. We need to see (T, \cdot) as topological group, so we equip (T, \cdot) with the t -topology [?]. Then we can write $T = U_1 \cup \dots \cup U_r \cup S$ where the sets U_i are definable, pairwise disjoint and there is a definable bijection between U_i and some open subset V_i in R^k and $\dim(S) < \dim(T)$.

Then by Lemma 2.5 and cell decomposition, there is a partition of U_j into cells $\{C_i^j : i \leq m_j\}$ such that each C_i^j is either k -dimensional t -open cell and $\text{ldim}(C_i^j \Delta (C_i^j \cap F)) < \dim(C_i^j)$ or $\text{ldim}(F \cap C_i^j) < k$ for $i = 1, \dots, m$. Then for some pair of indexes i, j , the cell $C = C_i^j$ is k -dimensional, $\text{ldim}(C \cap F) = k$ and $\text{ldim}(C \Delta (C \cap F)) < k$.

Let $f \in C \cap F$ and apply to T the left translation by f^{-1} , which is a continuous bijection in T . Then the set $D_F = f^{-1} \cdot (C \cap F)$ has large dimension k and contains the identity. On the other hand the set $D = f^{-1} \cdot C$ is t -open and $\text{ldim}(D \Delta D_F) < k$.

Claim $D_F = D$.

Assume otherwise and let $d \in D \setminus D_F$. Since F is a group, F is disjoint from dF . On the other hand $\text{ldim}(d \cdot D_F \Delta d \cdot D) < k$. Since D is a k -dimensional t -open set containing e , $d \cdot D$ is also a t -open set and they intersect in a non-empty t -open set (d belongs to the intersection), so $\dim(D \cap d \cdot D) = k$. Thus $\text{ldim}(D_F \cap d \cdot D_F) = k$, a contradiction.

Thus F contains an open set around the identity in the t -topology. Since translation is a homeomorphism, we get that F is t -open.

Assume now that that C_i^k is another k -dimensional t -open set such that $F \cap C_i^k \neq \emptyset$. Then the intersection is t -open non-empty and thus has dimension k . It follows as in the claim that the intersection must agree with C_i^k . We get that F is the union of the k -dimensional t -open sets C_i^k that it intersects and maybe some smaller dimensional pieces. Let X be the union of the sets C_i^j of dimension k such that $F \cap C_i^j \neq \emptyset$. Then $X \subset F$ and $\text{ldim}(X \Delta F) < k$. If $c \in F$ then choose $g \in F$ such that $\dim(g/c) = k$, then $g, c \cdot g^{-1} \in X$ and $c = (cg^{-1}) \cdot (g) \in X \cdot X$. This shows $F = X \cdot X$ and thus F is \mathcal{L} -definable. \square

Observation 3.6. *Note that in the middle of the argument above we get that F is t -open. If the dense/codense expansion has o -minimal open core then we get right away that F is \mathcal{L} -definable. This is the case for lovely (dense) pairs (see [8]).*

Corollary 3.7. *Let $M \models RCF$ and let (M, P) be a dense/codense expansion satisfyin TEA. Assume that (T, \cdot) is an \mathcal{L} -definable group in M^n . Write T^0 for the connected component of T in the sense of \mathcal{L} and T_P^0 for the connected component of T in the sense of \mathcal{L}_P . Then $T^0 = T_P^0$.*

Proof. Clearly $T_P^0 \subset T^0$. On the other hand, if $F \leq (T, \cdot)$ be \mathcal{L}_P -definable, we have two options. Either $\text{ldim}(F) < \dim(T)$ and thus there are unboundedly many

cosets of F in T or $\text{ldim}(F) = \dim(T)$ and then F is \mathcal{L} -definable and thus if it has finite index in the sense of \mathcal{L}_P then it has finite index in the sense of \mathcal{L} . \square

4. GEOMETRIC STRUCTURES, AMENABILITY AND GENERICS

The next result works in a more general setting: we assume $Th(M)$ is geometric. We will again consider \mathcal{L} -definable groups G and the subgroups definable in dense/codense expansions. We will see how to extend measures and we will prove that under mild assumptions G^{00} does not change in the expansion. We start by proving a variation of the last proposition from the previous section.

Proposition 4.1. *Let T be geometric and let (M, P) be a dense/codense expansion satisfying TEA. Assume that (G, \cdot) is an \mathcal{L} -definable group in M^n and let $F \leq (G, \cdot)$ be \mathcal{L}_P -type-definable. Then either $\text{ldim}(F) < \dim(G)$ or F is \mathcal{L} -type-definable.*

Proof. Let $k = \dim(T)$ and let $F \leq (G, \cdot)$ be \mathcal{L}_P -definable and assume that $k = \text{ldim}(F)$.

It is enough to consider the case where $F = \bigcap_{i \in \omega} F_i$ is a countable intersection of definable sets, since any type-definable group is an intersection of such groups. We may also assume all sets under consideration (G, F_0, F_1, \dots) are definable over a P -independent set B .

By Proposition 2.5, there are \mathcal{L} -definable sets A_i , also definable over B , such that $\text{ldim}(F_i \triangle A_i) < k$ and $A_i \subset G$. Let $A = \bigcap_{i \in \omega} A_i$, then A is \mathcal{L} -type definable over B . We will write $[A]$ for the set of types in \mathcal{L}_P with parameters in B that extend A and $[F]$ for the ones that extend F . We use a similar notation for $[A_i]$ and $[F_i]$.

Claim If $p \in [F] \triangle [A]$ then $\text{ldim}(p) < k$.

Let $p \in [F] \triangle [A]$ and first assume that $p \in [F] \setminus [A]$. Then for some i we have $p \in [F_i] \setminus [A_i]$ so $\text{ldim}(p) < k$. The argument is similar for $p \in [A] \setminus [F]$.

Let $X = \{p \in S(B) : A \in p, \dim(p) = k\}$. Since A is \mathcal{L} type-definable, X is also \mathcal{L} type-definable, we only need add to the type A the negation of all sets of smaller dimension with parameters in B (here we use the fact that T eliminates \exists^∞). As before write $[X]$ for the set of types in \mathcal{L}_P with parameters in B that extend X .

Claim If $p \in [F] \triangle [X]$ then $\text{ldim}(p) < k$.

Assume otherwise and let $p \in [F] \setminus [X]$ be such that $\text{ldim}(p) = k$. Let $p_0 = p \upharpoonright_{\mathcal{L}}$ and notice that $A \in p_0$. Then $\dim(p_0) = k$ so p_0 is generic in the sense of dimension and $p_0 \in X$, so $p \in [X]$, a contradiction. Assume now that there is $p \in [X] \setminus [F]$. Then $A \in p$. Let $p_0 = p \upharpoonright_{\mathcal{L}}$, then $\dim(p_0) = k$ and by the extension property $\text{ldim}(p_0) = k$. Then $p \in [A]$ and since $\text{ldim}([F] \triangle [A]) < k$ we have that $p \in [F]$, a contradiction.

Claim X is closed under generic multiplication.

Let $a_1, a_2 \in X$ and assume that $\dim(a_2/Ba_1) = k$. We may also assume that $a_1, a_2 \perp_B P$. Then $a_1, a_2 \in F$, $a_1 \cdot a_2 \in F$ and since a_1, a_2 are independent, we have $\dim(a_1 \cdot a_2/B) = k$, so $a_1 \cdot a_2 \in X$.

Claim X is closed under inverses.

This is clear, since F is closed under inverses and X contains of the generics (in the sense of large dimension) of F .

Claim $X \cdot X$ is a group.

Let $a_1 \cdot a_2 \in X \cdot X$ and let $c_1 \cdot c_2 \in X \cdot X$. Let $c'_1 \in X$ be such that $\dim(c'_1/\{a_1, a_2, c_1, c_2\}) = k$. Then $\dim(c'_1/\{a_1, a_2\}) = k$ and so $\dim(c'_1/a_1 \cdot a_2) = k$

and $(a_1 \cdot a_2 \cdot c'_1) \in X$. If we let $c'_2 = (c'_1)^{-1} \cdot c_1 \cdot c_2$ we also have that $c'_2 \in X$. Then $c_1 c_2 = c'_1 c'_2$ and $a_1 \cdot a_2 \cdot c_1 \cdot c_2 = (a_1 \cdot a_2 \cdot c'_1) \cdot c'_2 \in X \cdot X$ as we wanted.

Since every element in F is the product of two generics, it is easy to see that $F = X \cdot X$ and thus F is \mathcal{L} -type-definable. □

In what follows we will concentrate on definable amenability.

Proposition 4.2. *Let (M, P) be a dense/codense-structure satisfying TEA. Let G be a definably amenable group definable in $M \models T$, such that $\dim(G) = k$, and let μ is a left-invariant measure such that for all \mathcal{L} -definable subsets $A \subset G$ such that $\dim(A) < k$ we have $\mu(A) = 0$. Then the measure μ extends to a definable left-invariant measure on \mathcal{L}_P -definable subsets of G .*

Proof. Let $Y \subset G$ be \mathcal{L}_P -definable. By Proposition 2.5, there exists an \mathcal{L} -definable $X \subset G$ such that $\text{ldim}(X \Delta Y) < k$. Extend μ by defining $\mu(Y) = \mu(X)$.

Claim. The function μ is well-defined.

Suppose $X' \subset G$ is another \mathcal{L} -definable subset such that $\text{ldim}(X' \Delta Y) < k$. Let $S = Y \Delta X$ and $S' = Y \Delta X'$. Then $Y = X \Delta S$ and $Y = X' \Delta S'$

$$\emptyset = Y \Delta Y = (X \Delta S) \Delta (X' \Delta S') = (X \Delta X') \Delta (S \Delta S').$$

Then

$$\dim(X \Delta X') = \text{ldim}(X \Delta X') = \text{ldim}(S \Delta S') < k.$$

By assumption, $\mu(X \Delta X') = 0$, so $\mu(X) = \mu(X')$. □

Example 4.3. *Strongly minimal theories.* Assume T is a strongly minimal theory, $M \models T$ sufficiently saturated and $G \subset M^n$ a definable group. Assume $\text{MD}(G) = k$. Then G is definably amenable and for a $A \subset G$ definable, we have that $\mu(A)$ is the number of generic types in $\varphi(\vec{x})$ divided by k . Then whenever $\text{MR}(A) = \dim(A) < k$ the set A has no generics and $\mu(A) = 0$.

Example 4.4. *Pseudofinite fields* (see [16]). Assume T is the theory of pseudofinite fields, $F = \prod_U F_i$ is such an ultraproduct (in particular it is \aleph_1 -saturated) and $G \subset F^n$ a definable group. Assume $\dim(G) = k$. Then G has a measure associated to the counting measure: assuming $G = \varphi(M^n, \vec{b})$ and $\vec{b} = (\vec{b}_i)_U$ and $\psi(\vec{x}, \vec{b})$ defines a subset, we can define $\mu(\psi(\vec{x}, \vec{b}))$ as the ultralimit with respect to U of $|\psi(F_i^n, \vec{b}_i)| / |\varphi(F_i^n, \vec{b}_i)|$. This counting measure makes G a definably amenable group. For $A \subset G$ definable, whenever we have $\dim(A) < k$ we get that $\mu(A) = 0$. A similar argument holds for 1-dimensional asymptotic classes.

Example 4.5. Let T be an o-minimal expansion of the theory of densely ordered abelian groups. Now let $(M, +, <, \dots)$ be a sufficiently saturated model of T . Consider $G = (M, +)$. Let μ^∞ be the measure concentrated in the type $p^\infty(x) = \{x > a \mid a \in M\}$; let $\mu^{-\infty}$ be concentrated in the type $p^{-\infty}(x) = \{x < a \mid a \in M\}$. For every $0 \leq r \leq 1$ let $\mu^r(-) = r\mu^\infty(-) + (1-r)\mu^{-\infty}(-)$. Then whenever $A \subset M$ has dimension 0, it is finite, and for every $0 \leq r \leq 1$, $\mu^r(A) = 0$. A similar result holds for definable subsets of the group $G = (M^n, +)$: if μ is an invariant Keisler measure on definable subsets of G , then for any definable subset A of G such that $\dim(A) < n$, $\mu(A) = 0$.

Similarly, if we consider a normalized Haar measure on \mathbb{S}^1 in a model of RCF then definable subsets of dimension 0 are finite and have measure zero.

All the natural examples considered in this paper satisfy the “tameness” assumption from this section, namely, whenever a group is definably amenable, its definable subsets of smaller dimension have measure zero. This leads us to the following question.

Question 4.6. *Assume T is a geometric theory, $M \models T$ sufficiently saturated and $G \subset M^n$ a definable group. Assume $\dim(G) = k$ and G is definably amenable. Does it hold that whenever $A \subset G$ is definable and $\dim(A) < k$, we have that $\mu(A) = 0$.*

In [3] the authors showed that for $G \models T$ a strongly minimal, (G, P) an H -structure and $Z \leq G^n$ a \mathcal{L}_P -definable subgroup, we have that Z is \mathcal{L} -definable. The argument used stabilizers of generic elements in ω -stable theories using that the restriction of a \mathcal{L}_H -generic element to the old language is also generic. We will see how to use generics in amenable NIP groups (as presented in [9]) to recover a similar result.

Definition 4.7. Let M be a sufficiently saturated NIP structure, let $G \subset M^n$ be a definably amenable group and let μ be a left invariant measure. We say that $\varphi(x)$ is a f -generic formula if $\mu(\varphi(x)) > 0$ (see [9, Theorem 1.2]). We say that $q(x) \in S_G$ is f -generic if $\varphi(x)$ is f -generic for all $\varphi(x) \in q(x)$.

Corollary 4.8. *Let (M, P) be a dense/codense-structure satisfying TEA. Also assume (M, P) is sufficiently saturated and its theory is NIP. Let G be a \mathcal{L} -definably amenable group definable in M such that $\dim(G) = k$, and let μ is a left-invariant measure such that for all \mathcal{L} -definable subsets $A \subset G$ such that $\dim(A) < k$ we have $\mu(A) = 0$. Let μ_H be the extension from the previous proposition and let $q \in S_P(G)$ be f -generic. Then:*

- (1) $p = q \upharpoonright_{L \in S(G)}$ is f -generic.
- (2) $\text{Stab}(p) = \text{Stab}(q)$ and thus $G^{00} = G_P^{00}$.

Proof. In the following, all definable subsets under consideration are definable subsets of G .

(1) Let $\varphi(x) \in p$. Then $\varphi(x) \in q$ and since q is f -generic we have $\mu(\varphi(x)) = \mu_H(\varphi(x)) > 0$.

(2) Assume that $g \in G$ is such that $gq = q$. Then for every $\varphi(x) \in p$ we have that $g\varphi(x) \in q$, so $g\varphi(x) \in p$ and thus $\text{Stab}(p) \supset \text{Stab}(q)$.

Now let $g \in \text{Stab}(p)$ and let $\psi(x) \in q$. Since q is f -generic, the assumption on the measure μ implies that q can not contain sets of large dimension smaller than k , so we must have that $\text{ldim}(q) = k$. By Proposition 2.5 there is an \mathcal{L} -definable set $\psi'(x)$ such that $\text{ldim}(\psi(x) \Delta \psi'(x)) < k$. Thus $\psi'(x) \in q$ and $\psi'(x) \in p$. Since $g \in \text{Stab}(p)$, $g\psi'(x) \in p$ and so $g\psi'(x) \in q$. Since $\text{ldim}(\psi(x) \Delta \psi'(x)) < k$, $\text{ldim}(g\psi(x) \Delta g\psi'(x)) < k$ and thus if $g\psi(x) \notin q$ we must have that $g\psi(x) \Delta g\psi'(x) \in q$ which contradicts that the large dimension of q is k . Thus $g\psi(x) \in q$ and $g \in \text{Stab}(q)$.

Finally, since both types are f -generic, the last part follows from the fact that $\text{Stab}(p) = G^{00}$ and $\text{Stab}(q) = G_P^{00}$. \square

Corollary 4.9. *Let T be geometric and let (M, P) be a dense/codense expansion satisfying TEA. Assume that (S, \cdot) is an \mathcal{L} -definable group in M^n . Write S^{00} for the type-definable connected component of S in the sense of \mathcal{L} and S_P^{00} for the type-definable connected component of S in the sense of \mathcal{L}_P . Then $S^{00} = S_P^{00}$.*

Proof. Let $F \leq (S, \cdot)$ be \mathcal{L}_P -type-definable of bounded index. Then by the co-density property and saturation we must have $\text{ldim}(F) = \text{dim}(S)$. Now apply the previous proposition to show that F is \mathcal{L} -type-definable. \square

Remark 4.10. Let T be an o -minimal theory, let $M \models T$ and let (M, P) be a dense/codense expansion satisfying TEA. Assume that (G, \cdot) is an \mathcal{L} -definable group in M^n which is definably compact, so Pillay's conjecture holds [12]. Then $G^{00} = G_P^{00}$, the quotient $G/G^{00} = G/G_P^{00}$ is a Lie group and $\text{dim}_{\text{Lie}}(G/G_P^{00}) = \text{dim}(G) = \text{ldim}(G)$. On the other hand, the logic topology on G/G^{00} coming from the old language may differ from the logic topology on G/G_P^{00} , the predicate P may induce a new dense/codense closed subset in G/G_P^{00} . For example consider (M, P) a lovely pair and $G = S^1$ the unit circle group. Then G is definably compact and the collection $\{(x, y) \in S^1 : x \in P\}$ is a definable dense codense subset of the unit circle.

Remark 4.11. Let T be an NIP theory, let $M \models T$ and let (M, P) be a dense/codense expansion satisfying TEA. Assume that (G, \cdot) is an \mathcal{L} -definable group in M^n which is definably amenable and let μ be a left invariant measure on the definable subsets of G . Then G/G^{00} is the Ellis group associated to the dynamical system (G, S_G) . Furthermore, assume that for all \mathcal{L} -definable subsets $A \subset G$ such that $\text{dim}(A) < k$ we have $\mu(A) = 0$. Then μ extends to a left-invariant measure on \mathcal{L}_P -definable subsets of G and Newelski's conjecture also holds for $G/G_P^{00} = G/G^{00}$ in the extended language.

Question 4.12. With the same initial assumptions of the previous remark, assume now that there are \mathcal{L} -definable subsets $A \subset G$ such that $\text{dim}(A) < k$ but $\mu(A) > 0$. What can we say about the system (G, S_G) in the extended language?

5. SMALL GROUPS

Now we will study the definable subgroups of G inside pairs of the form (M, P) , where either (M, P) is a lovely pair of geometric theories or $M = R$ is a real closed field and G is either a dense subgroup of $R^{>0}$ with the Mann property (of infinite rank as in as in section ?? or of finite rank as studied by Günaydin and van den Dries) or G is a dense subgroup of $\mathbb{S}(R)$ with the Mann property (of infinite rank as in as in section ?? or of finite rank as studied by Belegardek and Zil'ber [2]).

5.1. Groups in lovely pairs with induced structure. Fix (M, P) a lovely pairs of geometric structures and $G \subset M^n$ \emptyset -definable. Then $G^P = G(P(M))$, the interpretation of G inside the predicate, is a definable group. The goal is to study G^P as a definable group in (M, P) .

We start with amenability and prove a result analogous to Proposition 4.2

Proposition 5.1. Let (M, P) be a sufficiently saturated lovely pair of geometric theories. Let $G \subset M^n$ be a \mathcal{L} - \emptyset -definably amenable group such that $\text{dim}(G) = k$, and let μ be a left-invariant measure such that for all \mathcal{L} -definable subsets $A \subset G$ with $\text{dim}(A) < k$ we have $\mu(A) = 0$. Then the measure induces a definable left-invariant measure on \mathcal{L}_P -definable subsets of G^P .

Proof. Let $Y \subset G^P$ be \mathcal{L}_P -definable. By Proposition, there exists an \mathcal{L} -definable $X \subset G$ such that $Y = X \cap G^P$. We define a measure μ_P in Y as $\mu_P(Y) = \mu(X)$. We will now prove that the function μ_P is well-defined. Suppose $X' \subset G$ is another \mathcal{L} -definable subset such that $Y = X' \cap G^P$.

Claim $\dim(X' \triangle X) < k$.

Assume otherwise, and suppose that $\dim(X' \setminus X) = k$. Choose $\vec{a} \in X' \setminus X$ and we may assume that $\vec{a} = (a_1, \dots, a_k, a_{k+1} \dots a_n)$, where a_1, \dots, a_k are independent and $a_{k+1} \dots a_n \in \text{acl}(a_1, \dots, a_k)$. By the density property we may assume that $a_1, \dots, a_k \in P(M)$ and since $P(M) \preceq M$ we get $\vec{a} \in P(M)$. But then $\vec{a} \in (X' \cap P(M)) \setminus (X \cap P(M)) = Y \setminus Y$, a contradiction.

Since $\dim(X' \triangle X) < k$ by the assumption on the measure we have that $\mu(X) = \mu(X')$ and μ_P is well-defined. Since the original measure was left-invariant, so is the extension. \square

Once we preserve amenability we can proceed as in the last section to compare generic elements in the original language with generics in the extended language. The result is analogous to Corollary 4.8

Proposition 5.2. *Let (M, P) be a pair of geometric theories and assume $Th(M)$ is NIP. Let G be a \mathcal{L} - \emptyset -definably amenable group such that $\dim(G) = k$, and let μ is a left-invariant measure such that for all \mathcal{L} -definable subsets $A \subset G$ such that $\dim(A) < k$ we have that $\mu(A) = 0$. Let μ_P be the left invariant measure on G^P defined in the previous proposition and let $q \in S_P(M)$ extending G^P be an f -generic type (in the extended language). Then:*

- (1) $p = q \upharpoonright_L S(M)$ is f -generic in the old language.
- (2) $Stab(p) \cap G^P = Stab(q)$, where we see the stabilizer $Stab(p)$ as a subgroup of $G(M)$ and $Stab(q)$ as a subgroup of G^P .

Proof. First note that since $Th(M)$ is NIP, so is $Th(M, P)$.

(1) Let $\varphi(\vec{x}, \vec{c}) \in p$, where $\varphi(\vec{x}, \vec{y})$ be a \mathcal{L} -formula. Then $\varphi(\vec{x}, \vec{c}) \in q$ and since q is f -generic we have $\mu(\varphi(\vec{x}, \vec{c}) \wedge G) = \mu_P(\varphi(\vec{x}, \vec{c}) \wedge G^P) > 0$.

(2) Assume that $g \in G^P$ is such that $gq = q$. Then for every $\varphi(\vec{x}) \in p$ with $\varphi(\vec{x}) \Rightarrow G(\vec{x})$ we have that $g\varphi(\vec{x}) \in q$, so $g\varphi(\vec{x}) \in p$ and thus $Stab(p) \cap G^P \supset Stab(q)$.

Now let $g \in Stab(p) \cap G^P$ and let $\psi(\vec{x}) \in q$. We may assume that $\psi(\vec{x}) \wedge G^P(\vec{x}) = \psi(\vec{x})$. Then there is $\varphi(\vec{x})$ \mathcal{L} -definable set such that $\psi(\vec{x}) = G^P \wedge \varphi(\vec{x})$. Again, we may assume that $\varphi(\vec{x}) \wedge G(\vec{x}) = \varphi(\vec{x})$. Thus $\varphi(\vec{x}) \in q$ and $\varphi(\vec{x}) \in p$. Since $g \in Stab(p)$, $g\varphi(\vec{x}) \in p$ and so $g\varphi(\vec{x}) \in q$. This implies that $g\psi(x) = gG^P \wedge g\varphi(x) = G^P \wedge g\varphi(x) \in q$. This shows that $g \in Stab(q)$. \square

Corollary 5.3. *Let T be an NIP theory, let $M \models T$ and let (M, P) be a lovely pair. Assume that (G, \cdot) is an \mathcal{L} -definable group over \emptyset in M^n which is definably amenable and let μ be a left invariant measure on the definable subsets of G . Note that $G(P(M))$ is also a saturated model of the group and that μ restricts to a left invariant measure on the definable subsets of $G(P(M))$. Then $G(P(M))/G^{00}(P(M))$ is the Ellis group associated to the dynamical system (G, S_G) . Furthermore, assume that for all \mathcal{L} -definable subsets $A \subset G$ such that $\dim(A) < k$ we have $\mu(A) = 0$. Then μ extends to a left-invariant measure μ_P on \mathcal{L}_P -definable subsets of $G(P(M))$ and Newelski's conjecture also holds for the dynamical system (G^P, S_{G^P}) , where S_{G^P} is the space of \mathcal{L}_P -types extending G^P . Furthermore, the Ellis group does not change in the expansion, i.e. $G(P(M))/G^{00}(P(M)) = G^P/(G^P)^{00}$ with the induced structure from (M, P) .*

Observation 5.4. *Let G, p, q be as in the previous corollary. Since both types are f -generic, we have that $Stab(p) = G(M)^{00}$ (interpreting the stabilizer in the*

model M) and $\text{Stab}(q) = (G^P)^{00}$ (interpreting the stabilizer in the group G^P in model $(M, P(M))$). So, since T is NIP, $G(P(M))^{00} = G(M)^{00} \cap G^P = (G^P)^{00}$. In particular, $G(P(M))^{00} = (G^P)^{00}$. Thus, the connected component of $G(P(M))$ viewed as a definable group in $P(M)$ is the same as its connected component in the sense of \mathcal{L}_P .

Proposition 5.5. *Let T be an o-minimal theory, let $M \models T$ and let (M, P) be a lovely pair. Assume that (G, \cdot) is an \mathcal{L} -definable group in M^n which is definably compact. Then the quotient $G(P(M))/G(P(M))^{00} = G^P/(G^P)^{00}$ is a Lie group and $\dim_{\text{Lie}}(G^P/(G^P)^{00}) = \dim(G)$, where $\dim(G)$ is in the sense of o-minimal theories. Furthermore the logic topology on $G(P(M))/G(P(M))^{00}$ agrees with the logic topology on $G^P/(G^P)^{00}$ and agrees with the Lie topology on the quotient.*

Proof. The first part follows from Proposition 5.2, Observation 5.4, and the fact that Pillay's conjecture holds for definably compact groups in an o-minimal theory [12].

For the second part, by the positive solution of Pillay's conjecture in [12], the logic topology on $G(P(M))/G(P(M))^{00}$ agrees with the Lie topology on the quotient. Since the logic topology on $G^P/(G^P)^{00}$ is compact and Hausdorff and refines the logic topology on $G(P(M))/G(P(M))^{00}$ which is also compact and Hausdorff, the two logic topologies coincide. Thus the two logic topologies on the quotient coincide with the Lie topology. \square

We concentrate now in o-minimal theories expanding the theory of dense ordered groups. The following result is very similar to Lemma 3.3, but in the dual setting of small groups.

Lemma 5.6. *Let T be an o-minimal theory in a language $\mathcal{L} = \{+, <, \dots\}$, let $M \models T$ and let (M, P) be a lovely pair. Then whenever $F \leq (P(M), +)$ is an \mathcal{L}_P -definable subgroup we have that F is also \mathcal{L} -definable.*

Proof. Since F is definable in the pair, $F = P \cap C$ for some $C \subset M$ which is \mathcal{L} -definable. We may write $C = C_1 \cup C_2 \cup \dots \cup C_n$, where each C_i is either a point or an open interval M . Then $F = (C_1 \cap P) \cup (C_2 \cap P) \cup \dots \cup (C_n \cap P)$. Note that whenever C_i is an open interval M , $C_i \cap P$ is a convex set in $P(M)$ possibly with non-standard endpoints.

If F is finite there is nothing to prove, so we may assume that F is infinite and fix some index i such that C_i is infinite. Let $a_i \in C_i \cap P$; since F is a group, $F - a_i = F$ and let $D_i = C_i - a_i \subset F$. The set D_i is again an open interval in M but now it contains 0. We may assume that $D_i = (c_1, d_1)$ where $c_1 < 0$ and $d_1 > 0$. By the density of P we can find $e_1 \in P$ with $d_1/2 < e_1 < d_1$ and $e_2 \in P$ with $c_1 < e_2 < c_1/2$. Since $(e_2, e_1) \cap P \subset F$, then $(2e_2, 2e_1) \cap P \subset F$ and thus $D_i \cap P \subset (2e_2, 2e_1) \cap P$ and $(2e_2, 2e_1) \cap P$ is an \mathcal{L} -definable subset of $P(M)$ contained in F . Let $E_i = (2e_2, 2e_1) + a_i$ and note that $E_i \cap P$ is again a \mathcal{L} -definable subset of P contained in F .

Thus every \mathcal{L} -definable subset $C_i \cap P$ of F can be covered with an \mathcal{L} -definable subset $E_i \cap P$ of P which is again contained in F . This shows F is \mathcal{L} -definable. \square

The main ingredient of the previous proof was that $(M, +, <)$ is an ordered group. A similar proof works for a multiplicative subgroup $F \leq (P(M), \cdot)$ if the theory is o-minimal and extends the theory of fields and also for subgroups $F \leq (\mathbb{S}^1(P(M)), \cdot)$.

Question 5.7. Let T be an o-minimal theory in a language $\mathcal{L} = \{+, <, \dots\}$, let $M \models T$ and let (M, P) be a lovely pair and let F be a 1-dimensional \mathcal{L} -definable group. Is every \mathcal{L}_P -definable subgroup of F also \mathcal{L} -definable?

5.2. Groups in Mann expansions with induced structure. In this section we study groups in pairs of the form (R, G) , where R is a real closed field and $G \leq R^{\geq 0}$ is a group with the Mann property. We will consider the case studied in [11] where G is a finite rank group and also G -structures, where the groups G are generated by a set H , where $H \subset R^{\geq 0}$ is dense and algebraically independent. We start by recalling some definitions from [11].

Definition 5.8. For $m \geq 1$, let $G^{[m]} = \{g \in G : \exists h \in G \ g = h^m\}$.

Let $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and for each $m \geq 1$ define $G_{m, \vec{k}} := \{(g_1, \dots, g_n) \in G^n : g_1^{k_1} \cdot \dots \cdot g_n^{k_n} \in G^{[m]}\}$ and $G_{\vec{k}} := \{(g_1, \dots, g_n) \in G^n : g_1^{k_1} \cdot \dots \cdot g_n^{k_n} = 1\}$

For each $n \geq 1$, let $\mathcal{D}(n)$ be the collection of finite intersections of the groups $G_{m, \vec{k}}$ and $G_{\vec{k}}$ where $\vec{k} \in \mathbb{Z}^n$.

We will also need the following characterization of types in G -structures (see Theorem 4.3 in [4]):

Lemma 5.9. Let (K, G) and (K', G') be sufficiently saturated models of T^G and let $g_1, \dots, g_n \in G$, $g'_1, \dots, g'_n \in G'$ be such that:

- (1) For every $\vec{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and every $m \geq 1$, we have $(g_1, \dots, g_m) \in G_{m, \vec{k}}$ if and only if $(g'_1, \dots, g'_m) \in G'_{m, \vec{k}}$.
- (2) For any semialgebraic set V definable over \emptyset , $\vec{g} \in V$ if and only if $\vec{g}' \in V$.

Then $\text{tp}_G(\vec{g}) = \text{tp}_G(\vec{g}')$. Moreover, if $\vec{b} \in K$ is such that $\vec{g}\vec{b}, \vec{g}'\vec{b}$ are G -independent, $G(\vec{g}\vec{b}) = \vec{g}$, $G(\vec{g}'\vec{b}) = \vec{g}'$ and for any semialgebraic set V definable over \vec{b} , $\vec{g} \in V$ if and only if $\vec{g}' \in V$ then $\text{tp}_G(\vec{g}/\vec{b}) = \text{tp}_G(\vec{g}'/\vec{b})$.

From this it follows that the definable subsets of G^n in the pair (R, G) are finite unions of intersections of semialgebraic sets with realizations of group-formulas $\theta(\vec{x}, \vec{c})$, where $\theta(\vec{x}, \vec{c})$ defines a finite boolean combination of cosets of subgroups of the form $F \in \mathcal{D}(n)$, where $n = |\vec{x}|$. Note that all groups $F \in \mathcal{D}(n)$ are definable and dense. This same description of definable subsets of the group applies to the case studied in [11], the proof provided in [4] gives a back and forth system that also applies for groups of finite rank.

This description above is the analogue, for tuples in the predicate, of the Type Equality Assumption that we used earlier in this paper. In this setting need to take into account the group-type. For dimension one, the structure of G is linear modulo some new cuts induced by R and thus we do not expect to have new definable groups.

We will try to simplify the picture when we deal with G -structures:

Lemma 5.10. Assume (R, G) is a model of the theory of G -structures. Then $G_{n, k} = G^{\lfloor \frac{n}{\gcd(n, k)} \rfloor}$. In particular, if $(n, k) = 1$ then $G_{n, k} = G^{[n]}$.

Proof. We may work in the model (R, G) , where $G(R)$ is generated by the independent set $H(R)$. Then $g = h_1^{s_1} \dots h_l^{s_l}$, where h_i are distinct elements of $H(R)$. Note that h_i are algebraically independent. Suppose $g \in G_{n, k}$. Then $g^k = e^n$ for some $e \in G(R)$. Write e as $h_1^{t_1} \dots h_m^{t_m}$ for some distinct $h_i \in H(R)$. Therefore,

$g^k = h_1^{ks_1} \dots h_l^{ks_l} = e^n = h_1^{mt_1} \dots h_m^{mt_m}$. Then clearly $m = l$ and (possibly, after a permutation) $h_i = h'_i$, $ks_i = nt_i$. Write $k = \gcd(n, k)k'$ and $n = \gcd(n, k)n'$. It follows that $s_i = \frac{n't_i}{k'}$, and since k' is coprime with n' , for any i the fraction $\frac{t_i}{k'}$ is an integer. Therefore n' divides s_i for all $i \leq m$, and hence $g \in G^{[n']} = G^{[\frac{n}{\gcd(n, k)}]}$.

Conversely, assume $g \in G^{[\frac{n}{\gcd(n, k)}]}$. Then $g^k \in G^{[\frac{nk}{\gcd(n, k)}]} = G^{[nk']} \subset G^{[n]}$, i.e. $g \in G_{n, k}$.

Hence, $G_{n, k} = G^{[\frac{n}{\gcd(n, k)}]}$, as needed. □

Lemma 5.11. *Whenever $n, k \geq 1$, $G^{[n]} \cap G^{[k]} = G^{[m]}$, where $m = \text{lcm}(n, k)$.*

Proof. Since this is an elementary property, we may work in the model (R, G) where G is generated by the independent set H . Clearly $G^{[m]} \subset G^{[n]} \cap G^{[k]}$. Assume that $g = h_1^{s_1} \dots h_l^{s_l}$ where h_i stand for generators of G which are algebraically independent. If $g \in G^{[n]} \cap G^{[k]}$ then each of s_1, \dots, s_l is divisible by n and by k , so they are also divisible by m . □

Lemma 5.12. *Every boolean combination of cosets of groups in $\mathcal{D}(1)$ is either finite or dense.*

Proof. As before, we may work in the model (R, G) where G is generated by the independent set H . By co-density of H , any coset of $G^{[n]}$ is co-dense. Since G is a dense ordered subgroup of $R^{>0}$, then for every $n > 1$, the group $G^{[n]}$ is also dense in $R^{>0}$, and so is every coset of $G^{[n]}$.

By Lemma 5.11, if $a, b \in G$ then $(aG^{[n]}) \cap (bG^{[k]})$ is either empty or a coset of $G^{[\text{lcm}(n, k)]}$. For any $n_1, \dots, n_m > 1$, $[G : G^{[n_i]}]$ is infinite and whenever $a_1, \dots, a_m \in G$, we have that $G \setminus \bigcup_{i \leq m} (a_i G^{[n_i]})$ is an infinite union of cosets of $G^{[l]}$, where $l = \text{lcm}(n_1, \dots, n_m)$, and, thus, is dense and co-dense. □

We now deal with the expansion (R, G) , where the pair is either a G -structure (as in [4]), or an expansion of R with a multiplicative subgroup $G \leq R^{>0}$ which is dense and has the Mann property and for every $n \geq 1$ $G^{[n]}$ has finite index in G (as in [11]).

Proposition 5.13. *Let $F \leq G$ be \mathcal{L}_G -definable. Then $F \in \mathcal{D}(1)$.*

Proof. We consider two cases.

Case 1. We first deal with the case of G -structures. Let $F \leq G$ be \mathcal{L}_G -definable, then by Lemma 5.9 we can write $F = ((a_1, a_2) \cap B_1) \cup ((a_3, a_4) \cap B_2) \cup \dots \cup ((a_{2n}, a_{2n+1}) \cap B_l) \cup F_0$ where each B_i is a boolean combination of cosets of groups in $\mathcal{D}(1)$, all $a_i > 0$, the intervals (a_{2i}, a_{2i+1}) are disjoint and F_0 is a finite set of points.

Assume that R is the standard real field and $F \leq R^{>0}$ is a subgroup. Then either F is just the identity, infinite cyclic or dense. Assume that $F \neq \{e\}$ so it must be infinite and thus for some i , $(a_{2i}, a_{2i+1}) \cap B_i$ is infinite. By Lemma 5.12, every boolean combination of cosets of groups in $\mathcal{D}(1)$ which is not finite is dense, and thus, $B_i \cap (a_{2i}, a_{2i+1})$ is dense in (a_{2i}, a_{2i+1}) . Therefore F must be dense in $R^{>0}$. Note that being dense is a first order property and thus this property is true in all models of the theory.

Let $c \in (a_1, a_2) \cap B_1$, then after multiplying by c^{-1} we may assume that $1 \in (a_1, a_2) \cap B_1$. Since F is dense in $R^{>0}$, then for every l , F^l is dense in $R^{>0}$ and in a saturated model F^{div} , the group consisting of the divisible elements of F , is also dense in $R^{>0}$.

Claim B_1 and B_2 differ by a finite set.

Let $t \in (a_3, a_4) \cap B_2$ be divisible. First observe that $t^{-1}G^{[n]} = G^{[n]}$ and that for every coset $dG^{[n]}$, we also get $t^{-1}dG^{[n]} = dG^{[n]}$, so $t^{-1}B_2$ differs from B_2 by at most a finite set. Then $t^{-1} \cdot (a_3, a_4) \cap B_2 = (a_3/t, a_4/t) \cap t^{-1}B_2$ also contains the identity and $(a_3/t, a_4/t)$ is an open set around the identity. Let $c_1 = \max(a_1, a_3/t)$, $c_2 = \min(a_2, a_4/t)$. Then on the open set (c_1, c_2) , we have $F \cap (c_1, c_2) = (c_1, c_2) \cap B_1 = (c_1, c_2) \cap t^{-1}B_2$, so B_1 and $t^{-1}B_2$ agree on an open interval and thus they differ by at most a finite set. The claim follows from this result.

Thus after modifying the finite set F , and using the density of S , we may assume all the sets B_i are the same and since the group is dense we can write $S = ((b_1, b_2) \cap B] \setminus F_1) \cup F_2$, where F_1, F_2 are finite sets and $b_1 = 0, b_2 = \infty$. Thus $S = B \cup F_2$ which is clearly an element in $\mathcal{D}(1)$.

Case 2. Now assume G is a dense multiplicative subgroup of $R^{>0}$ with the Mann property, and all the $G^{[n]}$ have finite index in G . Then any finite Boolean combination of cosets of elements of $\mathcal{D}(1)$ is a finite union of cosets of elements of $\mathcal{D}(1)$. □

Now we see how to modify the arguments in the previous proof to deal with the expansion (R, G) , where $G \leq \mathbb{S}(R)$ is dense, has the Mann property and it is generated by an independent collection of dense-codense elements. This includes the case studied in [4] and in [2]. We start with some notation.

Let G_{re} (or $G_{re}(K)$) be the projection of G (or $G(K)$) onto the first coordinate. The set G_{re} is a subset of the interval $(-1, 1]$. Since G is closed under complex conjugates (complex reciprocals), for each $a \in G_{re}$ both $a^+ = (a, \sqrt{1-a^2})$ and $a^- = (a, -\sqrt{1-a^2})$ belong to G . For a tuple $\vec{e} = (e_1, \dots, e_n)$ of elements of G_{re} , we write $\vec{e}^+ = (e_1^+, \dots, e_n^+)$ for the corresponding tuple of elements of G . We will also use the notation a_{re} for the first coordinate of any $a \in \mathbb{S}(K)$. Given a structure (K, G) as above, for any set $A \subset K$, we say that A is G_{re} -independent if A is independent from G_{re} over the intersection. Clearly, $\text{dcl}(G_{re}) = \text{dcl}(G)$ in K (in the ring language).

The main tool we will use is the following description of types from [4], Theorem 5.7.

Fact 5.14. *Let (K, G) and (K', G) be two models of the theory T^{GS} of G -structures in the circle. Then whenever $\vec{a} = \vec{b}\vec{g} \in K, \vec{a}' = \vec{b}'\vec{g}' \in K'$ are two G_{re} -independent tuples such that $\vec{g} = G_{re}(\vec{a}), \vec{g}' = G_{re}(\vec{a}')$, and (writing G for $G(K)$ and G' for $G'(K)$)*

- (1) $(G, \vec{g}^+) \equiv_{gr} (G', \vec{g}'^+)$ (the types of \vec{g}^+ and \vec{g}'^+ agree in the sense of multiplicative groups)
- (2) $\text{tp}_{RCF}(\vec{a}) = \text{tp}_{RCF}(\vec{a}')$ (their types agree in the sense of real closed fields)

Then $\text{tp}_G(\vec{a}) = \text{tp}_G(\vec{a}')$.

Lemma 5.15. *Let $U \leq G$ be \mathcal{L}_G -definable. Then $U \in \mathcal{D}(1)$ or U is finite.*

Proof. Let $U \leq G$ be \mathcal{L}_G -definable, then by Fact 5.14, we can write $U = (C_1 \cap B_1) \cup (C_2 \cap B_2) \cup \dots \cup (C_n \cap B_n) \cup F$ where each B_i is a boolean combination of cosets of groups in $\mathcal{D}(1)$ (see Definition 5.8 but now consider G as a subgroup of $\mathbb{S}(R)$ which is a stable 1-based group), the collection C_i is a family of 1-dimensional cells that are disjoint and F is a finite set of points. Assume first that R is the standard real field and $U \leq \mathbb{S}(R)$ is a subgroup. It is well known that the subgroups of $\mathbb{S}(R)$ are either finite or dense, since this is a first order property the same results holds in all models of the theory. If U is finite we get the desired result. If U is infinite, then it is dense and we proceed in a similar way as in the previous proof by showing that all B_i differ by at most a finite set and that the C_i form a partition of $\mathbb{S}(R)$ up to a finite set. \square

Question 5.16. *Assume $\Gamma \leq \mathbb{S}$ has finite rank and consider $(R, G, a)_{a \in \Gamma_{re} \cup \Gamma_{im}}$ as in [2]. Let $U \subset \Gamma$ be definable in this extended language. Can we write $U = (C_1 \cap B_1) \cup (C_2 \cap B_2) \cup \dots \cup (C_n \cap B_n) \cup F$ as we did in the proof of the previous lemma?*

If this is the case, the proof above also works for this expansion and whenever $U \leq \Gamma$, then $U \in \mathcal{D}(1)$ or U is finite.

6. CONNECTED COMPONENTS AND THE QUOTIENT G/G^{00} FOR GROUPS WITH THE MANN PROPERTY

In this section we study some dynamical properties of G inside the expansion (R, G) , where G is a dense codense subgroup of $R^{>0}$ (respectively a subgroup of $\mathbb{S}(R)$) and has the Mann property. We follow the presentation from [9]. First note that in all settings under consideration G is abelian, so it is definably amenable. Our goals are to find explicit invariant measures and f -generic types, characterize G^{00} and the quotient G/G^{00} . We also want to compare the connected component of G inside the pair (R, G) with the connected component of G seen as a pure ordered group, in all cases under consideration the connected component does not change.

6.1. Dynamics: the infinite index case. Assume first that $G \leq R^{>0}$ and $(R, G) \models T^G$ is a G -structure. In particular G is dense, has the Mann property and the groups $G^{[n]}$ have infinite index in G .

Lemma 6.1. *Let (K, G) be an \aleph_1 -saturated model model of T^G and consider the type $p(x) = x \in G \cup \{x > g : g \in G\} \cup \{x \notin gW : g \in G, W \in \mathcal{D}(1), W \neq G\}$. Let μ_p be the measure centered in p . Then μ_p is G -invariant. Furthermore $G^{00} = G$ and $\text{Stab}(p) = G$.*

Proof. For any $a \in K$, the formula $\varphi(x) = (x > a) \wedge (x \in G)$ has μ_p -measure 1. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x > a) \wedge (g^{-1}x \in G) = (x > ag) \wedge (x \in G)$ also has μ_p -measure 1. Similarly for $a \in G$ and $W \in \mathcal{D}(1) \setminus \{G\}$, the formula $\varphi(x) = (x \notin aW) \wedge (x \in G)$ has μ_p -measure 1. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x \notin aW) \wedge (g^{-1}x \in G) = (x \notin gaW) \wedge (x \in G)$ also has measure 1. By the description of definable subsets of G from Lemma 5.9 we get that μ_p is G -invariant.

Since for any $g \in G$, $g \cdot p = p$, the orbit of p under the action of G is a singleton, $\text{Stab}(p) = G^{00} = G$ and $p(x)$ is f -generic (see Theorem 1.2 [9] and also the proof of [18, Lemma 8.18]). \square

Corollary 6.2. *Let (K, G) be an \aleph_1 -saturated model model of T^G and consider $W \in \mathcal{D}(1)$ infinite. Then $W^{00} = W$.*

Proof. For any $n \geq 2$, the map that sends $g \in G$ to $g^n \in G^{[n]}$ is a definable group isomorphism, so the group $G^{[n]}$ is also definable amenable, $G^{[n]00} = G^{[n]}$ and has strong f-generics. Thus the lemma above also holds for all infinite groups $W \in \mathcal{D}(1)$ and by Proposition 5.13 for all definable subgroups of G . \square

Question 6.3. *Is the lemma above true for $W \in \mathcal{D}(1)$ just assuming that G has the Mann property and that all groups $G_{n,k}$ have infinite index in G ?*

As before, we want to compare the connected component of G seen as a pure ordered group and the connected component of G seen inside the pair (K, G) .

Lemma 6.4. *Let (K, G) be an \aleph_1 -saturated model model of T^G and consider G as a pure ordered group in the ordered group language \mathcal{L}_{ogr} . Consider the \mathcal{L}_{ogr} -type $p(x) = \{x > g : g \in G\} \cup \{x \notin gW : g \in G, W \in \mathcal{D}(1), W \neq G\}$. Let μ_p be the measure centered in p . Then μ_p is G -invariant. Furthermore $G^{00} = G$ and $Stab(p) = G$.*

The proof is the same as before. We can conclude:

$$G_{ogr}^{00} = G = G^{00}$$

where we write G_{ogr}^{00} for the connected component in the pure ordered group sense and G^{00} for the connected component in the model (K, G) .

6.2. Dynamics: the finite index case. We now assume that we are in the setting studied in [11]. So G has the Mann property and that all groups $G^{[n]}$ have finite index in G . There are several examples of Mann-multiplicative subgroups with this property, for example $2^{\mathbb{Q}}3^{\mathbb{Z}}$, $2^{\mathbb{Q}}3^{\mathbb{Z}}5^{\mathbb{Z}}$, etc. In this setting, since all groups $G^{[n]}$ have finite index in G , the connected component G^{00} is a subgroup of the group of divisible elements G^{div} .

Lemma 6.5. *Let (K, G) be an \aleph_1 -saturated model model of T^G and consider the type $p(x) = x \in G \cup \{x > g : g \in G\} \cup \{x \in G^{[n]} : n \geq 2\}$. Let μ be the measure defined by the conditions $\mu(x > g) = 1$ for any $g \in G$ (so bounded sets have measure zero) and $\mu(gG^{[n]}) = \frac{1}{[G:G^{[n]}]}$ for any $g \in G$. Then μ is G -invariant, $p(x)$ is f -generic and $Stab(p) = G^{div} = G^{00}$.*

Proof. For any $a \in K$, the formula $\varphi(x) = (x > a) \wedge (x \in G)$ has μ -measure 1. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x > a) \wedge (g^{-1}x \in G) = (x > ag) \wedge (x \in G)$ also has μ -measure 1. Similarly for $a \in G$ the formula $\varphi(x) = (x \in aG^{[n]})$ has μ -measure $\frac{1}{[G:G^{[n]}]}$. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x \in aG^{[n]}) = (x \in gaG^{[n]})$ also has μ -measure $\frac{1}{[G:G^{[n]}]}$. Since every definable set is a boolean combination of semialgebraic sets and elements in $\mathcal{D}(1)$, we get that μ is G -invariant. This also shows every formula in $p(x)$ has positive measure and we can conclude that $p(x)$ is f -generic. Finally, it is easy to see that $G^{div}p = p$, so $Stab(p) = G^{div} = G^{00}$. \square

Note that it follows from the previous proof for any $g \in G^{00}$, $g \cdot p = p$ and that the orbit of p under the action of G has cardinality at most 2^{\aleph_0} , the orbit is determined by choosing a coset of each $G^{[n]}$ (and there are finitely many such cosets) for every n .

We will again compare the connected component of G in the sense of T^G with that in the sense of the pure ordered group language.

Lemma 6.6. *Let (K, G) be an \aleph_1 -saturated model of T^G . Viewing $G = G(K)$ as a pure ordered abelian group, consider the type $p(x) = \{x > g : g \in G\} \cup \{x \in G^{[n]} : n \geq 2\}$. Let μ be the measure defined by the conditions $\mu(x > g) = 1$ for any $g \in G$ (so bounded sets have measure zero) and $\mu(gG^{[n]}) = \frac{1}{[G:G^{[n]}]}$ for any $g \in G$. Then μ is G -invariant, $p(x)$ is f -generic and $\text{Stab}(p) = G^{\text{div}} = G^{00}$.*

Proof. The proof is almost identical to that of Lemma 6.5 and we leave it to the reader. \square

Thus, G^{00} is the same whether we work in T^G or in the pure ordered group language.

The quotient group G/G^{00} is a profinite group. For every positive pair of integers n, m such that m divides n , we get a natural map $f_{nm} : G/G^{[n]} \rightarrow G/G^{[m]}$ defined by $f_{nm}(aG^{[n]}) = aG^{[m]}$. The group G/G^{00} is the inverse limit of this system. Note that the quotients $G/G^{[n]}$ are finite and thus they are not ordered groups. Similarly the profinite group G/G^{00} is not ordered and thus G/G^{00} , even when it is infinite, is not a model of the theory of G .

Proposition 6.7. *The profinite topology on $\lim G/G^{[n]}$ agrees with the logic topology on $G_{\text{ogr}}/G_{\text{ogr}}^{00}$ (in the sense of pure ordered groups) and agrees with the logic topology on G/G^{00} (in the sense of the pair (K, G))*

Proof. We can see the profinite group as a closed subgroup of the product group $\prod_{n \geq 2} G/G^{[n]}$, where each finite quotient $G/G^{[n]}$ has the discrete topology and the product has the product topology. Thus the topology on H is also compact. Every $a \in G$ and $k \geq 2$ define a basic clopen set $C_{a,k}$ in H given by the set of tuples $(g_n)_n \in H$ such that $g_n G^{[n]} = aG^{[n]}$ for $n \leq k$.

Let $\pi : G \rightarrow G_{\text{ogr}}/G_{\text{ogr}}^{00}$ be the projection. Note that $\pi^{-1}(C_{a,k}) = \bigcap_{n \leq k} (aG^{[n]})$ is definable and thus closed in the logic topology where we see G_{ogr} as a pure ordered group. Thus we have three topologies on G/G^{00} all of them Hausdorff and compact, the profinite one contained in the logic topology in the sense of pure ordered groups and this last one contained in the logic topology on G/G^{00} in the sense of the pair (K, G) . Thus all topologies agree. \square

Example 6.8. *Consider the multiplicative group $G = 2^{\mathbb{Q}}3^{\mathbb{Z}}$ that is dense in $\mathbb{R}^{>0}$ and has the Mann property (see [11]). Then $G/G^{[n]} = \mathbb{Z}/n\mathbb{Z}$ and G/G^{00} is the infinite compact group of profinite integers.*

6.2.1. *Dynamics: subgroups of \mathbb{S} with the Mann property.* Finally we study the dynamics in subgroups of \mathbb{S} in settings.

First we consider the case studied by Belegardek and Zil'ber in [2] which deals with dense subgroups of \mathbb{S} that have the Mann property that have finite rank. Under these assumptions, all groups $G^{[n]}$ have finite index in G and G^{00} contains the intersection $\bigcap_n G^{[n]}$ as well as the group of infinitesimals around 1.

Recall that the shortest arc in the circle between $e^{-i\pi/4}$ and $e^{i\pi/4}$ is homeomorphic to $(-1, 1)$ through a map that sends $(1, 0)$ to the point 0. Thus we can identify the elements infinitesimally close to $(1, 0)$ with the elements infinitesimally close to

0 and give the elements infinitesimally close to $(1, 0)$ in \mathbb{S} an order compatible with the group operation. We will use the order in the following proof:

Lemma 6.9. *Let (K, G) be an \aleph_1 -saturated model of T^G and consider the Haar measure μ of G that assigns to an arc its length normalized by 2π , where the parameters belong to K . Then μ is G -invariant and the measure of $G^{[n]}$ is $1/[G : G^{[n]}]$. Consider the type $p(x) = x \in G \cup \{x > a : a \in G, a \text{ infinitesimal}\} \cup \{x < 1/n : n \geq 1\} \cup \{x \in G^{[n]} : n \geq 2\}$, then p is f -generic and $\text{Stab}(p) = G^{00}$ is the set of infinitesimals which are divisible.*

Proof. The first part is clear, as the length of an arc with parameters in K (even if intersected with G) is preserved under rotations. Also, if the index of G^n on G is k , then the cosets can be permuted by multiplying by elements in G , so they all have the same measure and the result follows.

Now we will check that p is f -generic. Given a an infinitesimal, $k \geq 1$ and $n \geq 2$, $\mu((a < x < 1/k) \cap (x \in G^{[n]})) = \frac{1}{k[G:G^{[n]}}} > 0$ so all formulas have positive measure. It is easy to see that $\text{Stab}(p)$ is the set of the infinitesimals which are divisible. By Theorem 1.2 in [9] $\text{Stab}(p) = G^{00}$ is also the set of the infinitesimals which are divisible. \square

Example 6.10. *Consider the multiplicative group $G = \mathbb{U}$ of the roots of unity interpreted as a subset of \mathbb{R}^2 . It is a divisible group but it has torsion points, for every $n \in \mathbb{N}$, the collection of n torsion points are precisely the n -th roots of unity. In a saturated model, the interpretation of \mathbb{U} includes points without torsion that are dense inside the torsion points and \mathbb{U}^{00} are those points infinitesimally close to 1. In this setting, $\mathbb{U}/\mathbb{U}^{00}$ is the group \mathbb{S} .*

This is not surprising, as a pure group, \mathbb{U} is the interpretation of \mathbb{S} in the field $\overline{\mathbb{Q}^r}$ (the real closure of the rationals). In a saturated model $\tilde{\mathbb{R}}$, the quotient $\mathbb{U}(\tilde{\mathbb{R}})/\mathbb{U}^{00}(\tilde{\mathbb{R}})$ corresponds to the quotient of $\mathbb{S}(\tilde{\mathbb{R}})/\mathbb{S}^{00}(\tilde{\mathbb{R}})$ which is known to be $\mathbb{S}(\mathbb{R})$. One can also check directly using the work of Szmieliew [19] that as pure groups, \mathbb{U} is elementary equivalent to \mathbb{S} .

Lemma 6.11. *Let (K, G) be an \aleph_1 -saturated model of T^G and consider G in the language \mathcal{L}_{gr} of groups. Let μ Haar measure of G that assigns to an arc its length normalized by 2π . Then μ is G -invariant and the measure of $G^{[n]}$ is $1/[G : G^{[n]}]$. Furthermore the type $p(x) = x \in G \cup \{x > a : a \in G, a \text{ infinitesimal}\} \cup \{x < 1/n : n \geq 1\} \cup \{x \in G^{[n]} : n \geq 2\}$ is f -generic and $\text{Stab}(p) = G^{00}$ is the set of infinitesimals which are divisible.*

Proof. The proof is very similar to the one for Lemma 6.9. \square

Now we consider the case of G -structures as studied in [4] which deals with dense subgroups of \mathbb{S} that have the Mann property and have infinite rank. Now all groups $G^{[n]}$ have infinite index in G and G^{00} is the group of infinitesimals around 1.

Lemma 6.12. *Let (K, G) be an \aleph_1 -saturated model model of T^G and and consider the Haar measure μ of G that assigns to an arc its length normalized by 2π . Then μ is G -invariant and the measure of $G^{[n]}$ is zero. Consider the type $p(x) = x \in G \cup \{x > a : a \in G, a \text{ infinitesimal}\} \cup \{x < 1/n : n \geq 1\} \cup \{x \notin aG^{[n]} : n \geq 2, a \in G\}$. Then p is f -generic and $\text{Stab}(p)$ is the set of infinitesimal elements.*

Proof. The proof is very similar to the one of the previous lemma and we leave it to the reader. \square

Remark 6.13. *The logic topology on G_{gr}/G_{gr}^{00} (in the sense of pure ordered groups) is compact and refined by the logic topology on G/G^{00} (in the sense of the pair (K, G)) which is also compact and Hausdorff. Thus the two topologies agree.*

Question 6.14. *Assume F is small, bounded and definable in $(R, K) \models T^G$, where (R, K) is \aleph_1 -saturated and $G \leq \mathbb{S}$ has the Mann property (regardless if it has finite or infinite rank). Does it hold that $F \equiv F/F^{00}$ as pure groups?, i.e. Does a partial version of Pillay's conjecture hold in this setting?*

Note that since the groups G^n are definably amenable, by [9] we do have a positive answer for Newelski's conjecture.

7. IMAGINARIES IN PAIRS

7.1. Imaginaries in T^{ind} . Assume that in T we have $\text{acl}(\emptyset) = \emptyset$, and for any singleton a , $\text{acl}(a) = \{a\}$. That is, in model of T , acl induces a geometry. Note that any ω -categorical geometric structure can be transformed into one satisfying the above assumption by taking a quotient modulo the definable equivalence relation of interalgebraicity $\text{acl}(x) = \text{acl}(y)$. The assumption also holds in an affine space over any division ring.

We will be working in a sufficiently saturated H -structure (M, H) of T .

Proposition 7.1. *Let $E(x, y)$ be an L_H -definable over \emptyset equivalence relation on M . Then there exists an L -definable (over \emptyset) equivalence relation $E'(x, y)$ such that $\text{ldim}(E' \Delta E) \leq 1$.*

Proof. By Proposition 2.5, there exists an L -definable (over \emptyset) relation $E^*(x, y)$ such that $\text{ldim}(E^* \Delta E) \leq 1$. Let $E'(x, y) = E^*(x, y) \vee E^*(y, x) \vee x = y$.

Clearly $E'(x, y)$ is symmetric and reflexive.

Claim: $\text{ldim}(E' \Delta E) \leq 1$.

Indeed, suppose $\models E'(a, b) \wedge \neg E(a, b)$. Then at least one of $a = b$ or $E^*(a, b)$ or $E^*(b, a)$ holds in M . In each case, $\dim(ab/H(M)) \leq 1$: if $a \neq b$, then (a, b) or (b, a) satisfies $E^* \Delta E$. Next, suppose $\models \neg E'(a, b) \wedge E(a, b)$. Then, in particular, $\models \neg E^*(a, b)$ and therefore (a, b) satisfies $E^* \Delta E$. Then $\dim(ab/H(M)) \leq 1$. This proves the claim.

We will now show that E' is transitive. By reflexivity of E' , it suffices to show that for $a, b, c \in M$ pairwise distinct, such that both $E'(a, b)$ and $E'(b, c)$ hold in M , we have $\models E'(a, c)$. By the extension property, we may assume that $abc \downarrow_{\emptyset} H(M)$. Then a, b, c are pairwise independent over $H(M)$. By the Claim, $E(a, b)$ and $E(b, c)$ both hold in (M, H) . Then we also have $\models E(a, c)$. By the Claim again, $\models E'(a, c)$, as needed. \square

The above proposition suggests that, at least for \mathcal{L}_H -definable equivalence relations in *one* variable, new imaginaries can be expressed in terms of the “old” imaginaries. We know that geometric elimination of imaginaries holds in the case of SU-rank 1 [3] as well as in the case of o-minimal theories [10].

7.2. Imaginaries in trivial theories. Now assume that T is geometric and trivial (the pregeometry is disintegrated). That is, for $M \models T$ be sufficiently saturated and $A \subset M$, we have $\text{acl}(A) = \cup_{a \in A} \text{acl}(a)$.

Also assume that T weak elimination of imaginaries. Finally we will assume that the criterion from [17] holds for algebraic independence in M :

Criterion (*):

- (1) For all $E = \text{acl}(E) \subset M$, tuple $\vec{a} \in M$ and $C \subset M$, there exists \vec{a}^* such that $\text{tp}(\vec{a}^*/E) = \text{tp}(\vec{a}/E)$ and $\vec{a}^* \downarrow_E C$.
- (2) For all $E = \text{acl}(E) \subset M$ and tuples $\vec{a}, \vec{b}, \vec{c} \in M$, if $\vec{b} \downarrow_E \vec{a}$, $\vec{c} \downarrow_E \vec{a}\vec{b}$ and $\text{tp}(\vec{a}/E) = \text{tp}(\vec{b}/E)$, then there exists \vec{c}^* such that $\text{tp}(\vec{a}\vec{c}/E) = \text{tp}(\vec{a}\vec{c}^*/E) = \text{tp}(\vec{b}\vec{c}^*/E)$.

Note that part (1) always holds for algebraic independence (T is geometric and algebraic independence satisfies existence). It is proved in [17] that the criterion implies that T weak elimination of imaginaries. We will prove below that the criterion will transfer for some H -structures, thus showing that they also have weak elimination of imaginaries.

Assume now that (M, H) is a sufficiently saturated model of H -structures. For $\vec{a}, \vec{b} \in M$ (possibly infinite tuples) and $C \subset M$ we will write $\vec{a} \downarrow_C^H \vec{b}$ (and call this notion H -independence) if $\vec{a} \downarrow_{CH(M)} \vec{b}$ and $HB(\vec{a}/C\vec{b}) = HB(\vec{a}/C)$. This is a natural notion of independence that coincides with non-forking when T has SU-rank one (see [3]).

Lemma 7.2. *Suppose (M, H) is an H -structure, $\vec{a} \in M$ and $E = \text{acl}(E) \subset M$. Then $HB(\vec{a}/E) = H(\text{acl}(\vec{a})) \setminus E$.*

Proof. Recall that $HB(\vec{a}/E)$ is the smallest tuple $\vec{h} \in H(M)$ such that

$$\vec{a} \downarrow_{\vec{h}E} H(M)E.$$

Since acl is disintegrated and $E = \text{acl}(E)$, we have

$$\vec{a} \downarrow_{H(\text{acl}(\vec{a}))E} H(M)E.$$

Then $\vec{h} = \text{acl}(\vec{a}) \cap H(M) \setminus E$ is the smallest tuple in $H(M)$ such that $\vec{a} \downarrow_{\vec{h}E} H(M)E$. \square

Lemma 7.3. *Let $\vec{a}, \vec{b} \in M$ (possibly infinite tuples) and $C \subset M$. Then $\vec{a} \downarrow_C^H \vec{b}$ implies $\vec{a} \downarrow_C \vec{b}$.*

Proof. Assume that $\vec{a} \not\downarrow_C \vec{b}$, so we can write $\vec{b} = \vec{b}_1 b_2$ so that $b_2 \in \text{acl}(\vec{b}_1 C \vec{a}) \setminus \text{acl}(\vec{b}_1 C)$.

Case 1 Assume that $b_2 \in \text{acl}(\vec{b}_1 C \vec{a} H(M)) \setminus \text{acl}(\vec{b}_1 C H(M))$. Then $\vec{a} \not\downarrow_{CH(M)} \vec{b}$ as we wanted.

Case 2 Assume that $b_2 \in \text{acl}(\vec{b}_1 C H(M))$. Since $b_2 \notin \text{acl}(\vec{b}_1 C)$ by triviality there is $h \in H(M)$ such that $b_2 \in \text{acl}(h)$ and $h \in \text{acl}(b_2)$. Since $b_2 \in \text{acl}(\vec{b}_1 C \vec{a}) \setminus \text{acl}(\vec{b}_1 C)$, there is $a_2 \in \vec{a}$ such that $b_2 \in \text{acl}(a_2)$ and $a_2 \in \text{acl}(b_2)$. Then we also get $a_2 \in \text{acl}(h)$ and $h \in \text{acl}(a_2)$. This proves that $h \in HB(b_2/C)$, but $h \notin HB(b_2/C\vec{a})$, so $HB(\vec{a}/C\vec{b}) \neq HB(\vec{a}/C)$. \square

Lemma 7.4. *Suppose (M, H) is an H -structure, $\vec{a}, \vec{b} \in M$ and $E = \text{acl}(E) \subset M$. Suppose $\vec{a} \downarrow_E^H \vec{b}$. Let $\vec{a}' = \vec{a} HB(\vec{a}/E)$ and $\vec{b}' = \vec{b} HB(\vec{b}/E)$. Then $\vec{a}' \downarrow_E^H \vec{b}'$.*

Proof. We need to show that $\vec{a}' \downarrow_{EH(M)} \vec{b}'$ and $HB(\vec{a}'/E) = HB(\vec{a}'/E\vec{b}')$. The former follows the fact that $\text{acl}(\vec{a}') = \text{acl}(\vec{a})$, $\text{acl}(\vec{b}') = \text{acl}(\vec{b})$ (by Lemma 7.2) and

$\vec{a} \downarrow_{EH(M)} \vec{b}$. To show the latter, note that, by Lemma 7.2,

$$HB(\vec{a}'/E) = H(\text{acl}(\vec{a}')) \setminus E = H(\text{acl}(\vec{a})) \setminus E = HB(\vec{a}/E).$$

It remains to show that $HB(\vec{a}/E\vec{b}) = HB(\vec{a}'/E\vec{b}')$. First, note that

$$\begin{aligned} HB(\vec{a}'/E\vec{b}') &= H(\text{acl}(\vec{a}')) \setminus \text{acl}(E\vec{b}') = H(\text{acl}(\vec{a})) \setminus \text{acl}(E\vec{b} \cup (H(\text{acl}(\vec{b}))) \setminus E) = \\ &= H(\text{acl}(\vec{a})) \setminus (\text{acl}(E\vec{b}) \cup \text{acl}(H(\text{acl}(\vec{b}))) \setminus E) \subset H(\text{acl}(\vec{a})) \setminus (\text{acl}(E\vec{b})). \end{aligned}$$

Since $\vec{a} \downarrow_E \vec{b}$, we have $\text{acl}(\vec{a}) \cap \text{acl}(\vec{b}) \subset \text{acl}(E) = E$. Thus,

$$H(\text{acl}(\vec{a})) \setminus (E\vec{b} \cup (H(\text{acl}(\vec{b}))) \setminus E) = H(\text{acl}(\vec{a})) \setminus (E\vec{b}) = HB(\vec{a}/E\vec{b}).$$

Hence, $HB(\vec{a}'/E\vec{b}') = HB(\vec{a}/E\vec{b})$. □

Proposition 7.5. *Let (M, H) is a sufficiently saturated H -structure and assume the criterion (*) holds for algebraic independence in M . Assume also that T is trivial. Then the criterion (*) holds for H -independence in (M, H) .*

Proof. We first show that part (1) of criterion (*) holds. Let $E = \text{acl}_H(E) \subset M$, let $\vec{a} \in M$ and $C \subset M$. By making the tuple larger if necessary, we may assume that $\vec{a}E$ is H -independent and write $\vec{a} = \vec{a}_1\vec{a}_2\vec{a}_3$, where \vec{a}_1 is independent over $EH(M)$, \vec{a}_2 is independent over E but $\vec{a}_2 \in H(M)$ and $\vec{a}_3 \in \text{acl}(E\vec{a}_1a_2)$. Choose $\vec{a}' = \vec{a}'_1\vec{a}'_2\vec{a}'_3 \equiv_E \vec{a}_1\vec{a}_2\vec{a}_3$ independent from C over E (in the sense of algebraic closure). By the extension property, we may assume that $\vec{a}'_1 \downarrow_{ECH(M)}$. By the density property we may assume that $\vec{a}'_2 \in H(M)$. Then since $\vec{a}E$ and $\vec{a}'E$ are H -independent we get $\vec{a}'_1\vec{a}'_2\vec{a}'_3 \equiv_E^H \vec{a}_1\vec{a}_2\vec{a}_3$. We also get $\vec{a}' \downarrow_E^H C$.

We now show that part (2) of criterion (*) holds. Let $E = \text{acl}(E) \subset M$ and let $\vec{a}, \vec{b}, \vec{c} \in M$, be such that $\vec{b} \downarrow_E^H \vec{a}$, $\vec{c} \downarrow_E^H \vec{a}\vec{b}$ and $\text{tp}_H(\vec{a}/E) = \text{tp}_H(\vec{b}/E)$. By Lemma 7.4, we may assume that $\vec{b} = \vec{b} \cup HB(\vec{b}/E)$, $\vec{a} = \vec{a} \cup HB(\vec{a}/E)$ and $\vec{c} = \vec{c} \cup HB(\vec{c}/E)$.

By the previous lemma, $\vec{b} \downarrow_E \vec{a}$ and $\vec{c} \downarrow_E \vec{a}\vec{b}$. Since the criterion holds inside M , there exists \vec{c}^* such that $\text{tp}(\vec{a}\vec{c}/E) = \text{tp}(\vec{a}\vec{c}^*/E) = \text{tp}(\vec{b}\vec{c}^*/E)$. In particular $\vec{c}^* \downarrow_E \vec{a}$, $\vec{c}^* \downarrow_E \vec{b}$ and by triviality $\vec{c}^* \downarrow_E \vec{a}\vec{b}$. We may write $\vec{c} = \vec{c}_1\vec{c}_2\vec{c}_3$ where \vec{c}_1 is independent from $EH(M)$, \vec{c}_2 is independent from E but belongs to $H(M)$ and $\vec{c}_3 \in \text{acl}(E\vec{c}_2\vec{c}_1)$. Since $\vec{c} \equiv_E \vec{c}^*$, we may also write $\vec{c}^* = \vec{c}_1^*\vec{c}_2^*\vec{c}_3^*$. Using the extension property and the density property we may assume that \vec{c}_1^* is independent over $H(M)E\vec{a}\vec{b}$, \vec{c}_2^* is independent over $E\vec{a}\vec{b}$ and belongs to $H(M)$ and we still have $\vec{c}_3^* \in \text{acl}(E\vec{c}_2^*\vec{c}_1^*)$. Then the tuple \vec{c}^* is H -independent.

By triviality, both $\vec{a}\vec{c}E$ and $\vec{a}\vec{c}^*E$ are H -independent. Similarly $\vec{b}\vec{c}E$ and $\vec{b}\vec{c}^*E$ are H -independent. Thus

$$\text{tp}_H(\vec{a}\vec{c}/E) = \text{tp}_H(\vec{a}\vec{c}^*/E) = \text{tp}_H(\vec{b}\vec{c}^*/E)$$

as we wanted. □

Assume now that (M, P) is a sufficiently saturated lovely pair of models of T and that T is trivial. For $\vec{a} \in M$ a finite tuple we will write $PB(\vec{a})$ for some independent tuple $\vec{h} \in P(M)$ such that $\vec{a} \downarrow_{\vec{h}} P(M)$ which is minimal in length. Note that by triviality each element in \vec{h} will be interalgebraic with an element in \vec{a} . The choice for the elements in \vec{h} is not unique, but any other choice will be interalgebraic

with \vec{h} . Similarly, for $C \subset M$, we write $PB(\vec{a}/C)$ for some independent tuple $\vec{h} \in P(M)$, which is independent over C , such that $\vec{a} \downarrow_{\vec{h}C} P(M)$. The tuples $PB(\vec{a})$ and $PB(\vec{a}/C)$ work in a way similar to an H -basis for H -structures. For an infinite tuple \vec{a} we can define $PB(\vec{a}/C) = \cup\{PB(\vec{a}_0/C) : \vec{a}_0 \subset \vec{a} \text{ finite}\}$.

For $\vec{a}, \vec{b} \in M$ (possibly infinite tuples) and $C \subset M$ we will write $\vec{a} \downarrow_C^P \vec{b}$ (and call this notion P -independence) if $\vec{a} \downarrow_{CP(M)} \vec{b}$ and $PB(\vec{a}/C\vec{b}) = PB(\vec{a}/C)$. As before, this notion agrees with non-forking independence when T has SU -rank one and is trivial. The following proposition can be proved by modifying slightly the arguments from Proposition 7.5.

Proposition 7.6. *Let (M, P) is a sufficiently saturated lovely pair of models of T and assume the criterion (*) holds for algebraic independence in M . Assume also that T is trivial. Then the criterion (*) holds for P -independence in (M, H) .*

7.3. Imaginaries in the structure induced in the predicate. Here we follow the ideas from Eleftheriou [13]. We start with (M, P) a dense/codense pair satisfying TEA and study imaginaries in the induced structure on the predicate P from M . This is also related to generic trivializations [6], where the authors of this paper studied the induced structure on P *without parameters*. Does it matter if we consider parameters from a fixed model?.

Eleftheriou [13] proved that when T is an o-minimal theory extending the theory of a divisible abelian group, then $Th(P^*)$ has elimination of imaginaries.

We consider first the case of SU -rank one.

Theorem 7.7. *Let T is of SU -rank one and (M, H) is a model of the theory of H -structures. Then the theory of H with the induced structure from M has weak elimination of imaginaries.*

Proof. It is easy to see that $Th(H^*)$ has SU -rank one. Since the elements of H are algebraically independent, forking in H^* is trivial. Furthermore, for each $a \in H^*$ we have $\text{acl}_*(a) = \{a\}$ and for each $A \subset H^*$, $\text{acl}_*(A) = A$. Then for $A \subset H^*$ and $\vec{b} = (b_1, \dots, b_n)$ we have that $\vec{b} \downarrow_{\vec{b} \cap A}^* A$. Thus $Cb(\vec{b}/A) = \vec{b} \cap A$ so $Th(H^*)$ has weak elimination of imaginaries. \square

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