Hrushovski's Fusion

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Abstract

We present a detailed and simplified exposition of Hrushovki's fusion of two strongly minimal theories.

1 Introduction

A definable set whose definable subsets (in some saturated model) are either finite or cofinite is called *strongly minimal.* Examples of strongly minimal structures are the trivial one, infinite dimensional vector spaces or algebraically closed fields. It was observed that the geometrical behaviour of these archetypical examples could be generalized to the pregeometry of algebraic closure on strongly minimal sets, exhibiting a first example of *regular types.* This led B. Zilber to conjecture that all strongly minimal sets could be classified according to these three basic ones, motivated by his work on *Zariski geometries* with E. Hrushovski. This conjectured was refuted in a genial way by E. Hrushovski [9], which in a genial way adapted Fraïssé's construction in order to *collapse* to finite rank a candidate for the counter-example. The amalgamation procedure can be described in the following way: the goal is to construct a countable universal model starting from a given collection of finitely generated structures. In this model there is a unique type of rank ω. The decisive part (or *collapse*) is to modify this construction in order to algebraize types of finite rank. In order to do so, a collection of representatives (or *codes*) of these types needs to be chosen and one assigns a maximal length of an independence sequence of realisations to each code. The structured obtained after amalgamating again has now finite rank. Note that the prescribed maximal length must reflect any interaction between different codes, since some realizations of one code may yield realizations for another.

Using the same procedure, E. Hrushovski also merged two strongly minimal theories over a trivial geometry into a new one, their *fusion.* This answered negatively a question of G. Cherlin on the existence of a maximal strongly minimal theory. More precisely, consider two countable strongly minimal theories T_1 and T_2 with the definable multiplicity property (in short, DMP) whose respective languages L_1 and L_2 are disjoint. Recall that a ω-stable theory has the DMP if Morley rank and degree are definable on the parameters of any given definable set.

In this survey we give a detailed and slightly simplified exposition of the following theorem proved by E. Hrushovski.

Theorem ([8]). *$T_1 \cup T_2$ admits a strongly minimal completion T^μ. Its models satisfy the following: Let tr_i denote the transcendence degree in T_i. Given a finite subset A of M, then*

$$|A| \leq \mathrm{tr}_1(A) + \mathrm{tr}_2(A).$$

Our presentation grew out of a seminar held at the Humboldt-Universität Berlin in 2003. Several articles on this topic (among others [7] and [6]) have been published, however we believe that this survey will be beneficial for the mathematical community in order to become more acquainted with Hrushovski's fusion method. The authors used the simplified approach in three subsequent articles: In [4] to reprove a theorem of Poizat and Baldwin–Holland ([10], [1]) about the existence of a fields of Morley rank 2 with a distinguished subset in any characteristic, in [5] to construct fields of Morley rank 2 with a distinguished additive subgroup, and in [3] to the fusion over a vector space over a finite field, which had been proposed by E. Hrushovski.

The simplified technique was crucial in [2], where a bad field was constructed: a field of Morley rank 2 with a distinguished multiplicative subgroup. This solved a long standing open problem.

Finally our techniques were used in [11] to prove the following generalization of Theorem 1: *Let T_1 and T_2 be two countable complete theories in disjoint languages of finite Morley rank and of the same Morley degree. Assume that in T_1 and T_2 Morley rank and Morley degree are definable. Then $T_1 \cup T_2$ has a "nice" completion of any rank which is a common multiple of the ranks of T_1 and T_2.*

2 Codes

All throughout the following sections (and until specified otherwise) T denotes a countable strongly minimal theory with DMP.

First, let us fix some notation: $\mathrm{tr}(a/B)$ is the transcendence degree of a over B[1] and $\mathrm{MR}(p)$ denotes Morley rank of the type p. Note that

$$\mathrm{tr}(a/B) = \mathrm{MR}(\mathrm{tp}(a/B)).$$

We write

$$\phi(x) \sim^k \psi(x)$$

or $\phi(x) \sim^k_x \psi(x)$, if the symmetric difference of ϕ and ψ has smaller Morley rank than k.

A formula $\chi(x, b)$ is *simple* if it has Morley degree 1 and the components of a generic realization are pairwise different and not in $\mathrm{acl}(b)$. If a is an n–tuple and s is some subset of $\{1, \ldots, n\}$, then a_s is $\{a_i \mid i \in s\}$.

A *code* c is a parameter-free formula

$$\phi_c(x, y),$$

where $|x| = n_c$ and y lies in some sort of T^{eq}, with the following properties.

[1] The maximal length of a B-independent subtuple of a.

(i) $\phi_c(x,b)$ is either empty[2] or simple. Furthermore, $\phi(x,b)$ implies that the components of x are pairwise different.

(ii) All non-empty $\phi_c(x,b)$ have Morley rank k_c and Morley degree 1.

(iii) For each subset s of $\{1,\ldots,n_c\}$ there exists an integer $k_{c,s}$ such that for every realization a of $\phi_c(x,b)$

$$\mathrm{tr}(a/ba_s) \leq k_{c,s}.$$

Equality holds for generic a.

(iv) If both $\phi_c(x,b)$ and $\phi_c(x,b')$ are non-empty and $\phi_c(x,b) \sim^{k_c} \phi_c(x,b')$, then $b = b'$.

For $\phi_c(x,b)$ to have Morley rank k_c in (ii) is equivalent to $k_{c,\emptyset} = k_c$. The simplicity of $\phi_c(x,b)$ in (i) is equivalent to Morley degree 1 and $k_{c,\{i\}} = k_c - 1$ for all i.

Corollary 2.1. *Let $p \in \mathrm{S}(b)$ be the unique type of Morley rank k_c containing $\phi_c(x,b)$. Then b is the canonical basis of p.*

Proof. This follows easily from (iv). □

Lemma 2.2. *Let $\chi(x,d)$ be a simple formula. Then there is some code c and some $b_0 \in \mathrm{dcl}^{\mathrm{eq}}(d)$ such that $\chi(x,d) \sim^{k_c} \phi_c(x,b_0)$.*

We say that c *encodes* $\chi(x,d)$.

Proof. Set $k_c = \mathrm{MR}(\chi(x,d))$ and $n_c = |x|$. Let p be the global type of rang k_c containing $\chi(x,d)$, with canonical basis b_0. We find a formula $\phi(x,b_0)$ in p of rank k and degree 1. Choose a generic realization a_0 of $\phi(x,b_0)$. For each $s \subset \{1,\ldots,n_c\}$ set $k_{c,s} = \mathrm{MR}(a_0/b_0 a_{0s})$. By strengthening $\phi(x,b_0)$ appropriately we may assume that $\phi_c(a,b_0)$ implies that the components of a are pairwise different, and that $\mathrm{tr}(a/b_0 a_s) \leq k_{c,s}$ and $\mathrm{tr}(a_s/b_0) \leq (k_c - k_{c,s})$ for all realizations a of $\phi(x,b_0)$.

Consider now the following property $\mathsf{E}(b,b')$:

- $\phi(x,b)$ implies that the components of x are pairwise disjoint.
- $\phi(x,b)$ has Morley rank k_c and degree 1.
- $\mathrm{tr}(a/ba_s) \leq k_{c,s}$ and $\mathrm{tr}(a_s/b) \leq (k_s - k_{c,s})$ for all realizations a of $\phi(x,b)$.
- $\phi(x,b) \sim^{k_c} \phi(x,b')$ implies that $b = b'$.

E holds for all b, b' realizing the type of b_0. Moreover, E is equivalent to an infinite disjunction of formulae $\epsilon(y,y')$. Therefore, there is some $\theta(y) \in \mathrm{tp}(b_0)$ such that $\models \theta(y) \wedge \theta(y') \to \mathsf{E}(y,y')$. Set

$$\phi_c(x,y) = \phi(x,y) \wedge \theta(y).$$

Let a be a generic realization of $\phi_c(x,b)$. Then $\mathrm{tr}(a/ba_s) = k_{c,s}$ follows from $\mathrm{tr}(a/b_a s) \leq k_{c,s}$, $\mathrm{tr}(a_s/b) \leq (k_c - k_{c,s})$ and $\mathrm{tr}(a/b) = k_c$. By simplicity of $\chi(x,d)$ we have $k_{c,\{i\}} < k_c$ for all i, which in turn implies that all non–empty $\phi_c(x,b)$ are simple. □

[2]We assume that $\phi_c(x,b)$ is non-empty for some b.

Let c be a code, $\phi_c(x, b)$ non-empty and $p \in \mathrm{S}(b)$ the type of rank k_c determined by $\phi(x, b)$. Hence, b lies in the definable closure of a sufficiently large segment of a Morley sequence of p. Let m_c be some upper bound for the length of such a segment.

Lemma 2.3. *For every code c and every integer $\mu \geq m_c - 1$ there exists some formula $\Psi_c(x_0, \ldots, x_\mu, y)$ without parameters satisfying the following:*

(v) Given a Morley sequence $e_0, \ldots, e_\mu$ of $\phi(x, b)$, then $\models \Psi_c(e_0, \ldots, e_\mu, b)$.

(vi) For all $e_0, \ldots, e_\mu, b$ realizing Ψ_c the e_i's are pairwise disjoint realizations of $\phi_c(x, b)$.

(vii) Let $e_0, \ldots, e_\mu, b$ realize Ψ_c. Then b lies in the definable closure of any m_c many e_i's.

We say that "$x_0, \ldots, x_\mu$ is a pseudo Morley sequence of c over y".

Proof. The statement "(e_i) is a Morley sequence of $\phi_c(x, b)$" can be described by a partial type $\mathsf{M}(e_0, \ldots, b)$. Likewise, properties (vi) and (vii) can be described by an infinite disjunction $\mathsf{D}(e_0, \ldots, b)$. Since non-empty $\phi_c(x, b)$ are simple, it follows that $\models \mathsf{M} \to \mathsf{D}$. Hence we may choose a sufficiently strong formula Ψ_c in M with the desired properties. □

Choose now for every code (and every μ)[3] a formula Ψ_c as above.

Let c be a code and σ some permutation of $\{1, \ldots, n_c\}$. Then c^σ defined by

$$\phi_{c^\sigma}(x^\sigma, y) = \phi_c(x, y)$$

is also a code. Similarly,

$$\Psi_{c^\sigma}(\bar{x}^\sigma, y) = \Psi_c(\bar{x}, y)$$

defines a pseudo Morley sequence of c^σ.

We consider two codes c and c' to be *equivalent* if $n_c = n_{c'}$, $m_c = m_{c'}$, and

- For every b there is some b' such that $\phi_c(x, b) \equiv \phi_{c'}(x, b')$ and $\Psi_c(\bar{x}, b) \equiv \Psi_{c'}(\bar{x}, b')$ in T.
- Similarly permuting c and c'.

Theorem 2.4. *There is a collection of codes C such that:*

(viii) Every simple formula can be encoded by exactly one $c \in C$.

(ix) For every $c \in C$ and every permutation σ, we have that c^σ is equivalent to a code in C.[4]

In [8] it was stated that one could find such a set C closed under permutations, which is stronger than (ix). This is not true.

[3] In the proof of 2.4 this choice may be modified.

[4] In fact, we find a collection C such that every c^σ is equivalent to some permutation of c which lies in C.

Proof. Work inside some countable ω–saturated model M of T and list all simple formulae χ_i, $i = 1, 2, \ldots$, with parameters in M. We need only show that every χ_i may be encoded by some $c \in C$. We build up C as the union of an increasing sequence $\emptyset = C_0 \subset C_1 \subset \cdots$ of finite sets. Suppose by induction on i that C_{i-1} has already been constructed and satisfies (ix). If χ_i may be encoded by some element in C_{i-1}, then set $C_i = C_{i+1}$. Otherwise, choose some code c and some b_0 with $\phi_c(x, b_0) \sim^{k_c} \chi_i$. Replace ϕ_c by

$$\phi_c(x, y) \wedge \text{“}\phi_c(x, y) \text{ cannot be encoded by any element of } C_{i-1}.\text{”}$$

We obtain a new code which still encodes χ_i. Therefore, we may assume that no permutation of c encodes a formula which may be also encoded by some element of C_{i-1}. Let G be now the group of all permutations $\sigma \in \mathrm{Sym}(n_c)$ with

$$\phi_c(x, b_0) \sim^{k_c} \phi_{c^\sigma}(x, b_0'))$$

for some realization b_0' of p, the type of b_0. It follows that b_0' is uniquely determined and hence given by a $\emptyset$-definable function of b_0. Write $b_0' = b_0^\sigma$.

After strengthening $\phi_c(x, y)$ with an appropriate subset[5] of p, we may assume that for all b with non-empty $\phi_c(x, b)$ and all σ there is a b^σ with $\phi_c(x, b) \sim^{k_c} \phi_{c^\sigma}(x, b^\sigma)$ iff $\sigma \in G$.

It is easy to see that

$$\phi_d(x, y) = \bigwedge_{\sigma \in G} \phi_{c^\sigma}(x, y^\sigma)$$

defines a code which still encodes χ_i. Likewise,

$$\Psi_d(\bar{x}, y) = \bigwedge_{\sigma \in G} \Psi_{c^\sigma}(\bar{x}, y^\sigma)$$

defines pseudo Morley sequences of d.

Moreover, $\phi_d(x, y) \equiv \phi_{d^\sigma}(x, y^\sigma)$ and $\Psi_d(\bar{x}, y) \equiv \Psi_{d^\sigma}(\bar{x}, y^\sigma)$ for all $\sigma \in G$. Hence d and d^σ are equivalent. Finally, choose representatives $\rho_1, \ldots, \rho_r$ of the right cosets of G in $\mathrm{Sym}(n_c)$ and set $C_i = C_{i-1} \cup \{d^{\rho_1}, \ldots, d^{\rho_r}\}$. □

3 The δ–function

From now on, T_1 and T_2 are two strongly minimal theories[6], formulated in two disjoint languages L_1 and L_2.

We assume the following

QE-Hypothesis. T_1 and T_2 have quantifier elimination, and L_1 and L_2 are pure relational languages.

[5] Choose $\rho'(y) \in \mathrm{tp}(b_0)$ such that $\models \neg\rho'(b_0^\sigma)$ for all $\sigma \notin G$. The aforementioned subset of p is

$$\rho(y) = \bigwedge_{\sigma \in G} \rho'(b^\sigma) \wedge \bigwedge_{\sigma \notin G} \neg\rho'(b^\sigma)$$

[6] Countability and DMP of T_i will not be used in this section.

by considering Morleyizations of T_1 and T_2. We may also assume that ϕ_c and Ψ_c are quantifier-free for the T_1–Codes and the T_2–Codes. Types $\mathrm{tp}_i(a/B)$ in each theory T_i are always quantifier-free types. These assumptions will be dropped in section 7.

Let $\mathcal{K}$ be the class of all models of $T_1^\forall \cup T_2^\forall$. We also allow $\emptyset$ to be in $\mathcal{K}$. If $\mathbb{C}_i$ is some monster model of T_i, we may see elements of $\mathcal{K}$ simultaneously as subsets of $\mathbb{C}_1$ and $\mathbb{C}_2$.

Given a finite $A \in \mathcal{K}$, define

$$\delta(A) = \mathrm{tr}_1(A) + \mathrm{tr}_2(A) - |A|.$$

The following hold

(1) $\delta(\emptyset) = 0$

(2) $\delta(\{a\}) \leq 1$

(3) $\delta(A \cup B) + \delta(A \cap B) \leq \delta(A) + \delta(B)$

If $A \setminus B$ is finite, we set

$$\delta(A/B) = \mathrm{tr}_1(A/B) + \mathrm{tr}_2(A/B) - |A \setminus B|.$$

For B finite, it follows that $\delta(A/B) = \delta(A \cup B) - \delta(B)$.

We say that B is *strong* in A if $B \subset A$ and $\delta(A'/B) \geq 0$ for all finite $A' \subset A$. Denote this by

$$B \leq A.$$

An element $a \in A$ is *algebraic* over $B \subset A$ if a is algebraic over B either in the sense of T_1 or in the sense of T_2. A is transcendental over B if no $a \in A \setminus B$ is algebraic over B.

$B \lneqq A$ is *minimal* if $B \leq A' \leq A$ for no A' properly contained between B and A.

Lemma 3.1. *A proper strong extension $B \leq A$ is minimal if and only if $\delta(A/A') < 0$ for all A' properly contained between B and A.*

Proof. One direction is clear, since $A' \leq A$ implies that $\delta(A/A') \geq 0$. On the other hand, if $\delta(A/A') \geq 0$ for some A', we may choose A' such that $\delta(A/A')$ is maximal. Hence $A' \leq A$, which implies that A is not minimal over B. □

Note that $A \setminus B$ is finite for minimal extensions.

Lemma 3.2. *Let $B \leq A$ be a minimal extension of elements in $\mathcal{K}$. Then one of the following hold:*

(I) $\delta(A/B) = 0$ *and* $A = B \cup \{a\}$ *for some element* $a \in A \setminus B$ *algebraic over* B. (algebraic minimal extension)

(II) $\delta(A/B) = 0$ *and* A *is transcendental over* B. (prealgebraic minimal extension)

(III) $\delta(A/B) = 1$ *and* $A = B \cup \{a\}$ *for some* a *in* A *transcendental over* B. (transcendental minimal extension)

Note that $|A \setminus B| \geq 2$ in the prealgebraic case.

Proof. If $A \setminus B$ contains some algebraic element a, then $\delta(a/B) = 0$. Hence $B \cup \{a\} = A$.

Otherwise, two cases apply: if $\delta(A/B) = 0$, then the extension is prealgebraic minimal. Otherwise, $\delta(A'/B) \geq 1$ for each $B \subsetneq A' \subset A$. Given any $a \in A \setminus B$, we have that $B \cup \{a\} \leq A$ and hence $B \cup \{a\} = A$.

$\square$

Define $K^0 \subset \mathcal{K}$ as the subclass

$$\mathcal{K}^0 = \{M \in \mathcal{K} \mid \emptyset \leq M\}.$$

It is easy to see that $\mathcal{K}^0$ may be described by a collection of universal $L_1 \cup L_2$–sentences. The following lemmas follow easily from (1), (2) and (3).

Lemma 3.3. *Let M in K^0 and A a finite subset of M. Set*

$$\mathrm{d}(A) = \min_{A \subset A' \subset M} \delta(A').$$

Then d *is the dimension function a pregeometry, i.e.* d *satisfies* (1), (2), (3) *and*

(4) $\mathrm{d}(A) \geq 0$

(5) $A \subset B \Rightarrow \mathrm{d}(A) \leq \mathrm{d}(B)$ $\square$

Lemma 3.4. *Let $M \in \mathcal{K}^0$ and A a finite subset of M. Take a minimal superset A' of A with $\delta(A') = \mathrm{d}(A)$. Then A' is the smallest strong subset* $\mathrm{cl}(A)$ *of M containing A, called the* closure *of A.* $\square$

4 Prealgebraic codes

From now on, T_1 and T_2 are two countable strongly minimal theories with DMP as in Theorem 1. The **QE-Hypothesis** from section 3 holds all throughout this and the next sections (5,6).

Fix for each T_i a collection C_i of codes as in Theorem 2.4. A *prealgebraic* code $c = (c_1, c_2)$ consists of a code $c_1 \in C_1$ and a code $c_2 \in C_2$ with the following properties:

- $n_c := n_{c_1} = n_{c_2} = k_{c_1} + k_{c_2}$
- For each proper non-empty subset s of $\{1, \ldots, n_c\}$,

$$k_{c_1,s} + k_{c_2,s} - (n_c - |s|) < 0$$

Set $m_c = \max(m_{c_1}, m_{c_2})$. Note that simplicity of $\phi_{c_i}(x, b)$ implies that

$$n_c \geq 2.$$

For each permutation σ, the code

$$c^\sigma = (c_1^\sigma, c_2^\sigma)$$

is also prealgebraic.

Some explanatory remarks: T_1^{eq} and T_2^{eq} share only their home sort. An element $b \in \mathrm{dcl}^{\mathrm{eq}}(B)$ is a pair $b = (b_1, b_2)$ with $b_i \in \mathrm{dcl}^{\mathrm{eq}}{}_i(B)$ for $i = 1, 2$. Likewise for $\mathrm{acl}^{\mathrm{eq}}(B)$. A *generic realization* of $\phi_c(x, b)$ (over B) is a generic realization of $\phi_{c_i}(x, b_i)$ (over B) in T_i for $i = 1, 2$. A *Morley sequence* of $\phi_c(x, b)$ is a Morley sequence both of $\phi_{c_1}(x, b_1)$ and $\phi_{c_2}(x, b_2)$. A *pseudo Morley sequence* of c over b is a realization of both $\Psi_{c_1}(\bar{x}, b_1)$ and $\Psi_{c_2}(\bar{x}, b_2)$. We say that M is *independent* from A over B if M is independent from A over B both in T_1 and T_2.

Lemma 4.1. *Let $B \leq B \cup \{a_1, \ldots, a_n\}$ be a prealgebraic minimal extension and $a = (a_1, \ldots, a_n)$. Then there is some prealgebraic code c and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$ such that a is a generic realization of $\phi_c(a, b)$.*

Proof. Fix $i \in \{1, 2\}$ and choose $d_i \in \mathrm{acl}^{\mathrm{eq}}{}_i(B)$ such that $\mathrm{tp}_i(a/Bd_i)$ is stationary, and $\chi_i(x, d_i) \in \mathrm{tp}_i(a/Bd_i)$ with Morley rank $\mathrm{MR}_i(a/Bd_i)$ and degree 1. Since A/B is transcendental, the formula $\chi_i(x, d_i)$ is simple. Choose some T_i–code $c_i \in C_i$ and some $b_i \in \mathrm{dcl}^{\mathrm{eq}}{}_i(d_i)$ with $\chi_i(x, d_i) \sim^{k_{c_i}} \phi_{c_i}(x, b_i)$. It follows from $\delta(A/B) = 0$ that $k_{c_1} + k_{c_2} = n$. Moreover, $k_{c_1,s} + k_{c_2,s} - (n - |s|) < 0$ holds by 3.1.

□

The following lemma is proved similarly.

Lemma 4.2. *Let $B \in \mathcal{K}$, c a prealgebraic code and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$. Take a generic realization $a = (a_1, \ldots, a_{n_c})$ of $\phi_c(x, b)$ over B. Then $B \cup \{a_1, \ldots, a_{n_c}\}$ is a prealgebraic minimal extension of B.* □

Note that the isomorphism type of a over B is uniquely determined.

Lemma 4.3. *Let $B \subset A$ in $\mathcal{K}$, c a prealgebraic Code, b in $\mathrm{acl}^{\mathrm{eq}}(B)$ and $a \in A$ a realization of $\phi_c(x, b)$ which does not lie completely in B. Then*

1. *$\delta(a/B) \leq 0$.*
2. *If $\delta(a/B) = 0$, then a is a generic realization of $\phi_c(x, b)$ over B.*

Proof. Let $s = \{i \mid a_i \in B\}$. Since a is not completely contained in B, then s is a proper subset of $\{1, \ldots, n_c\}$. Therefore

$$\delta(a/B) = \mathrm{tr}_1(a/B) + \mathrm{tr}_2(a/B) - (n - |s|) \leq k_{c_1,s} + k_{c_2,s} - (n - |s|)$$

If $s \neq \emptyset$, then the right-hand side is negative. If $s = \emptyset$, we have that

$$\delta(a/B) = \mathrm{tr}_1(a/B) + \mathrm{tr}_2(a/B) - n \leq k_{c_1} + k_{c_2} - n = 0$$

So, $\delta(a/B) = 0$ implies that $\mathrm{tr}_i(a/B) = k_{c_i}$. □

Lemma 4.4. *Let $M \leq N$ be a strong extension of structures in $\mathcal{K}$ and $e_0, \ldots, e_\mu$ a pseudo Morley sequence of c in N over b. Then one of the following hold:*

- $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$
- *At least $\mu - n_c m_c + 1$ many e_i's lie in $N \setminus M$.*

Proof. Permute the e^i's so that $e^0, \ldots, e^{r_0-1}$ are in M and $e^{r_1}, \ldots, e^{\mu(c)}$ lie in $M' \setminus M$. Hence $0 \leq r_0 \leq r_1 \leq \mu(c) + 1$. Possibly the e_i's do not form a pseudo Morley sequence of c after permutation, however they are still disjoint realizations of $\phi_c(x, b)$. Assume $b \notin \mathrm{dcl}^{\mathrm{eq}}(M)$. Then (vii) implies $r_0 < m_c$. We need only to show that $r_1 \leq m_c n_c$. Suppose that $m_c \leq r_1$.

Define $\delta(i) = \delta(e^i / M e^0 \cdots e^{i-1})$. For $i < r_1$ the following upper bound holds[7] $\delta(i) \leq (n_c - 1)$. If $m_c \leq i < r_1$, then $\delta(i) < 0$ since $b \in \mathrm{dcl}^{\mathrm{eq}}(M e^0 \cdots e^{i-1})$ by 4.3. Therefore

$$\begin{aligned} 0 \leq \delta(e^0 \cdots e^{r_1 - 1}/M) &= \sum_{i<r_1} \delta(i) = \sum_{i<m_c} \delta(i) + \sum_{m_c \leq i < r_1} \delta(i) \\ &\leq m_c(n_c - 1) - (r_1 - m_c). \end{aligned}$$

The above inequality proves the claim. □

5 The class $\mathcal{K}^\mu$

Let μ^* be a function that assigns to each prealgebraic code c some natural number $\mu^*(c)$. We suppose that

- $\mu^*(c) \geq m_c - 1$
- For all triples l, m, n with $m > 0$ there are only finitely many c's with $\mu^*(c) = l$, $m_c = m$ and $n_c = n$ for each $m > 0$ and n. (Such μ^* exist since there are only countably many codes.)
- $\mu^*(c) = \mu^*(d)$ if c is equivalent to some permutation of d[8].

Define
$$\mu(c) = m_c n_c + \mu^*(c).$$
Note that $\mu(c) \geq m_c$.

From now on, a *pseudo Morley sequence* denotes a pseudo Morley sequence of length $\mu(c) + 1$ for a prealgebraic code c. Given such a pseudo Morley sequence (e_i), so is every (e_i^σ) for every permutation σ by (ix).

The class $\mathcal{K}^\mu$ consists of the elements $M \in \mathcal{K}^0$ which do not contain any pseudo Morley sequence.

Lemma 5.1. *Let B be a finite strong subset of $M \in \mathcal{K}^\mu$ and B/A a prealgebraic minimal extension. Then there are only finitely many B–isomorphic copies of A in M.*

[7] Note that $\delta(A/B) \leq |A/B|$ in general.
[8] Note that each permutation is equivalent to at most one prealgebraic code.

Proof. Let $A = B \cup \{a\}$ for some tuple a and choose $d \in \mathrm{acl}^{\mathrm{eq}}(B)$ with $\mathrm{tp}_i(a/Bd_i)$ stationary. We need only show that $\mathrm{tp}_1(a/Bd_i) \cup \mathrm{tp}_2(a/Bd_i)$ has only finitely many realizations in M. Choose a prealgebraic code c by 4.1 and $b \in \mathrm{acl}^{\mathrm{eq}}(B)$ with $\models \phi_c(a, b)$. We show that $\phi_c(x, b)$ cannot be infinitely often realized in M. Otherwise, we obtain at least $(\mu(c)+1)$ many such realizations e_i with $e_i \notin B \cup \{e_0, \ldots, e_{i-1}\}$. It follows from 4.3 that the e_i's form a Morley sequence of $\phi_c(x, b)$ over B and hence a pseudo Morley sequence of c over b by (v), which contradicts that M is in $\mathcal{K}^\mu$. □

Corollary 5.2. *Let $B \leq M \in \mathcal{K}^\mu$, $B \subset A$ finite with $\delta(A/B) = 0$. Then there are only finitely many $B \leq A' \subset M$ which are B-isomorphic to A.*

Note that $A' \leq M$ automatically.

Proof. Decompose the extension A/B into a finite sequence of minimal ones. □

Corollary 5.3. *Let B be a finite subset of $M \in \mathcal{K}^\mu$. Then the d–*closure *of B:*

$$\mathrm{cl_d}(B) = \{x \in M \,|\, \mathrm{d}(Bx) = \mathrm{d}(B)\}$$

is countable.

Proof. Recall that $\mathrm{cl_d}(B)$ is the union of all finite $A' \subset M$ with $\mathrm{cl}(B) \subset A'$ and $\delta(A'/\mathrm{cl}(B)) = 0$. □

Lemma 5.4. *If $M \in \mathcal{K}^\mu$, $M \leq N$ and $|N \setminus M| = 1$, then N is also in $\mathcal{K}^\mu$.*

Proof. Let (e_i) be a pseudo Morley sequence of c over b in N. There is at most one e_i not in M. Now $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$ since $\mu(c) \geq m_c$. It follows that every e_i is either in M or in $N \setminus M$ by 4.3. The latter cannot hold, since $n_c \geq 2$. Hence (e_i) is completely contained in M. Contradiction. □

Theorem 5.5. *$\mathcal{K}^\mu$ (and hence the class of all finite structures in $\mathcal{K}^\mu$) has the amalgamation property with respect to strong embeddings.*

Proof. Let $B \leq M$ and $B \leq A$ be structures in $\mathcal{K}^\mu$. We need to find a strong extension $M' \in \mathcal{K}^\mu$ of M and some $B \leq A' \leq M'$ isomorphic to A over B. We may assume that both A/B and M/B are minimal. We will show that either a "free amalgam" M' of M and A over B is in $\mathcal{K}^\mu$ or that M and A are B–isomorphic.

Case 1: A/B is an algebraic minimal extension. Suppose that $A = B \cup \{a\}$ for some a algebraic over B in T_1 and transcendental over B in T_2. Two possible (non-exclusive) cases may arise.

Subcase 1.1: $\mathrm{tp}_1(a/B)$ is realized in M by some a'. Then a'/B is transcendental in T_2. Hence $B \cup \{a'\}$ is B–isomorphic to A and strong in M. By minimality, $M = B \cup \{a'\}$.

Subcase 1.2: There is some $a' \notin M$ realizing $\mathrm{tp}_1(a/B)$. Define $M' = M \cup \{a\}$ by letting a have the type of a' over M in the sense of T_1 and be transcendental over M in the sense of T_2. Then M' is a *free amalgam* of M and A over B, i.e. $M \cap A = B$ and M is independent from A over B. It is easy to see that

$M \leq M'$ and $A \leq M'$ for such amalgams. Now, $M' \in \mathcal{K}^\mu$ by 5.4.

Case 2: A/B is transcendental. Then there is a free amalgam M' of M and A as above. Suppose that M' is not in $\mathcal{K}^\mu$. Then M' contains a pseudo Morley sequence (e_i) of c over b. Apply Lemma 4.4 to the extension M'/M to obtain one of the following cases.

Subcase 2.1: $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$. Since M is in $\mathcal{K}^\mu$, not all members of the pseudo Morley sequence lie in M. Let $e_i \notin M$. By 4.3 e_i is a generic realization of $\phi_c(x,b)$ over M. Independence of M and e_i over B yields that b in $\mathrm{acl}^{\mathrm{eq}}(B)$ by 2.1. Since $B \in \mathcal{K}^\mu$, there is some e_j not completely contained in B. Again, e_j is a generic realization of $\phi_c(x,b)$ over B. It follows that $M = B \cup \{e_j\}$ and $A = B \cup \{e_i\}$ are isomorphic over B.

Subcase 2.2: More than $\mu^*(c)$ many e_i's lie in $M' \setminus M$. Since $\mu^*(c) + 1 \geq m_c$, we have that $b \in \mathrm{dcl}^{\mathrm{eq}}(A)$. Proceed now as in subcase 2.1. □

A structure $M \in \mathcal{K}^\mu$ is *rich* if for every finite $B \leq M$ and every finite $B \leq A \in \mathcal{K}^\mu$ there is some B-isomorphic copy of A in M. We will show in the next section that rich structures are models of $T_1 \cup T_2$.

Corollary 5.6. *There is a unique (up to isomorphism) countable rich structure K^μ. Any two rich structures are $(L_1 \cup L_2)_{\infty,\omega}$–equivalent.* □

6 The theory T^μ

Lemma 6.1. *Let $M \in \mathcal{K}^\mu$, $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$, $a \models \phi_c(x,b)$ generic over B and M' the prealgebraic minimal extension $M \cup \{a_1, \cdots a_{n_c}\}$. If M' is not in $\mathcal{K}^\mu$, then one of the following hold.*

(a) M' contains a pseudo Morley sequence of c over b, all whose elements but possibly one are contained in M

(b) M' contains a pseudo Morley sequence for some code c' with more than $\mu^*(c')$ many elements in $M' \setminus M$. Moreover, $m_{c'} > 0$.

Proof. Let (e'_i) be a pseudo Morley sequence of c' over b'. If (b) does not hold, it follows that $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$ by 4.4. There must be some e_i not completely contained in M, which is a B–generic realization of $\phi_{c'}(x,b')$ by 4.3. Minimality of M'/M yields that e_i is some permutation of a. After permutation of the pseudo Morley sequences, we may assume that $e_i = a$. Hence $\phi_{c'}(x,b') \sim^{k_c} \phi_c(x,b)$, so $c = c'$ and $b = b'$. □

Corollary 6.2.

1. *Let c be a prealgebraic code. The statement "M contains no pseudo Morley sequence for c" can be expressed by a universal $L_1 \cup L_2$–sentence.*

2. *Let c be as above, $M \in \mathcal{K}^\mu$ a model of $T_1 \cup T_2$. The statement "For no $b \in \mathrm{dcl}^{\mathrm{eq}}(M)$ and generic realization a of $\phi_c(x,b)$ is $M \cup \{a_1, \ldots, a_{n_c}\}$ in $\mathcal{K}^\mu$" can be expressed by an inductive $L_1 \cup L_2$–sentence.*

Proof. 1. Let $\Psi^i(\bar{x})$ be quantifier-free and T_i–equivalent to $\exists y\, \Psi_{c_i}(\bar{x}, y)$. Hence, the desired sentence is

$$\neg\exists\bar{x}\ (\Psi^1(\bar{x}) \wedge \Psi^2(\bar{x}))$$

2. Let $i \in \{1, 2\}$ and M be some elementary substructure of $\mathbb{C}_i$. Take $m \in M$ and $\phi(x, m)$ be some L_i–formula of rank k and degree 1. Pick some M–generic realization $a \in \mathbb{C}_i$ of $\phi(x, m)$. Then every quantifier-free property $\psi(a, m)$ of a, m is equivalent to some quantifier-free property $\psi^*(m)$ of m: Set

$$\psi^*(y) = \mathrm{MR}_x\big(\phi(x, y) \wedge \psi(x, y)\big) \doteq k.$$

The above shows that for all $M \in \mathcal{K}$ and for all M–generic realization a of $\phi_c(x, b)$ every $L_1 \cup L_2$–sentence over $M \cup \{a_1 \ldots, a_{n_c}\}$ can be transformed into one $L_1 \cup L_2$–sentence over M, b.

The claim follows now by 6.1, since only finitely many codes c' need to be considered in case (b), namely those with $m_{c'} > 0$ and

$$(\mu^*(c') + 1)n_{c'} \leq |M' \setminus M| = n_c.$$

□

Models M of the $L_1 \cup L_2$–theory T^μ will be described by the following properties. Lemmas 4.1 and 4.2 and the above show that the axioms can be first-order described.

Axioms of T^μ.

(a) $M \in \mathcal{K}^\mu$

(b) $T_1 \cup T_2$

(c) No prealgebraic minimal extension of M lies in $\mathcal{K}^\mu$.

It is easy to see that M is a model of (b) if and only if M is infinite and has no algebraic minimal extensions. Hence, M is a model of (b) and (c) if and only if M is infinite and has no minimal (or proper) extensions $M' \in \mathcal{K}^\mu$ with $\delta(M'/M) = 0$.

Theorem 6.3. *An $L_1 \cup L_2$ structure is rich if and only if it is an ω–saturated model of T^μ.*

Proof. Let $M \models T^\mu$ be ω–saturated. To show that M is rich, we need only consider a finite strong subset B of M and a minimal strong extension A of B in $\mathcal{K}^\mu$. We aim to show that M contains a B-isomorphic copy of A.

Case (I/II): A/B is algebraic or prealgebraic. We can amalgam M and A in $\mathcal{K}^\mu$, however M has no proper algebraic or prealgebraic extensions in $\mathcal{K}^\mu$. Therefore there must be a B–copy of A in M.

Case (III): $A = B \cup \{a\}$ is transcendental. We want some $a' \in M$ transcendental over B with $B \cup \{a'\} \leq M$. By saturation of M (note that a' satisfies a given partial type over B), it suffices to find a' in some elementary extension M' of

M. If M' is uncountable, there is some $a' \in M' \setminus \mathrm{cl_d}(B)$ by 5.3. Equivalently, $B \cup \{a'\} \leq M'$.

Let M be now a rich structure.

Axiom (b): Let a be some element in $\mathrm{acl}_1(M)$ transcendental over M in T_2. There is some finite subset B of M witnessing 1–algebraicity of a. We may assume that $B \leq M$. By lemma 5.4, $B \leq B \cup \{a\} \in \mathcal{K}^\mu$, so there exists some copy of a over B in M. It follows that M is acl_1–closed. Since M is infinite[9] it is a model of T_1. Likewise for T_2.

Axiom (c): Let a be a generic realization of $\phi_c(x, b)$ over M such that $M \cup \{a\}$ is in $\mathcal{K}^\mu$. Choose some finite strong subset C of M with $b \in \mathrm{dcl}^{\mathrm{eq}}(C)$. Then $C \leq C \cup \{a\}$, so there is a copy a' of a in M over C with $C' = C \cup \{a'\} \leq M$ by richness of M. Iterate to obtain a sufficiently large Morley sequence $a', a'', \ldots$ of $\phi_c(x, b)$ in M. This contradicts that $M \in \mathcal{K}^\mu$.

Choose now some ω–saturated $M' \equiv M$. The first part of the proof yields that M' is rich. So $M' \equiv_{\infty,\omega} M$. So M is also ω–saturated. □

7 Proof of the Main Theorem

In this section we drop the QE-Hypothesis of section 3. Hence in our class $\mathcal{K}$ we replace isomorphic embeddings by *bi-elementary maps*, i.e. maps which are both T_1 and T_2 elementary.

Corollary 7.1. *T^μ is complete. Two tuples a and a' in models M and M' have the same type if and only if there is some bi-elementary bijection*

$$f : \mathrm{cl}(a) \to \mathrm{cl}(a')$$

with $f(a) = a'$.

Proof. The structure K^μ is a model of T^μ, so T^μ is consistent. Let M be any model of T^μ. By 6.3 there is some rich $M' \equiv M$. Since $M' \equiv_{\infty,\omega} K^\mu$, we have that T^μ is complete.

Let $M \prec N$ and $M' \prec N'$ be two ω–saturated elementary extensions. It is easy to see[10] that $M \leq N$ and $M' \leq N'$, i.e. closure does not change. An isomorphism $f : \mathrm{cl}(a) \to \mathrm{cl}(a')$ belongs to some back-and-forth system of partial isomorphisms between finite strong subsets of M' and N'. Hence f is elementary.

For the other direction, suppose that a and a' have the same type. Then there is some isomorphic embedding $f : \mathrm{cl}(a) \to M'$ mapping a to a'. Write $A' = f(\mathrm{cl}(a))$. Then $\mathrm{d}(a) = \delta(\mathrm{cl}(a)) = \delta(A')$. Therefore $\mathrm{d}(a') \leq \mathrm{d}(a)$ and by symmetry $\mathrm{d}(a') = \mathrm{d}(a)$. Note that A' has no proper subset A'' containing a' with $\delta(A'') = \mathrm{d}(a')$ since $\mathrm{cl}(a)$ does not. Hence $A' = \mathrm{cl}(a')$. □

[9] This follows also from 5.4.

[10] If $M \not\leq N$, there is some $a \in N$ with $\delta(a/M) < 0$. Find some finite $B \leq M$ with $\delta(a/B) < 0$. a realizes some $L_1 \cup L_2$–formula witnessing this fact. However $\phi(x, b)$ is not realized in M, so $M \not\prec N$.

Theorem 7.2. *T^μ is strongly minimal and* d *is the dimension function of the natural pregeometry on models of T^μ. In particular*

$$\mathrm{MR}(\bar{a}/B) = \mathrm{d}(\bar{a}/B)$$

.

Proof. All types $\mathrm{tp}(a/B)$ with $\mathrm{d}(a/B) = 0$ are algebraic by 5.2. It follows from 7.1 that there is only one type with $\mathrm{d}(a/B) = 1$[11]. Therefore T is strongly minimal. The rest follows easily since d describes the algebraic closure. $\square$

The above proves Theorem 1.

8 Remarks

It is easy to prove that T^μ has the following properties:

- T^μ has the DMP.
- For each $i = 1, 2$ every L_i–formula $\phi(x, b)$ preserves its Morley rank and degree from T_i in T^μ.
- ([7]) Let M be a model of T^μ which is an elementary substructure of N according to both T_1 and T_2. Then M is an elementary substructure of N.

We prove the last property. Let M and N be models of T^μ with $M \restriction L_i \prec N \restriction L_i$ for $i = 1, 2$. We show that $M \leq N$. It follows then from 7.1 that $N \prec M$.

Recall that M has no extension M' in N with $\delta(M'/M) = 0$. If M is not strong, there is some $a \in N$ with $\delta(a/M) = -1$ which is algebraic over M in both T_1 *and* T_2. Contradiction.

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