Cell decomposition for P-minimal fields

Marie-Hélène Mourgues

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Abstract In [S-vdD] P. Scowcroft and L. van den Dries prove a Cell Decomposition Theorem for p-adically closed fields. We work here with the notion of P-minimal fields defined by D. Haskell and D. Macpherson in [H-Mph]. We prove that a P-minimal field K admits cell decomposition if and only if K has definable selection. A preprint version in French of this result appeared as a prepublication [M].

1 Introduction

A *p-valued field* is a valued field (K, v) of characteristic 0 such that $v(p) = 1$ with valuation group vK and residue field K/v of characteristic p. The residue field is a finite algebraic extension of $\mathbb{F}_p$; the degree of this extension, denoted by d is called the rank of (K, v). A p-valued field of rank d is said to be *p-adically closed* if it does not admit any proper algebraic extension to a p-valued field of the same rank. A characterisation of the p-adically closed fields of rank d is given in [P-R]: a p-valued field is p-adically closed if and only if it is Henselian and its value group is a $\mathbb{Z}$-group.

We denote by $L_d = \{+, -, ., 0, 1, Div, (P_n)_{n>1}, c_1, \cdots, c_d\}$ Macintyre's language for p-adically closed fields of rank d. If (K, v) is a p-valued field whose value group is a $\mathbb{Z}$-group, the language is interpreted as follows. For each $n > 1$, $K \models P_n(x)$ if and only if $\tilde{K} \models \exists y(x = y^n)$ where $\tilde{K}$ is the p-adic closure of K. We will use $P_n^\star$ to abbreviate the formula $P_n(x) \wedge x \neq 0$. The binary predicate Div is interpreted by $Div(a, b)$ if and only if $v(a) \leq v(b)$. The c_i are interpreted in K as a basis of the residue field over $\mathbb{F}_p$. A. Prestel and P. Roquette [P-R], generalizing the theorem of Macintyre [Ma], have shown that in this language, the theory of p-adically closed fields of rank d admits elimination of quantifiers. Then, in this language, the definable subsets of K^n are exactly the semi-algebraic.

Let L'_d be any language extending L_d, we recall from [H-Mph] the definition of a *P-minimal* L'_d-structure, which is the analogue in the p-adic case of *o-minimality* in the real case.

Definition 1.1 *Let K be an L'_d-structure. We say that K is P-minimal if for every K' elementary equivalent to K, every definable subset of K' is quantifier free definable by an L_d-formula.*

Haskell and Macpherson carry on the analogy by showing that any P-minimal field is p-adically closed. In the same paper, they ask whether P-minimal fields admit cell decomposition.

We prove here, (3.5) and (4), that a P-minimal field K admits a cell decomposition if and only if K has definable selection (3.2).

R. Cluckers [C1] and [C2] proved a Cell Decomposition Theorem for subanalytic sets of finite field extensions of $\mathbb{Q}_p$ which also gives a preparation result for definable functions.

Hans Schoutens in [Sc] introduced a notion of t-minimality and proved independently a Cell Decomposition Theorem for strongly t-minimal structures with definable selection which include the P-minimal case. He also proves that in some t-minimal structures cell decomposition implies definable selection.

2 Preliminaries

The starting point of our work is the Cell Decomposition Theorem given for p-adically closed fields in [S-vdD]:

Proposition 2.1 *[S-vdD] Let K be a p-adically closed field of p-rank d. Let S be a semi-algebraic subset of K^n and $f : S \mapsto K$ a definable function. Then there is a partition of S into finitely many definable sets on each of which f is continuous. Each set in the partition either is open in K^n or has no interior and is homeomorphic by a bicontinuous projection onto certain of the coordinate axes to an open subset of K^l, where $l < n$.*

Then, they obtain for the field $\mathbb{Q}_p$ of p-adic numbers a result of cylindric algebraic decomposition using Denef's Theorem (here $|x|$ means $p^{-v(x)}$):

Theorem 2.2 *[D2]: Let $f_i(x,t) \in \mathbb{Q}_p[x,t], i = 1, \cdots, r,\ x = (x_1, \cdots x_m), t$ one variable. Let $n \in \mathbb{N}$, $n > 0$, be fixed. Then there exists a finite partition of $\mathbb{Q}_p^m$ into subsets A of the form*

$$A = \{(x,t) \in \mathbb{Q}_p^{m+1}; x \in C, |a_1(x))|\square_1|t - c(x)|\square_2|a_2(x)|\}$$

where C is a definable subset of $\mathbb{Q}_p^m$ and $\square_1$ (resp. $\square_2$) denotes either $<$, $\leq$, or no condition, and a_1, a_2, c are definable functions from $\mathbb{Q}_p^m$ to $\mathbb{Q}_p$ such that for all $(x,t) \in A$, we have

$$f_i(x,t) = u_i(x,t)^n h_i(x)(t - c(x))^{\nu_i},\ \textit{for}\ i = 1, \cdots, r$$

with $u_i(x,t)$ a unit in $\mathbb{Z}_p$, h_i a definable function from $\mathbb{Q}_p^m$ to $\mathbb{Q}_p$ and $\nu_i \in \mathbb{N}$

As noticed in [S-vdD] Denef's Theorem is still true for finite extensions of $\mathbb{Q}_p$ and therefore for every p-adically closed fields of rank d which are elementary equivalent to a finite extension of $\mathbb{Q}_p$. So it follows that the cylindric algebraic decomposition is again true for any p-adically closed field.

We will add the hypothesis of definable selection (3.2) to obtain a Cell Decomposition Theorem in the P-minimal case.

The next two results come from [H-Mph] and will be of great use in what follows. The topological dimension $topdim(S)$ of a definable subset S of K^n is the greatest integer $k \leq n$ for which there is a projection $\pi : K^n \mapsto K^k$ such that $\pi(S)$ has non empty interior in K^k. Haskell and Macpherson show that topological dimension is *well-behaved*, i.e.

$$topdim(S_1 \cup \cdots \cup S_m) = max\{topdim(S_1), \cdots, topdim(S_m)\}.$$

Proposition 2.3 *[H-Mph] Let K be a P-minimal field and $f : K \mapsto K$ be a definable partial function. Then, there is an open subset U of $dom(f)$ such that $dom(f) - U$ is finite and $f_{\restriction U}$ is continuous.*

Proposition 2.4 *[H-Mph] Let $n > 0$ and $f : K^n \mapsto K$ be a definable partial function, and let $X = dom(f)$.*
Let $Y = \{y \in X : f$ is defined and continuous in a neighbourhood of $y\}$. Then $topdim\ (X \backslash Y) < n$.

We will use in the last section the following version of Hensel's Lemma.

Lemma 2.5 *Let K be a p-adically closed field and let $\mathcal{O}$ be its valuation ring. Let $f(X) \in \mathcal{O}[X]$ and $f'(X)$ denotes its derivative. Suppose that there exists $a \in \mathcal{O}$ such that $v(f(a)) > 2v(f'(a))$. Then there exists a unique $b \in K$ such that $f(b) = 0$ and $v((b - a)) > v(f'(a))$.*

3 Cell decomposition for P-minimal fields

In the following, we consider P-minimal fields of fixed rank d. For simplification we write L instead of L_d and L' instead of L'_d. Definable will always mean definable with parameters. An L-definable subset of K^n will be called *semi-algebraic* and *definable* will always means L'-definable.

From P-minimality by a classical model-theoretic compactness argument we get:

Lemma 3.1 *For any L'-definable set $S' \subset K^{n+1}$ there exists m and a semi-algebraic subset S of K^{m+1} such that for each $y \in K^n$ there is $z \in K^m$ with $S'_y = S_z$, where S'_y denotes the fiber at y of S'.*

In other words, if $\phi(y, x)$ is an L'-formula defining S' then there exists a quantifier free L-formula $\psi(z, x)$ such that $K \models \forall y \exists z \forall x(\phi(y, x) \Leftrightarrow \psi(z, x))$.

Definition 3.2 *Let K be a structure over a language L. We say that K admits definable selection if for any definable set $S \subset K^{n+m}$ there exists a definable function $g : \pi(S) \mapsto K^m$ whose graph is contained in S (where $\pi : K^{n+m} \mapsto K^n$ is the projection map).*

Throughout this section, K will denote a P-minimal L'-structure with definable selection. Then, the following lemma holds:

Lemma 3.3 *Let S' be a L'-definable subset of K^{n+1}. Let $\pi_n : K^{n+1} \mapsto K^n$ be the projection map. There exists m and a semi-algebraic subset S of K^{m+1} and a L'-definable function f from $\pi_n(S')$ to K^m such that for any $y \in \pi_n(S')$,*

$$\{x \in K; (y, x) \in S'\} = \{x \in K; (f(y), x) \in S\}.$$

Proof: Let $\phi(y, x)$ be a L'-formula defining S'. Let S be a semi-algebraic subset of K^{m+1} given by (3.1) and $\psi(z, x)$ a L-formula defining S. Let $F(y, z) = \forall x(\phi(y, x) \Leftrightarrow \psi(z, x))$. We apply definable selection to the L'-definable set $A = \{(y, z) \in K^{n+m}; K \models F(y, z)\}$. Let $\pi : K^{n+m} \mapsto K^n$ as in (3.2). Then there exists a definable function $f : \pi(A) \mapsto K^m$ whose graph is contained in A. By (3.1), $\pi_n(S') \subset \pi(A)$, hence, for any $y \in \pi_n(S')$, $\{x \in K; (y, x) \in S'\} = \{x \in K; (f(y), x) \in S\}$. $\Box$

Now, let us formulate a precise definition of cells in the sense of [vdD]

Definition 3.4 *Let $(i_1, \cdots, i_n)$ be a sequence of zeros and ones of length n. An $(i_1, \cdots, i_n)$-cell is a definable subset of K^n defined by induction on n as follows:*

1. *A (0)-cell is a point of K and a (1)-cell is of the form*

   $$\{x \in K; \gamma_1 < v(x - c) < \gamma_2 \wedge P_k^\star(\lambda(x - c))\}$$

   where $\gamma_1, \gamma_2 \in v(K) \cup \{-\infty, \infty\}$; c the center of the cell, is in K; $k \in \mathbb{N}$ and λ is chosen from a fixed finite set of coset representatives of $P_k^\star$ in K^*.

2. *Suppose that $(i_1, \cdots, i_n)$-cells are already defined.*
   Then an $(i_1, \cdots, i_n, 0)$-cell is the graph of a definable continuous function from an $(i_1, \cdots, i_n)$-cell to K. And an $(i_1, \cdots, i_n, 1)$-cell is a set of the form

   $$\{(y, x) \in C \times K; v(a_1(y))\Box_1 v(x - c(y))\Box_2 v(a_2(y)) \wedge P_k^\star(\lambda(x - c(y)))\}$$

   where C is an $(i_1, \cdots, i_n)$-cell, a_1, a_2, c are definable continuous functions on C, λ is as in (1) and $\Box_1$ and $\Box_2$ are either $\leq$, $<$ or no condition.

Note that the $(1, \cdots, 1)$-cells are exactly the cells which are open in their ambient space K^n and are called **open cells**. Let C be a $(i_1, \cdots, i_n)$-cell. Then *topdim*$(C) = k$ where k is the number of i_l equal to 1. If $\pi : K^n \mapsto K^k$ is the projection onto the k axes corresponding to indexes i_l equal to 1, then π maps C homeomorphically onto an open cell of K^k. Each cell is locally closed.

Theorem 3.5 *Let K be a P-minimal L'-structure with definable selection. For each $n \in \mathbb{N}$,*

I_n *If S' is a definable subset of K^n, then S' can be partitioned in finitely many cells of K^n.*

II_n *Given a definable function $f : S' \mapsto K$ where S' is a definable subset of K^n, there exists a finite partition of S' into cells such that the restriction of f to each cell is continuous.*

Remark 3.6 *When each occurrence of the word "definable" is replaced by "semi-algebraic", Theorem (3.5) follows easily from Denef and Scowcroft-van den Dries results recalled in the above preliminaries. In this case we will speak of* **semi-algebraic cell decomposition** *and we will refer to this result by SACD.*

Proof: We will prove I_n and then II_n by induction on n.

I_1 follows from P-minimality and the cell decomposition for p-adically closed fields (2.1) and II_1 follows from I_1 and (2.3).

Assume I_i and II_i for $i \leq n$. So let S' be a definable subset of K^{n+1}. Let π_n be the usual projection $\pi_n : K^{n+1} \mapsto K^n$ onto the first n axes. Let $S \in K^{m+1}$ be a semi-algebraic set and $f : \pi_n(S') \mapsto K^m$ a definable function given by (3.3), i.e. such that for any $y \in \pi_n(S')$

$$\{x \in K; (y, x) \in S'\} = \{x \in K; (f(y), x) \in S\}.$$

By SACD, S is a finite partition of semi-algebraic cells. We call B any such cell and we denote by C the projection of B onto the first m axes. Now, by our inductive hypothesis II_n, for each co-ordinate function f_i of f, there is finite decomposition into cells of $\pi_n(S')$, such that the restriction of f_i to each cell is continuous. Thus we can find a finite decomposition of $\pi_n(S')$ into cells C' such that the restriction of f to each cell is continuous. For each C and C' in the previous partitions, consider the set $T = \{y \in K^n; y \in C' \text{ and } f(y) \in C\}$. Since T is a definable set of K^n, the inductive hypothesis I_n tell us that T is a finite union of cells of K^n. Take A' a fixed cell of this partition of T, then we will show that the set $B' = \{(y, x) \in A' \times K; (f(y), x) \in B\}$ is a cell of K^{n+1} contained in S'.

Assume first that B is an $(i_1, \cdots, i_m, 1)$-cell of K^{m+1}, i.e.

$$B = \{(z, x) \in C \times K; v(a_1(z)) \square_1 v(x - c(z)) \square_2 v(a_2(z)) \wedge P_k^\star(\lambda(x - c(z)))\}$$

where C is here a semi-algebraic $(i_1, \cdots, i_m)$-cell, and a_1, a_2, c are semi-algebraic continuous functions on C. Then,

$$B' = \{(y, x) \in A' \times K; v(a_1(f(y))) \square_1 v(x - c(f(y))) \square_2 v(a_2(f(y))) \wedge P_k^\star(\lambda(x - c(f(y))))\}.$$

Since f is continuous on A' and $f(A') \subset C$, $a_2 \circ f$, $a_1 \circ f$ and $c \circ f$ are definable continuous functions, thus B' is a cell of K^{n+1}.

Assume now that B is the graph of a semi-algebraic function $g : C \mapsto K$. Then B' is the graph of the definable function $h : A' \mapsto K$ defined by $h(y) = g(f(y))$. Hence B' in this case again is a cell of K^{n+1}.

Morever, it is clear that S' is the finite union of the cells B' obtained from the cells B which partition S, the cells C' which partition $\pi_n(S')$, and for each corresponding T, the cells A' which partition T. Therefore I_{n+1} is established.

We will now derive II_{n+1} from I_i, II_i, $i \leq n$ and I_{n+1}.

Let again S' be a definable subset of K^{n+1} and $g : S' \mapsto K$ be a definable function. Because of I_{n+1} it suffices to show that S' can be partitioned into finitely many definable sets such that the restriction of g to each set is continuous. Again by I_{n+1} we can assume without loss of generality that S' is already a cell. If the cell S' is not open in K^{n+1} we are done by using our inductive hypothesis on $\pi(S')$ where π is the projection on the $k = topdim(S')$ axes defined in section 2. Since $k < n+1$, the set $\pi(S')$ can be partitioned into finitely many cells on which $g \circ \pi^{-1}$ is continuous which leads directly to the conclusion.

Suppose now that S' is an open cell of K^{n+1}. Let

$$U' = \{y \in S'; g \text{ is continuous at a neighbourhood of } y\}.$$

By (2.3) and (2.4) we have $topdim(S' \backslash U') < n+1$. As above, by inductive hypothesis II_i, $i \leq n$, $S' \backslash U'$ can be partitioned into cells on which the restriction of g is continuous. Since U' is definable, the conclusion holds. $\square$

Assertion II_2 can be refined as follows:

Proposition 3.7 *Let C be a 1-cell and $f : C \times K \mapsto K$ an L'-definable function such that for any $x \in C$, the function $y \mapsto f(x,y)$ is continuous on K. Then there are 1-cells $C_1, \cdots, C_n$ whose union is co-finite in C such that f is continuous on each $C_i \times K$.*

Proof: By Theorem (3.5) there is a finite partition of C in points and 1-cells $C_1, \cdots, C_n$ and for any i a partition of $C_i \times K$ in cells which are either the graphs $\Gamma_{i,j}$ of K-definable functions $c_{i,j}$ continuous on C_i or sets of the form

$$D_{i,j} = \{(x,y) \in C_i \times K; v(a_{i,j}(y)) \square_1 v(x - c_{i,j}) \square_2 v(b_{i,j}(y)) \wedge P_k(\lambda(x - c_{i,j}(y)))\}$$

such that the restriction of f to each $\Gamma_{i,j}$ and $D_{i,j}$ is continuous. In order to prove that f is continuous on every $C_i \times K$, it suffices to show that f is continuous at each point $(x, c_{i,j}(x))$. So let c be one of the functions $c_{i,j}$ and $(x, c(x))$ a point of the graph $\Gamma_{i,j}$. Using the facts that the function $y \mapsto f(x,y)$ is continuous on K for any $x \in C_i$, the function f is continuous on the graph $\Gamma_{i,j}$, and the function c is continuous on C_i, we get that for any $\beta \in v(K)$ there is $\alpha \in v(K)$ such that, if $min(v(x - x'), v(c(x) - y') > \alpha$ then
$v(f(x, c(x)) - f(x', y'))$
$\geq min\{v(f(x,c(x)) - f(x', c(x'))), v(f(x', c(x')) - f(x', c(x))), v(f(x', c(x)) - f(x', y'))\}$
$\geq \beta$.

This gives the continuity of f at $(x, c(x))$. $\square$

4 The converse

The hypothesis of definable selection might seem too strong. However, we can verify that it is necessary:

Proposition 4.1 *Let K be an L'_d-structure which is P-minimal. If K satisfies I_n for all $n \in \mathbb{N}$ then K admits definable selection.*

Proof: It suffices to adapt the proof of the existence of semi-algebraic selection given in the appendix of [D-vdD]. Let S be a definable subset of K^{n+m}. Without loss of generality, we may assume that $m = 1$, because the general case then follows by induction on m. By I_{n+1}, S is a finite union of cells. So, it is enough to prove that, for any definable cell C of K^{n+1}, there exists a definable function $f : \pi(C) \mapsto K$, whose graph is included in C. In the case where C is an $(i_1, \cdots, i_n, 0)$-cell of the form $\{(y, x) \in B \times K; x = c(y)\}$, the function c is suitable. Let us now consider the case where C is a $(i_1, \cdots, i_n, 1)$-cell, i.e.

$$C = \{(y, x) \in B \times K; v(a_1(y))\square_1 v(x - c(y))\square_2 v(a_2(y)) \wedge P_k^\star(\lambda(x - c(y)))\}.$$

Let $t^k = \lambda(x - c(y))$, $b_1(y) = \lambda a_1(y)$ and $b_2(y) = \lambda a_2(y)$. We have to prove the existence of a definable function $g : B \mapsto K$ whose graph is included in the set $\{(y, t) \in B \times K; v(b_1(y))\square_1 kv(t)\square_2 v(b_2(y))\}$. Let M be a family of coset representatives modulo P_k, such that for any $\mu \in M$, $0 \leq v(\mu) \leq k$. Then the sets $B_\mu = B \cap P_k^\star(\mu b_1(y))$ partition B into definable sets on which $v(b_1(y)) = -v(\mu)$ modulo k. Put $b(y) = \mu b_1(y)$, then $v(b(y))$ is a multiple of k on B_μ. Hence we may suppose that for all $y \in B$, $v(b(y))$ is a multiple of k.

Now we follow the lines of the proof of Lemma (2.4) of [D1]. Let π be a fixed element of K such that $v(\pi) = 1$, and for $x \neq 0$, let $ac(x) = x\pi^{-v(x)}$. By Lemma (2.1) of [D1], there exists a definable function $\theta(y)$ from B to K such that $v(\theta(y)) = v(b(y))$ and $v(ac(\theta(y)) - 1) > 2v(k)$ (here θ is definable instead of semi-algebraic since b is definable). By applying (2.5) with $f(X) = X^k - ac(\theta(y))$ and the approximate solution 1, for every $y \in B$ there exists a unique $\eta(y) \in K$ such that $\eta(y)^k = ac(\theta(y))$ and $v(\eta(y) - 1) > v(k)$. The function g defined from B to K by $g(y) = \eta(y)\pi^{v(b(y))/k}$ is clearly definable and, forall $y \in B$,

$$v(g(y)) = \frac{v(b(y))}{k}.$$

Therefore, the function g is suitable in the case where $\square_1$ is $\leq$.

The other cases are similar. $\square$

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Equipe de Logique Mathématique
UFR de Mathématiques
Université Denis-Diderot Paris 7
2 place Jussieu
75251 Paris Cedex 05 France
E-mail: mhm@logique.jussieu.fr