Countable imaginary simple unidimensional theories are supersimple

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Abstract

We prove that a countable simple unidimensional theory that eliminates hyperimaginaries is supersimple. This solves a problem of Shelah in the more general context of simple theories under weak assumptions.

1 Introduction

The notion of a unidimensional theory already appeared (in a different form) in Baldwin-Lachlan characterization of $\aleph_1$-categorical theories; a countable theory is $\aleph_1$-categorical iff it is ω-stable and unidimensional (equivalently, T has no Vaught pairs and is ω-stable). Later, Shelah defined a unidimensional theory to be a stable theory T in which every two $|T|^+$-staurated models of the same power are isomorphic, and proved that in the stable context a theory is unidimensional iff every two non-algebraic types are non-orthogonal. A problem posed by Shelah was whether any unidimensional stable theory is superstable. This was answered positively by Hrushovski around 1986 first in the countable case [H0] and then in full generality [H1]. Taking the right hand side of Shelah characterization of unidimensional stable theories seems natural for the simple case. Shelah's problem extended to this context seems much harder. In [S3] it was observed that a small simple unidimensional theory is supersimple. Later, Pillay [P] gave a positive answer for countable imaginary simple theories with wnfcp (the weak non finite cover property) building on the arguments in [H0] and using some machinary from [BPV]. Then using the result on elimination of $\exists^\infty$ in simple unidimensional theories [S1] completed his proof for countable imaginary low theories. In this paper we prove the result for any countable imaginary simple theory. In general, the main idea is the dividing line "T is essentially 1-based" which means that every type is coordinatized by essentially 1-based types in the sense of the forking topology (the τ^f-topology). In case T is not essentially 1-based we prove there is an unbounded τ^f-open set of finite SU-rank (possibly with no finite bound); this is a general dichotomy for countable imaginary simple theories. Then one applies the result in [S2] and a property (PCFT) proved in the current paper to conclude T is supersimple. If T is essentially 1-based the proof first reduces the problem to finding a type-definable τ^f-open set of bounded finite SU_s-rank (the foundation rank with respect to forking with stable formulas). In order to show the existence of such sets, we introduce the notion of a $\tilde{\tau}^f_{st}$-set and prove a theorem saying that in any simple theory in which the extension property is first-order any unbounded $\tilde{\tau}^f_{st}$-set contains an unbounded type-definable τ^f-open set (over parameters). Then, in the proof of the main result we show how unidimensionality of T implies there is a $\tilde{\tau}^f_{st}$-set that by a more precise version of the theorem is in fact a type-definable τ^f-open set over some finite set and is of bounded finite SU_s-rank.

2 Preliminaries

We will assume basic knowledge of simple theories as in [K1],[KP],[HKP]. A good text book on simple theories that covers much more is [W]. We recall here some other definitions and facts relevant for this paper. In this section T will be a simple theory and we work in $\mathcal{C}^{eq}$, the monster model of T^{eq}.

2.1 Interaction

For the rest of this section let $\mathcal{P}$ be an A-invariant set of small types and $p \in S(A)$. We say that *p is (almost-) $\mathcal{P}$-internal* if there exists a realization a of p and there exists $B \supseteq A$ with $a \underset{A}{\downarrow} B$ such that for some tuple c of realizations of types in $\mathcal{P}$ over B we have $a \in dcl(B, c)$ (respectively, $a \in acl(B, c)$). We say that *p is analyzable in $\mathcal{P}$* if there exists a sequence

$I = \langle a_i | i \leq \alpha \rangle \subseteq dcl^{eq}(a_\alpha A)$ where $a_\alpha \models p$ and $tp(a_i/A \cup \{a_j | j < i\})$ is $\mathcal{P}$-internal for every $i \leq \alpha$. We say that p is *foreign* $\mathcal{P}$ if for every $B \supseteq A$ and $a \models p$ with $a \underset{A}{\downarrow} B$ and a realization c of a type in $\mathcal{P}$ over B, $a \underset{B}{\downarrow} c$. Also, recall that $p \in S(A)$ is said to be *orthogonal* to some $q \in S(B)$ if for every $C \supseteq A \cup B$, for every $\bar{p} \in S(C)$, a non-forking extension of p , and every $\bar{q} \in S(C)$, a non-forking extension of q, for every realization a of $\bar{p}$ and realization b of $\bar{q}$, $a \underset{C}{\downarrow} b$.

We say that T is *imaginary (or has elimination of hyperimaginaries)* if for every type-definable over $\emptyset$ equivalence relation E on a complete type p (of a possibly infinite tuple of elements), E is equivalent on p to the intersection of some definable equivalence relations $E_i \in L$.

Fact 2.1 *Let T be an imaginary simple theory. Then*
1)If p is not foreign to $\mathcal{P}$, then for $a \models p$ there exists $a' \in dcl(Aa) \backslash acl(A)$ such that $tp(a'/A)$ is $\mathcal{P}$-internal.
2) p is analyzable in $\mathcal{P}$ iff every non-algebraic extension of p is non-foreign to $\mathcal{P}$.

An easy fact we will be using is the following.

Fact 2.2 *Assume $tp(a_i)$ are $\mathcal{P}$-internal for $i < \alpha$. Then $tp(\langle a_i | i < \alpha \rangle)$ is $\mathcal{P}$-internal.*

An important characterization of almost-internality is the following fact [S0, Theorem 5.6.] (a similar result obtained independently in [W, Proposition 3.4.9]).

Fact 2.3 *Let $p \in S(A)$ be an amalgamation base and let $\mathcal{U}$ be an $\emptyset$-invariant set. Suppose p is almost-$\mathcal{U}$-internal. Then there is a Morley sequence $\bar{a}$ in p and there is a definable relation $R(x, \bar{y}, \bar{a})$ (over $\bar{a}$ only) such that, for every tuple $\bar{c}$, $R(\mathcal{C}, \bar{c}, \bar{a})$ is finite and for every a' realizing p, there is some tuple $\bar{c}$ from $\mathcal{U}$ such that $R(a', \bar{c}, \bar{a})$ holds.*

T is said to be *unidimensional* if whenever p and q are complete non-algebraic types, p and q are non-orthogonal. From Fact 2.3 and Fact 2.1 it is easy to deduce the following (using compactness).

Fact 2.4 *Let T be a simple theory. Let $p \in S(\emptyset)$ and let $\theta \in L$. Assume p is analyzable in $\theta^{\mathcal{C}}$. Then p is analyzable in $\theta^{\mathcal{C}}$ in finitely many steps. In particular, if T is an imaginary simple unidimensional theory and there exists a non-algebraic supersimple definable set (i.e. a definable set with global D-rank), then T has finite SU-rank i.e., every complete type has finite SU-rank (in fact, for every given sort there is a finite bound on the SU-rank of all types in that sort).*

2.2 The forking topology

Definition 2.5 *Let $A \subseteq \mathcal{C}$. An invariant set $\mathcal{U}$ over A is said to be* a basic τ^f-open set over A *if there is $\phi(x, y) \in L(A)$ such that*

$$\mathcal{U} = \{a | \phi(a, y) \text{ forks over } A\}.$$

Note that the family of basic τ^f-open sets over A is closed under finite intersections, thus form a basis for a unique topology on $S_x(A)$.

Definition 2.6 We say that *the τ^f-topologies over A are closed under projections (T is PCFT over A)* if for every τ^f-open set $\mathcal{U}(x, y)$ over A the set $\exists y \mathcal{U}(x, y)$ is a τ^f-open set over A. We say that *the τ^f-topologies are closed under projections (T is PCFT)* if they are over every set A.

We will make an essential use of the following facts from [S2].

Fact 2.7 *Let $\mathcal{U}$ be a τ^f-open set over a set A and let $B \supseteq A$ be any set. Then $\mathcal{U}$ is τ^f-open over B.*

Fact 2.8 *Let $\mathcal{U}$ be an unbounded τ^f-open set over some set A. Assume $\mathcal{U}$ has bounded finite SU-rank. Then there exists a set $B \supseteq A$ and $\theta(x) \in L(B)$ of SU-rank 1 such that $\theta^{\mathcal{C}} \subseteq \mathcal{U} \cup acl(B)$.*

The following theorem ([S2], Theorem 3.11]) generalizes Fact 2.4 but at the price of PCFT.

Fact 2.9 *Assume T is a simple theory with PCFT. Let $p \in S(A)$ and let $\mathcal{U}$ be a τ^f-open set over A. Suppose p is analyzable in $\mathcal{U}$. Then p is analyzable in $\mathcal{U}$ in finitely many steps.*

3 Unidimensionality and PCFT

We show that if T is any simple theory in which the extension property is first-order (more generally, if T is semi-PCFT) then T is PCFT. Recall that the extension property is first-order for T iff for every formulas $\phi(x,y), \psi(y,z) \in L$ the relations $Q_{\phi,\psi}$ defined by: for all a, $Q_{\phi,\psi}(a)$ iff "$\phi(x,b)$ does not divide over a for every $b \models \psi(y,a)$" are type-definable (see Definition 3.9, part 4). In this section T is assumed to be simple.

Definition 3.1 *We say that T is semi-PCFT over A if for every formula $\psi(x,yz) \in L(A)$ the set $\{a|\ \psi(x,ab)$ forks over Aa for some $b\}$ is τ^f-open over A.*

Lemma 3.2 *If the extension property is first-order then the extension property is first-order over every set A. Thus if the extension property is first-order, then T is semi-PCFT over every set A.*

Proof: First, note that the type-definability of the $Q_{\phi,\psi}$-s implies that the κ-extension property is first-order for every $\kappa \geq |T|^+$. Now, if the κ-extension property is first-order for every $\kappa \geq |T|^+$ then, given a $(|T|+|A|)^+$-saturated model $M^* = (M_A, P(M_A))$ of the common theory of all pairs of $T_A (= Th(\mathcal{C}, a)_{a\in A})$ satisfying the extension property, $M^*|L_P$ is also a model of the common theory of all pairs of T satisfying the $(|T|+|A|)^+$-extension property (indeed, every pair of T satisfying the extension property has an elementary extension which is sufficiently saturated and thus has an expansion to a pair of models of T_A). Thus, since we assume the κ-extension property is first-order for every $\kappa \geq |T|^+$, $M^*|L_P$ has the $(|T|+|A|)^+$-extension property, and thus M^* has the extension property.

Lemma 3.3 *Assume T is semi-PCFT over A. Then T is PCFT over A.*

Proof: We may clearly assume $A = \emptyset$. Let $\psi(x,yz) \in L$. We need to show that Γ, defined by $\Gamma(a)$ iff $\forall b(\psi(x,ab)$ dnf over $\emptyset)$ is a τ^f-closed set. Let Γ^* be defined by: for all a:

$$\Gamma^*(a) \ \text{iff} \bigwedge_{\phi(x,y)\in L} [FK_\phi(a) \rightarrow \forall b(\psi(x,ab) \wedge \neg\phi(x,a) \text{ dnf over a})].$$

To finish it is sufficient to prove:

Subclaim 3.4 *Γ^* is τ^f-closed and $\Gamma = \Gamma^*$.*

Proof: First, by our assumption Γ^* is τ^f-closed. To prove the second part, first assume $\Gamma(a)$. Then for any b there is c such that $c \underset{}{\perp} ab$ and $\psi(c, ab)$. Thus $\Gamma^*(a)$. Assume now $\Gamma^*(a)$. Let $p^{ind}_a(x) = \bigwedge\{\neg\phi(x,a)|\ \phi(x,y) \in L,\ FK_\phi(a)\}$. Let b be arbitrary and let $q(x) = p^{ind}_a(x) \wedge \psi(x, ab)$. It is enough to show that $q(x)$ doesn't fork over a (since any realization of q is independent of a). Indeed, by $\Gamma^*(a)$, every finite subset of $q(x)$ doesn't fork over a, so we are done.

Corollary 3.5 *Suppose the extension property is first-order in T. Then T is PCFT.*

Now we conclude that every unidimensional simple theory is PCFT. Recall the following two facts and their corollary. First, let $\mathcal{P}^{SU\leq 1}$ denote the class of complete types over sets of size $\leq |T|$, of SU-rank ≤ 1.

Fact 3.6 *(Pillay) Let T be a simple theory that eliminates $\exists^\infty$. Moreover, assume every non-algebraic type is non-foreign to $\mathcal{P}^{SU\leq 1}$. Then the extension property is first-order in T.*

Fact 3.7 [$S1$] *Let T be any unidimensional simple theory. Then T eliminates $\exists^\infty$.*

Corollary 3.8 *In any unidimensional simple theory the extension property is first-order.*

We give now an easy generalization of Fact 3.6. Let us first fix some notations and definitions. By a pair (M, P^M) we mean an elementary pair of models of T, where P is a new predicate symbol. For the rest of this section, by a small type we mean a complete hyperimaginary type in $\leq |T|$ variables over a hyperimaginary of length $\leq |T|$.

Definition 3.9 *Let $\mathcal{P}_0, \mathcal{P}_1$ be $\emptyset$-invariant families of small types.*
1) We say that a pair (M, P^M) satisfies the extension property for $\mathcal{P}_0$ if for every L-type $p \in S(A)$, $A \in dcl(M)$ with $p \in \mathcal{P}_0$ there is $a \in p^M$ such that $a \underset{A}{\perp} P^M$.

2) Let

$$T_{Ext,\mathcal{P}_0} = \bigcap \{Th_{L_P}(M, P^M) |\ \textit{the pair } (M, P^M) \textit{ satisfies the extension property w.r.t. } \mathcal{P}_0 \}.$$

3) We say that $\mathcal{P}_0$ dominates $\mathcal{P}_1$ w.r.t. the extension property if (M, P^M) satisfies the extension property for $\mathcal{P}_1$ for every $|T|^+$-saturated pair $(M, P^M) \models T_{Ext,\mathcal{P}_0}$. In this case we write $\mathcal{P}_0 \trianglerighteq_{Ext} \mathcal{P}_1$.
4) We say that the extension property is first-order for $\mathcal{P}_0$ if $\mathcal{P}_0 \trianglerighteq_{Ext} \mathcal{P}_0$. The extension property is first-order if the extension property is first-order for the family of all small types.

For an $\emptyset$-invariant family $\mathcal{P}_0$ of small types we say that $\mathcal{P}_0$ is extension-closed if for all $p \in \mathcal{P}_0$ if $\bar{p}$ is any extension of p to a small type, then $\bar{p} \in \mathcal{P}_0$.

Lemma 3.10 *Let $\mathcal{P}_0$ be an $\emptyset$-invariant family of small types. Assume $\mathcal{P}_0$ is extension-closed and that the extension property is first-order for $\mathcal{P}_0$. Let $\mathcal{P}^*$ be the maximal class of small types such that $\mathcal{P}_0 \trianglerighteq_{Ext} \mathcal{P}^*$. Then $\mathcal{P}^* \supseteq An(\mathcal{P}_0)$ (where $An(\mathcal{P}_0)$ is the class of all small types analyzable in $\mathcal{P}_0$).*

Proof: Note that if the pair (M, P^M) satisfies the extension property for the family of $\emptyset$-conjugates of $tp(b/A)$ and for the family of $\emptyset$-conjugates of $tp(a/bA)$ then (M, P^M) satisfies the extension property for the family of $\emptyset$-conjugates of $tp(ab/A)$. Thus, since $\mathcal{P}_0$ is extension-closed (and the extension property is first-order for $\mathcal{P}_0$) we conclude that if B is any small hyperimaginary and $a_0, a_1, ..., a_n$ are realizations of some types from $\mathcal{P}_0$ over B, then if (M, P^M) is a $|T|^+$-saturated pair and $(M, P^M) \models T_{Ext,\mathcal{P}_0}$ then (M, P^M) satisfies the extension property for the family of $\emptyset$-conjugates of $tp(a_0a_1...a_n/B)$. Therefore, $\mathcal{P}^* \supseteq Int(\mathcal{P}_0)$, where $Int(\mathcal{P}_0)$ denotes the family of small types internal in $\mathcal{P}_0$. By induction we conclude that $\mathcal{P}^* \supseteq An(\mathcal{P}_0)$.

Remark 3.11 *Note that if T eliminates $\exists^\infty$ then the extension property is first-order for $\mathcal{P}^{SU \leq 1}$ (this was proved in [V, Proposition 2.15]). Thus Lemma 3.10 implies Fact 3.6.*

We conclude:

Theorem 3.12 *Let T be any unidimenisonal simple theory. Then T is PCFT.*

Corollary 3.13 *Let T be an imaginary simple unidimensional theory. Let $p \in S(A)$ and let $\mathcal{U}$ be an unbounded τ^f-open set over A. Then p is analyzable in $\mathcal{U}$ in finitely many steps. In particular, for such T the existence of an unbounded supersimple τ^f-open set over some set A implies T is supersimple.*

Proof: By Theorem 3.12 every unidimensional theory is $PCFT$. Thus by Fact 2.9 and the assumption that T is imaginary and unidimensional, if $\mathcal{U}$ is an unbounded τ^f-open set over A, then $tp(a/A)$ is analyzable in $\mathcal{U}$ in finitely many steps for every $a \in \mathcal{C}$. Thus, if $\mathcal{U}$ is supersimple, $SU(a/A) < \omega$ for all $a \in \mathcal{C}$. Thus T is supersimple.

4 Definability of being in the canonical base

In this section we show that in suitable setting the relation R defined by $R(e, a)$ iff $e \in acl(Cb(C/a))$ is Stone open over C (for a fixed set C).

Definition 4.1 *Let C be any set. We say that a set $\mathcal{U}$ is a basic τ^f_*-open set over C if there exists $\psi(x, y, C) \in L(C)$ such that $\mathcal{U} = \{a \mid \psi(x, aC)$ forks over $a\}$.*

First, we note the following claim:

Claim 4.2 *For every e, C, a, we have $e \in acl(Cb(C/a))$ iff for every Morley sequence $(C_i|i < \omega)$ of $Lstp(C/a)$ we have $e \in acl(C_i|i < \omega)$.*

Proof: Right to left follows form the well known fact that $Cb(C/a) \in dcl(C_i|i < \omega)$ for every Morley sequence $(C_i|i < \omega)$ of $Lstp(C/a)$. For the other direction, assume the right hand side. Let $(C_i|i < \omega \cdot 2)$ be a Morley sequence of $Lstp(C/a)$. Let $e^* = Cb(C/a)$. Then $e^* \in bdd(a)$ and thus clearly $(C_i|i < \omega \cdot 2)$ is a Morley sequnce of $Lstp(C/e^*)$. In particular, $(C_i|i < \omega)$ is independent from $(C_i|\omega \leq i < \omega \cdot 2)$ over e^*. By our assumption, $e \in acl(C_i|i < \omega)$ and $e \in acl(C_i|\omega \leq i < \omega \cdot 2)$. Thus $e \in acl(e^*)$.

Lemma 4.3 *Let C be any set and let (in a given sort) $\mathcal{W} = \{(e, a) \mid e \in acl(Cb(C/a))\}$. Then $\mathcal{W}$ is a τ^f_*-open set over C.*

Proof: First note that since T is simple, for any two sorts, if x, x' has the first sort, and y has the second sort, there exists a type-definable relation

$E_L(x, x', y)$ such that for all a, a', b with the right sorts we have $E_L(a, a', b)$ iff $Lstp(a/b) = Lstp(a'/b)$. By Claim 4.2, $(e, a) \notin \mathcal{W}$ iff there exists an a-indiscernible sequence $(C_i|i < \omega)$ which is independent over a with $E_L(C_0, C, a)$ such that $e \notin acl(C_i|i < \omega)$. Let $\mathcal{F}$ be the collection of all finite sets of formulas of the form $\psi(Y_0, ..., Y_i, y) \in L$ where each of $\{Y_i\}_{i<\omega}$ have the sort of C and y has the sort of a. For $\Delta \in \mathcal{F}$ let $n(\Delta)$ be the maximal natural number n such that Y_n appears in one of the ψ-s in Δ. Also, for $\psi(Y_0, ..., Y_i, y) \in L$ as above let $\psi_C = \psi(Y_0, ..., Y_{i-1}, C, y)$ and let $i(\psi) = i$. For a formula $\phi(x, y, C) \in L(C)$, let FK^*_ϕ be the relation over C defined by $FK^*_\phi(a)$ iff $\phi(x, a, C)$ forks over a. Thus, by compactness $(e, a) \notin \mathcal{W}$ iff

$$\bigwedge_{\Delta\in\mathcal{F}} [(\bigwedge_{\psi\in\Delta} FK^*_{\psi_C}(a)) \to \exists Y_0, ...Y_{n(\Delta)} \Theta(e, a, Y_0, Y_1, ..., Y_{n(\Delta)}, C)].$$

where Θ is the partial type over C defined by $\Theta(e, a, Y_0, Y_1, ..., Y_{n(\Delta)}, C) =$

$$E_L(Y_0, C, a) \wedge I(Y_0, ...Y_{n(\Delta)}, a) \wedge (\bigwedge_{\psi\in\Delta} \neg\psi(Y_0, Y_1, ..., Y_{i(\psi)}, a)) \wedge e \notin acl(Y_0, Y_1, ..., Y_{n(\Delta)}).$$

and where $I(Y_0, ...Y_{n(\Delta)}, a)$ is the partial type saying $Y_0, , , , Y_{n(\Delta)}$ is a-indiscernible. Note that the complement of $\mathcal{W}$ is an intersection of τ^f_*-closed sets over C (note that clearly every Stone-closed set over C is τ^f_*-closed over C.)

Proposition 4.4 *Let $q(x, y) \in S(\emptyset)$ and let $\chi(x, y, z) \in L$ be such that $\models \forall y \forall z \exists^{<\infty} x \chi(x, y, z)$. Then $\mathcal{U} = \{(e, c, b, a)|\ e \in acl(Cb(cb/a))\}$ is relatively Stone-open inside the Stone-closed set*

$$F = \{(e, c, b, a)|\ \ b \mathop{\smile\!\!\!\!\!\!|}\ a\ , \models \chi(c, b, a), tp(cb) = q\}.$$

Proof: Note that since $q \in S(\emptyset)$, it is enough to show that for any fixed $c^*b^* \models q$ the set $\mathcal{U}^* = \{(e, a)|\ e \in acl(Cb(c^*b^*/a))\}$ is relatively stone-open inside

$$F^* = \{(e, a)|\ \ b^* \mathop{\smile\!\!\!\!\!\!|}\ a\ , \models \chi(c^*, b^*, a)\}.$$

Now, by Lemma 4.3 we know $\mathcal{U}^*$ is a τ^f_*-open set over b^*c^*. Thus, for some $\psi_i(z; x, y, c^*b^*) \in L(c^*b^*)$ $(i \in I)$ we have $\mathcal{U}^* = \bigcup_i \mathcal{U}^*_{\psi_i}$ where $\mathcal{U}^*_{\psi_i} = \{(e, a)|\ \psi_i(z; e, a, c^*b^*)$ forks over $ea\}$.

Subclaim 4.5 *For every $(e, a) \in F^*$ we have $(e, a) \in \mathcal{U}^*_{\psi_i}$ iff*

$$\forall d(\psi_i(d; e, a, c^*b^*) \to\ da \not\mathop{\smile\!\!\!\!\!\!|}\ b^*\ ) \wedge e \in acl(a).$$

Proof: Assuming the left hand side we know $e \in acl(Cb(c^*b^*/a))$, hence $e \in acl(a)$. Let $d \models \psi_i(z; e, a, c^*b^*)$. If $da \underset{}{\smile\!\!\!\!\!\!|}\ b^*$, then $d \underset{a}{\smile\!\!\!\!\!\!|}\ b^*$. Since $(e, a) \in F^*$, $c^* \in acl(b^*a)$ implies $d \underset{ea}{\smile\!\!\!\!\!\!|}\ b^*c^*$, contradicting $(e, a) \in \mathcal{U}^*_{\psi_i}$. Assume now the right hand side. Let $d \models \psi_i(z; e, a, c^*b^*)$. Assume by a way of contradiction $d \underset{ea}{\smile\!\!\!\!\!\!|}\ b^*c^*$. Since $e \in acl(a)$, this equivalent to $d \underset{a}{\smile\!\!\!\!\!\!|}\ b^*c^*$. Since $(e, a) \in F^*$ this is equivalent to $da \smile\!\!\!\!\!\!|\ b^*$, contradiction.

By Subclaim 4.5 we see that each of $\mathcal{U}^*_{\psi_i}$ and hence $\mathcal{U}^*$ is Stone-open relatively inside F^*.

5 A dichotomy for projection closed topologies

We consider a general family of topologies on the Stone spaces $S_x(A)$ that refines the Stone topology and is closed under projections. For any such family of topologies we will prove a dichotomy saying that either there exists an unbounded invariant set $\mathcal{U}$ that is open in this topology and is supersimple OR for any SU-rank 1 type p_0 every type analyzable in p_0 is analyzable in p_0 by essentially 1-based types by mean of our family of topologies. In this section T is assumed to be a simple theory with elimination of hyperimaginaries

Definition 5.1 A family

$$\Upsilon = \{\Upsilon_{x,A} |\ x \text{ is a finite sequence of variables and } A \subset \mathcal{C} \text{ is small}\}$$

is said to be *a projection closed family of topologies* if each $\Upsilon_{x,A}$ is a topology on $S_x(A)$ that refines the Stone-topology on $S_x(A)$, this family is invariant under automorphisms of $\mathcal{C}$ and change of variables by variables of the same sort, and the family is closed under product by the full Stone space $S_y(A)$ (where y is a disjoint tuple of variables) and closed by projections, namely whenever $\mathcal{U}(x, y) \in \Upsilon_{xy,A}$, $\exists y \mathcal{U}(x, y) \in \Upsilon_{x,A}$.

There are two natural examples of projections-closed families of topologies; the Stone topology and the τ^f-topology of a PCFT theory. From now on fix a projection closed family Υ of topologies.

Definition 5.2 1) A type $p \in S(A)$ is said to be *essentially 1-based over $A_0 \subseteq A$, by mean of* Υ if for every finite tuple $\bar{c}$ from p and for every type-definable Υ-open set $\mathcal{U}$ over $A\bar{c}$, with the property that a is independent from A over A_0 for every $a \in \mathcal{U}$, the set $\{a \in \mathcal{U} \mid Cb(a/A\bar{c}) \notin bdd(aA_0)\}$ is nowhere dense in the Stone-topology of $\mathcal{U}$. We say $p \in S(A)$ is *essentially 1-based by mean of* Υ if p is essentially 1-based over $\emptyset$ by mean of Υ.
2) Let V be an A_0-invariant set and let $p \in S(A_0)$. We say that p is *analyzable in V by essentially 1-based types by mean of* Υ if there exists $a \models p$ and there exists a sequence $(a_i \mid i \leq \alpha) \subseteq dcl^{eq}(A_0 a)$ with $a_\alpha = a$ such that $tp(a_i/A_0 \cup \{a_j \mid j < i\})$ is V-internal and essentially 1-based over A_0 by mean of Υ for all $i \leq \alpha$.

Example 5.3 *The unique non-algebraic 1-type over $\emptyset$ in ACF_p is not essentially 1-based by mean of the τ^f-topologies for all p.*

Remark 5.4 *Note that $p \in S(\emptyset)$ is essentially 1-based by mean of Υ iff for every finite tuple $\bar{c}$ from p and for every type-definable Υ-open set $\mathcal{U}$ over $\bar{c}$, there exists a relatively Stone-open non-empty subset χ of $\mathcal{U}$ such that $a \underset{acl^{eq}(a) \cap acl^{eq}(\bar{c})}{\downarrow} \bar{c}$ for all $a \in \chi$.*

Remark 5.5 *Assume $d \in dcl(c)$. Then $Cb(d/a) \in dcl(Cb(c/a))$ for all a.*

One of the key ideas for proving the main result is the following theorem.

Theorem 5.6 *Let T be a countable simple theory that eliminates hyperimaginaries. Let Υ be a projection-closed family of topologies. Let p_0 be a partial type over $\emptyset$ of SU-rank 1. Then, either there exists an unbounded finite-SU-rank Υ-open set over some countable set, or every type $p \in S(A)$, with A countable, that is internal in p_0 is essentially 1-based by mean of Υ. In particular, either there exists an unbounded finite-SU-rank Υ-open set, or whenever A is countable, $p \in S(A)$ and every non-algebraic extension of p is non-foreign to p_0, p is analyzable in p_0 by essentially 1-based types by mean of Υ.*

Proof: Υ will be fixed and we'll freely omit the phrase "by mean of Υ". To see the "In particular" part, work over A and assume that every $p' \in S(A')$, with $A' \supseteq A$ countable, that is internal in p_0, is essentially 1-based over A. Indeed, assume $p \in S(A)$ is such that every non-algebraic extension of p is non-foreign to p_0. Then, for $a \models p$ there exists $a' \in dcl^{eq}(Aa) \backslash acl^{eq}(A)$ such that $tp(a'/A)$ is p_0-internal and thus essentially 1-based over A by our assumption. Since L and Aa are countable so is $dcl^{eq}(Aa)$ and thus by repeating this process we get that p is analyzable in p_0 by essentially 1-based types. We prove now the main part. Assume there exist a countable A and $p \in S(A)$ that is internal in p_0 and p is not essentially 1-based. By Fact 2.2, we may assume there exists $d \models p$, and b that is independent from d over A, and a finite tuple $\bar{c} \subseteq p_0$ such that $d \in dcl(Ab\bar{c})$, and there exists a type-definable Υ-open set $\mathcal{U}$ over Ad such that a is independent from A for all $a \in \mathcal{U}$ and $\{a \in \mathcal{U} | Cb(a/Ad) \not\subseteq acl^{eq}(a)\}$ is not nowhere dense in the Stone-topology of $\mathcal{U}$. So, since Υ refines the Stone-topology, by intersecting it with a definable set, we may assume that $\{a \in \mathcal{U} | Cb(a/Ad) \not\subseteq acl^{eq}(a)\}$ is dense in the Stone-topology of $\mathcal{U}$. Now, for each disjoint partition $\bar{c} = \bar{c}_0\bar{c}_1$ and formula $\chi(\bar{x}_1, \bar{x}_0, y, z) \in L(A)$ such that (*) $\forall \bar{x}_0, y, z \exists^{<\infty} \bar{x}_1 \chi(\bar{x}_1, \bar{x}_0, y, z)$, let

$F_{\chi,\bar{c}_0,\bar{c}_1} = \{a \in \mathcal{U} | \ \exists b', \bar{c}'_0, \bar{c}'_1$ s.t. $tp(b'\bar{c}'_0\bar{c}'_1/Ad) = tp(b\bar{c}_0\bar{c}_1/Ad)$ and a is independent from $b'\bar{c}'_0\bar{c}'_1$ over Ad and $\models \chi(\bar{c}'_1, \bar{c}'_0, b', a)$ and a is independent from $Ab'\bar{c}'_0$ over $\emptyset\}$.

Let $\mathcal{P}_{\bar{c}}$ be the (finite) set of partitions of $\bar{c}$ into two subsets. Note that since d is independent from b over A, any $a \in \mathcal{U}$ is independent from Ab' whenever $tp(b'/Ad) = tp(b/Ad)$ and $a \underset{Ad}{\downarrow} b'$. Thus, since p_0 is a partial type over $\emptyset$ of SU-rank ≤ 1 we have

$$\mathcal{U} = \bigcup_{(\bar{c}_0,\bar{c}_1)\in\mathcal{P}_{\bar{c}},\ \chi\models(*)} F_{\chi,\bar{c}_0,\bar{c}_1}.$$

Note that since we are fixing the type of $b'\bar{c}'_0\bar{c}'_1$ over Ad, the sets $F_{\chi,\bar{c}_0,\bar{c}_1}$ are type-definable over Ad. Since L and A are countable, by the Baire category theorem for the Stone-topology of the closed set $\mathcal{U}$, there exists $(\bar{c}^*_0, \bar{c}^*_1) \in \mathcal{P}_{\bar{c}}$ and there is $\chi^* \models (*)$ such that $F_{\chi^*,\bar{c}^*_0,\bar{c}^*_1}$ has non-empty interior in the Stone-topology of $\mathcal{U}$. Thus, we may assume that $\mathcal{U}$ is a type-definable Υ-open set over Ad such that $\{a \in \mathcal{U} | Cb(a/Ad) \not\subseteq acl^{eq}(a)\}$ is dense in the Stone-topology of $\mathcal{U}$ and for every $a \in \mathcal{U}$ there exists $b'\bar{c}'_0\bar{c}'_1 \models tp(b\bar{c}^*_0\bar{c}^*_1/Ad)$ that is independent from a over Ad and such that $\models \chi^*(\vec{c}_1', \vec{c}_0', b', a)$ and a is independent from$Ab'\vec{c}_0'$ over $\emptyset$. Let us now define a set V over Ad by

$$V = \{(\vec{c}_0', \vec{c}_1', b', a', e')| \text{ if } tp(b'\vec{c}_0'\vec{c}_1'/Ad) = tp(b\vec{c}_0^*\vec{c}_1^*/Ad) \text{ and } a' \text{ is independent from } b'\vec{c}_0'\vec{c}_1' \text{ over } Ad \text{ and } a' \text{ is independent from } Ab'\vec{c}_0' \text{ over } \emptyset \text{ and } \models \chi^*(\vec{c}_1', \vec{c}_0', b', a') \text{ then } e' \in acl(Cb(Ab'\vec{c}_0'\vec{c}_1'/a'))\}.$$

Let

$$V^* = \{e'|\exists a' \in \mathcal{U}\ \forall b', \vec{c}_0', \vec{c}_1'\ V(\vec{c}_0', \vec{c}_1', b', a', e')\}.$$

Subclaim 5.7 *V^* is a Υ-open set over Ad.*

Proof: By Proposition 4.4, we see that V is a Stone-open set over Ad. Since Stone-open sets are closed under the $\forall$ quantifier and the Υ topology refines the Stone-topology and closed under product by a full Stone-space and closed under projections, we conclude that V^* is a Υ-open set.

Subclaim 5.8 *For appropriate sort for e', the set V^* is unbounded and has finite SU-rank over Ad.*

Proof: Let $a^* \in \mathcal{U}$ be such that $Cb(a^*/Ad) \not\subseteq acl^{eq}(a^*)$. Then $Cb(Ad/a^*) \not\subseteq acl^{eq}(Ad)$. By Remark 5.5, there exists $e^* \notin acl^{eq}(Ad)$ such that $e^* \in acl^{eq}(Cb(Ab'\vec{c}_0'\vec{c}_1'/a^*))$ for all $b'\vec{c}_0'\vec{c}_1' \models tp(b\vec{c}_0^*\vec{c}_1^*/Ad)$. In particular, $e^* \in V^*$. Thus, if we fix the sort for e' in the definition of V^* to be the sort of e^*, then V^* is unbounded. Now, let $e' \in V^*$. Then for some $a' \in \mathcal{U}$, $\models V(\vec{c}_0', \vec{c}_1', b', a', e')$ for all $b', \vec{c}_0', \vec{c}_1'$. By what we saw above, there exists $b'\vec{c}_0'\vec{c}_1' \models tp(b\vec{c}_0^*\vec{c}_1^*/Ad)$ that is independent from a' over Ad such that $\models \chi^*(\vec{c}_1', \vec{c}_0', b', a')$ and a' is independent from $Ab'\vec{c}_0'$ over $\emptyset$. Thus, by the definition of V^*, $e' \in acl(Cb(Ab'\vec{c}_0'\vec{c}_1'/a'))$. Since Ab' is independent from a' over $\emptyset$, $tp(e')$ is almost-p_0-internal, and thus $SU(e') < \omega$. In particular, $SU(e'/Ad) < \omega$.

Thus V^* is the required set.

6 Stable dependence

The goal of this section is to prove the symmetry of the relation stable-dependence. In this section T is assumed to be a complete theory unless otherwise stated, and we work in $\mathcal{C}^{eq}$.

Definition 6.1 *Let $a \in \mathcal{C}$, $A \subseteq B \subseteq \mathcal{C}$. We say that a is stably-independent from B over A if for every stable $\phi(x, y) \in L$, if $\phi(x, b)$ is over B (i.e. the canonical parameter of $\phi(x, b)$ is in $dcl(B)$) and $a' \models \phi(x, b)$ for some $a' \in dcl(Aa)$, then $\phi(x, b)$ doesn't divide over A. In this case we denote it by* $a \underset{A}{\overset{s}{\perp\!\!\!\!\smile}} B$.

We will need some basic facts from local stability [HP]. From now on we fix a stable formula $\phi(x, y)$. A formula $\psi \in L(\mathcal{C})$ is said to be a ϕ-formula over A if it is a finite boolean combination of instances of ϕ, that is equivalent to a formula with parameters from A. A complete ϕ-type over A is a consistent complete set of ϕ-formulas over A. $S_\phi(A)$ denotes the set of complete ϕ-types over A. Note that if M is a model then every $p \in S_\phi(M)$ is determined by the set $\{\psi \in p |\ \psi = \phi(x, a)$ or $\psi = \neg\phi(x, a)$ for $a \in M\}$ (in fact, it is easy to see that every ϕ-formula over M is equivalent to a ϕ-formula whose parameters are from M). Recall the following well known facts.

Fact 6.2 *Let $\phi(x, y) \in L$ be stable. Then*
1) For any model M, every $p \in S_\phi(M)$ is definable.
2) Let A be any set, let $p \in S(A)$, and let $M \supseteq A$ be a model. Then there exists $q \in S_\phi(M)$ that is consistent with p and is definable over $acl^{eq}(A)$.
3) Let $A = acl(A)$. Let $p \in S_\phi(A)$. Then for every model $M \supseteq A$, there is a unique $\bar{p} \in S_\phi(M)$ that extends p and such that $\bar{p}$ is definable over A (i.e. its ϕ-definition is over A). Moreover, there is a canonical formula over A that is the definition of any such $\bar{p}$ over any such model M.
4) Assume $p, q \in S_\phi(acl(A))$ are such that $p|A = q|A$. Then there exists $\sigma \in Aut(\mathcal{C}/A)$ such that $\sigma(p) = q$.

The following definition is standard.

Definition 6.3 *Let $p \in S_\phi(B)$ and let $A \subseteq B$. We say that p doesn't fork over A in the sense of local stability (=LS) if for some model M containing B and some $\bar{p} \in S_\phi(M)$ that extends p, $\bar{p}$ is definable over $acl(A)$.*

The following lemma is easy but important.

Lemma 6.4 *Assume T is a simple theory in which Lstp=stp over sets and let $\phi(x,y) \in L$ be stable. Then for all a and $A \subseteq B \subseteq \mathcal{C}$, $tp_\phi(a/B)$ doesn't fork over A in the sense of LS iff $tp_\phi(a/B)$ doesn't fork over A.*

Proof: Assume $p_\phi = tp_\phi(a/B)$ doesn't fork over A in the sense of LS. Extend it to a complete ϕ-type $\bar{p}_\phi$ over a sufficiently saturated and sufficiently strongly-homegeneous model $\mathcal{M}$ that is definable over $acl(A)$. If $tp_\phi(a/B)$ divide over A, there is an $acl(A)$-indiscernible sequence $(B_i|i<\omega) \subseteq \mathcal{M}$ such that if $p^\phi_{B_i}$ are the corresponding $acl(A)$-conjugates of p_ϕ, then $\bigwedge_i p^\phi_{B_i} = \emptyset$. By the uniqueness of non-forking extensions (in the sense of LS) of complete ϕ-types over algebraically closed sets (and the fact that $\mathcal{M}$ is sufficiently strongly-homegeneous) we conclude that $\bar{p}_\phi$ extends each $p^\phi_{B_i}$, a contradiction. For the other direction, assume $p_\phi = tp_\phi(a/B)$ doesn't fork over A. Let $\mathcal{M} \supseteq B$ be a sufficiently saturated and sufficiently strongly homogeneous model. Let $\bar{p} \in S(\mathcal{M})$ be an extension of p_ϕ that doesn't fork over A. Let $\psi(x,c) \in L(\mathcal{M})$ be the definition of $\bar{p}|\phi$ (where c is the canonical parameter of ψ). We claim that $c \in acl(A)$. Indeed, otherwise let $\sigma \in Aut(\mathcal{M}/acl(A))$ be such that $\sigma c \neq c$. So, $\bar{p}, \sigma(\bar{p})$ have different ϕ-definitions, contradiction to the following claim:

Claim 6.5 *Let T be simple. Let $\phi(x,y) \in L$ be stable. Assume $a \underset{A}{\downarrow} b$ and $a' \underset{A}{\downarrow} b$ and $Lstp(a/A) = Lstp(a'/A)$. Then $\phi(a,b)$ iff $\phi(a',b)$.*

Corollary 6.6 *Let T be a simple theory in which Lstp=stp over sets. Then for all $a, A \subseteq B \subseteq \mathcal{C}$ we have $a \underset{A}{\downarrow^s} B$ iff $tp_\phi(a'/B)$ doesn't fork over A in the sense of LS for every stable $\phi(x,y) \in L$ and every $a' \in dcl(aA)$.*

Given $a, A \subseteq B \subseteq \mathcal{C}$, we will say that $tp(a/B)$ *doesn't fork over A in the sense of LS* if the right hand side of Corollary 6.6 holds.

Lemma 6.7 *Let T be a simple theory in which Lstp=stp over sets. Then*
1) stable independence is a symmetric relation, that is, for all a, b, A we have $a \underset{A}{\downarrow^s} Ab$ iff $b \underset{A}{\downarrow^s} Aa$.

2) For all $a, A \subseteq B \subseteq C$, if $a \underset{A}{\downarrow^s} B$ *and* $a \underset{B}{\downarrow^s} C$*, then* $a \underset{A}{\downarrow^s} C$*. In fact, in any theory the same is true in the sense of LS.*

Proof: To prove 1), first note the following.

Subclaim 6.8 *Let $\phi(x, y) \in L$ be stable and let $a, a' \in \mathcal{C}$ and let $A \subseteq \mathcal{C}$. Assume $tp_\phi(a/A) = tp_\phi(a'/A)$. Then $\phi(a, y)$ forks over A iff $\phi(a', y)$ forks over A.*

Proof: Otherwise, there are $p, q \in S(\mathcal{C})$, both extends $tp_\phi(a/A) = tp_\phi(a'/A)$, and do not fork over A such that p represent $\phi(x, y)$ (namely, for some $b \in M$, $\phi(x, b) \in p$) and q doesn't represent $\phi(x, y)$. By Fact 6.2 (4), $(p|\phi)|acl(A)$ and $(q|\phi)|acl(A)$ are A-conjugate. Let $\sigma \in Aut(\mathcal{C}/A)$ be such that $\sigma((p|\phi)|acl(A)) = (q|\phi)|acl(A)$. Now, both $\sigma(p|\phi)$ and $q|\phi$ extend $(q|\phi)|acl(A)$ and doesn't fork over $acl(A)$, and therefore by Lemma 6.4, both doesn't fork over $acl(A)$ in the sense of LS. By Fact 6.2 (3), $\sigma(p|\phi) = q|\phi$, which is a contradiction.

We prove symmetry. Assume $a \underset{A}{\downarrow^s} Ab$. To show $b \underset{A}{\downarrow^s} Aa$, let $\phi(x, y) \in L$ be stable such that $\phi(b', a')$ for some $b' \in dcl(Ab)$ and some $a' \in dcl(Aa)$. By the assumption, $tp_{\tilde{\phi}}(a'/Ab)$ doesn't fork over A (in the usual sense), so there exists $a'' \models tp_{\tilde{\phi}}(a'/Ab)$ such that $a'' \underset{A}{\downarrow} Ab$. Let $(a''_i | i < \omega)$ be a Morley sequence of $tp(a''/Ab)$. Now, $b' \models \bigwedge_{i<\omega} \phi(x, a''_i)$. Thus $\phi(x, a'')$ doesn't fork over A. By Subclaim 6.8, $\phi(x, a')$ doesn't fork over A. 2) is immediate by Corollary 6.6 and fact that the relation of being a non-forking extension in the LS sense is a transitive relation on complete ϕ-types (where ϕ is a fixed stable formula).

7 An unbounded τ^f_∞-open set of bounded finite SU_s-rank is sufficient

In this section T is an imaginary simple theory. We work in $\mathcal{C}^{eq}$.

Definition 7.1 *For $a \in \mathcal{C}$ and $A \subseteq \mathcal{C}$ the SU_s-rank is defined by induction on α by $SU_s(a/A) \geq \alpha + 1$ if there exists $B \supseteq A$ such that $a \underset{A}{\not\downarrow^s} B$ and $SU_s(a/B) \geq \alpha$.*

Definition 7.2 *The τ^f_∞-topology* on $S(A)$ is the topology whose basis is the family of type-definable τ^f-open sets over A.

Lemma 7.3 *For $a \in \mathcal{C}$ and $A \subseteq B \subseteq \mathcal{C}$, assume $tp(a/B)$ doesn't fork over $acl(aA) \cap acl(B)$ and $a \underset{A}{\not\downarrow} B$. Then $a \underset{A}{\not\downarrow^s} B$.*

Proof: It will be sufficient to show that whenever $a \underset{A}{\not\downarrow} B$ and $a \underset{acl(a) \cap acl(B)}{\downarrow} B$ for some (possibly infinite) tuple a and some $A \subseteq B$, there exists a stable $\phi(x, y) \in L$ such that $\phi(a, B)$ and $\phi(x, b)$ forks over A (indeed, the above implies the following: if $aA \underset{A}{\not\downarrow} B$ and $aA \underset{acl(aA) \cap acl(B)}{\downarrow} B$ then there exists a stable formula $\phi(x, y) \in L$ such that $\phi(aA, B)$ and $\phi(x, B)$ forks over A, i.e. $a \underset{A}{\not\downarrow^s} B$). To prove this, let $E = Cb(a/B)$. Then $E \subseteq acl(a) \cap acl(B)$. By the assumption, there is $e^* \in dcl(E) \backslash acl(A)$, so $e^* \in (acl(a) \cap acl(B)) \backslash acl(A)$. Hence there are $n_0, n_1 \in \omega$ and formulas $\chi_0(x, y), \chi_1(x, z) \in L$ such that $\forall y \exists^{<n_0} x \chi_0(x, y)$ and $\forall z \exists^{<n_1} x \chi_0(x, z)$ and $\chi_0(e^*, a)$ and $\chi_1(x, B)$ isolates $tp(e^*/B)$. Let

$$\phi(y, z) \equiv \exists x(\chi_0(x, y) \wedge \chi_1(x, z)).$$

Note that $\phi(y, z)$ is stable. Indeed, otherwise there are $a \in \mathcal{C}$ and an $\emptyset$-indiscernible sequence $B = (b_i | i \in Z)$ (Z=the integer numbers) such that $i \geq 0$ iff $\phi(a, b_i)$. Since B is indiscernible, and $\chi_1(x, b_0)$ is algebraic, $\bigcap_{i \in I} \chi_1(\mathcal{C}, b_i) = \bigcap_{i \in Z} \chi_1(\mathcal{C}, b_i)$ for every infinite $I \subseteq Z$. But since $\chi_0(x, a)$ is algebraic, for some infinite $I^* \subseteq \omega$, $\chi_0(\mathcal{C}, a) \cap \bigcap_{i \in I^*} \chi_1(\mathcal{C}, b_i) \neq \emptyset$. A contradiction to $\neg \phi(a, b_i)$ for $i < 0$. To see that $\phi(y, B)$ forks over A, note that otherwise there exists $a' \models \phi(y, B)$ such that $a' \underset{A}{\downarrow} B$, so if $e' \models \chi_0(x, a') \wedge \chi_1(x, B)$, then on one hand $e' \underset{A}{\downarrow} B$ and on the other hand since $\chi_1(x,B)$ isolates $tp(e^*/B)$, $e' \in acl(B)\backslash acl(A)$ which is a contradiction.

Lemma 7.4 *Assume $\mathcal{U}$ is an unbounded τ^f_∞-open set of bounded finite SU_s-rank over some finite set A. Then there exists a τ^f_∞-open set $\mathcal{U}^* \subseteq \mathcal{U}$ over some finite set $B^* \supseteq A$ of SU_s-rank 1.*

Proof: We may clearly assume $\mathcal{U}$ is a basic τ^f_∞-open set. Let $n = SU_s(\mathcal{U})$ ($\mathcal{U}$ is over A, and $n < \omega$). Let $a^* \in \mathcal{U}$ with $SU_s(a^*/A) = n$. Let $B \supseteq A$ be finite such that $a^* \underset{A}{\not\downarrow^s} B$, and $SU_s(a^*/B) = n-1$. So, there exists $a' \in dcl(a^*A)$ and stable $\phi(x,y) \in L$ such that $\phi(a',B)$ and $\phi(x,B)$ forks over A. Let f an $\emptyset$-definable function such that $a' = f(a^*,A)$. Let

$$\mathcal{U}' = \{a \in \mathcal{U} |\ \phi(f(a,A),B)\ \} \text{ (as a set over } B).$$

Since $a^* \in \mathcal{U}'$, $SU_s(\mathcal{U}') \geq n-1$. If $a \in \mathcal{U}'$, then $\phi(f(a,A),B)$ implies $a \underset{A}{\not\downarrow^s} B$ and therefore $SU_s(\mathcal{U}') \leq n-1$. We conclude $SU_s(\mathcal{U}') = n-1$. Clearly, $\mathcal{U}' \subseteq \mathcal{U}$ and $\mathcal{U}'$ is type-definable. By Fact 2.7, $\mathcal{U}'$, is a τ^f-open set over B. We finish by induction.

Lemma 7.5 *Let T be a countable imaginary simple unidimensional theory. Assume there is $p_0 \in S(\emptyset)$ of SU-rank 1 and there exists an unbounded τ^f_∞-open set over some finite set of bounded finite SU_s-rank. Then T is supersimple.*

Proof: By Lemma 7.4, there exists a finite set A_0 and a τ^f_∞-open set $\mathcal{U}$ over A_0 of SU_s-rank 1. Clearly, we may assume $\mathcal{U}$ is type-definable. By Theorem 3.12, T is PCFT. Thus, working over A_0, by Theorem 5.6 for the τ^f-topology either (i) there exists an unbounded τ^f-open set of finite SU-rank over some countable set or (ii) every non-algebraic type over A_0 is analyzable in p_0 by essentially 1-based types by mean of τ^f. By Corollary 3.13, we may assume (ii). We claim $SU(\mathcal{U}) = 1$. Indeed, otherwise there exists a and $d \in \mathcal{U}$ such that $d \underset{A_0}{\not\downarrow} a$ and $d \notin acl(aA_0)$. By (ii), there exists $(a_i|i \leq \alpha) \subseteq dcl^{eq}(aA_0)$ with $a_\alpha = a$ such that $tp(a_i/A_0 \cup \{a_j|j<i\})$ is essentially 1-based over A_0 for all $i \leq \alpha$. Now, let $i^* \leq \alpha$ be minimal such that there exists

$d' \in \mathcal{U}$ satisfying $\underset{A_0}{d' \not\perp \{a_i|i \leq i^*\}}$, and $d' \notin acl(A_0 \cup \{a_i|i \leq i^*\})$. Pick $\phi(x, a') \in L(A_0 \cup \{a_i|i \leq i^*\})$ that forks over A_0 and such that $\phi(d', a')$. Let

$$V = \{d \in \mathcal{U}|\ \phi(d, a') \text{ and } d \notin acl(A_0 \cup \{a_i|i \leq i^*\})\ \}.$$

By minimality of i^*, d is independent from $\{a_i|i < i^*\}$ over A_0 for all $d \in V$. Clearly V is type-definable and by Fact 2.7, V is a τ^f-open set over $A_0 \cup \{a_i|i \leq i^*\}$. Now, since $tp(a_{i^*}/A_0 \cup \{a_i|i < i^*\})$ is essentially 1-based over A_0, the set

$$\{d \in V|\ Cb(d/A_0 \cup \{a_i|i \leq i^*\}) \in bdd(dA_0)\}$$

contains a relatively Stone-dense and open subset of V. In particular, there exists $d^* \in V$ such that $tp(d^*/A_0 \cup \{a_i|i \leq i^*\}$ doesn't fork over $acl(A_0d^*) \cap acl(A_0 \cup \{a_i|i \leq i^*\})$. Since we know $\underset{A_0}{d^* \not\perp A_0 \cup \{a_i|i \leq i^*\}}$, Lemma 7.3 implies $\underset{A_0}{d^* \not\perp^s A_0 \cup \{a_i|i \leq i^*\}}$. Hence $d^* \in V$ implies $SU_s(d^*/A_0) \geq 2$, which contradict $SU_s(\mathcal{U}) = 1$. Thus we have proved $SU(\mathcal{U}) = 1$. Now, by Fact 2.8 there exists a definable set of SU-rank 1, and thus by Fact 2.4, T is supersimple.

Remark 7.6 *Note that if X is any Stone-closed subset of the Stone-space $S_x(T)$ and $B = \{F_i\}_{i \in I}$ is a basis for a topology τ on X that consists of Stone-closed subsets of X, then (X, τ) is a Baire space (i.e. the intersection countably many dense open sets in it is dense). In particular, the τ^f_∞-topology on $S(A)$ is Baire.*

Remark 7.7 *If we could show that for all $a, A \subseteq B \subseteq C$,*

$$\underset{A}{a \perp^s C} \Rightarrow \underset{B}{a \perp^s C},$$

then this would imply that for $A \subseteq B$, $\underset{A}{a \perp^s B}$ implies $SU_s(a/A) = SU_s(a/B)$. Thus by Remark 7.6, a Baire categoricity argument, applying Theorem 3.12, will imply the existence of a bounded finite SU_s-rank unbounded τ^f_∞-open set in any countable imaginary unidimensional simple theory and thus supersimplicity will follow by Lemma 7.5. Unfortunately, this seems to be false for a general simple theory without stable forking.

8 $\tilde{\tau}^f$ and $\tilde{\tau}^f_{st}$-sets

In this section T is assumed to be a simple theory. We work in $\mathcal{C}$.

Definition 8.1 A relation $V(x, z_1, ...z_l)$ is said to be a *pre-$\tilde{\tau}^f$-set relation* if there are $\theta(x, \tilde{x}, z_1, z_2, ..., z_l) \in L$ and $\phi_i(\tilde{x}, y_i) \in L$ for $0 \leq i \leq l$ such that for all $a, d_1, ..., d_l \in \mathcal{C}$ we have

$$V(a, d_1, ..., d_l) \text{ iff } \exists \tilde{a}\ [\theta(a, \tilde{a}, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 d_2 ... d_i)]$$

(for $i = 0$ the sequence $d_1 d_2 ... d_i$ is interpreted as $\emptyset$). If each $\phi_i(\tilde{x}, y_i)$ is assumed to be stable, $V(x, z_1, ...z_l)$ is said to be a *pre-$\tilde{\tau}^f_{st}$-set relation.*

Definition 8.2 1) A *$\tilde{\tau}^f$-set (over $\emptyset$)* is a set of the form

$$\mathcal{U} = \{a \mid \exists d_1, d_2, ...d_l\ V(a, d_1, ..., d_l)\}$$

for some *pre-$\tilde{\tau}^f$-set relation* $V(x, z_1, ...z_l)$. The minimal natural number l for which the set $\mathcal{U}$ above can be defined is called the degree of the set $\mathcal{U}$ (e.g. τ^f-open sets have degree 0).
2) A $\tilde{\tau}^f_{st}$-set is defined in the same way as a $\tilde{\tau}^f$-set but we add the requirement that $V(x, z_1, ...z_l)$ is a *pre-$\tilde{\tau}^f_{st}$-set relation.*

We will say that the formula $\phi(x, y) \in L$ is *low in x* if there exists $k < \omega$ such that for every $\emptyset$-indiscernible sequence $(b_i | i < \omega)$, the set $\{\phi(x, b_i) | i < \omega\}$ is inconsistent iff every subset of it of size k is inconsistent. Note that every stable formula $\phi(x, y)$ is low in both x and y.

Remark 8.3 *Note that if $\phi(x, y) \in L$ is low in x then the relation F_ϕ defined by $F_\phi(b, A)$ iff $\phi(x, b)$ forks over A is type-definable. Thus every pre-$\tilde{\tau}^f_{st}$-set relation is type-definable and every $\tilde{\tau}^f_{st}$-set is type-definable.*

Lemma 8.4 *Assume the extension property is first-order in T. Let $\theta(x, z_1, ..., z_n)$ be a Stone-open relation over $\emptyset$ and let $\phi_j(x, y_j) \in L$ for $j = 0, .., n$. Let U be the following invariant set. For all $d_1 \in \mathcal{C}$, $U(d_1)$ iff*

$$\exists a, d_2, d_3, ...d_n [\theta(a, d_1, ...d_n) \wedge \bigwedge_{j=0}^{n} \phi_j(a, y_j)\ forks\ over\ d_1 ... d_j].$$

Then U is a τ^f-open set over $\emptyset$. If each $\phi_j(x, y_j)$ is assumed to be low in y_j and θ is assumed to be definable, then U is a basic τ^f_∞-open set.

Proof: We prove the lemma by induction on $n \geq 1$. Consider the negation Γ of U:

$$\Gamma(d_1) \text{ iff } \forall a \forall d_2 ... d_n (\theta(a, d_1, ... d_n) \rightarrow \bigvee_{j=0}^{n} \phi_j(a, y_j) \text{ dnfo } d_1 ... d_j)$$

(where "dnfo"=doesn't fork over).

Subclaim 8.5 *Let Γ' be defined by $\Gamma'(d_1)$ iff*

$$\bigwedge_{\{\eta_j\}_{j=0}^{n-1} \in L} \forall d_2 ... d_n [(\bigwedge_{j=0}^{n-1} \eta_j(d_1 ... d_n, y_j) \textit{ forks over } d_1 ... d_j) \rightarrow \forall a \Lambda(a, d_1, ..., d_n)].$$

where Λ is defined by

$$\Lambda(a, d_1, ... d_n) \;\; \textit{iff} \;\; \theta(a, d_1, ... d_n) \rightarrow \bigvee_{j=0}^{n} \phi_j(a, y_j) \wedge \neg \eta_j(d_1 ... d_n, y_j) \textit{ dnfo } d_1 ... d_n$$

where η_n denotes a contradiction. Then $\Gamma' = \Gamma$.

Proof: Assume $\Gamma(d_1)$. Let $\eta_0, ... n_{n-1} \in L$ and let $d_2, ... d_n \in \mathcal{C}$. Assume $\eta_j(d_1 ... d_n, y_j)$ forks over $d_1 ... d_j$ for all $0 \leq j \leq n-1$, and let $a \in \mathcal{C}$ be such that $\theta(a, d_1, ... d_n)$. By the assumption, $\phi_{j_0}(a, y_{j_0})$ doesn't fork over $d_1 ... d_{j_0}$ for some $0 \leq j_0 \leq n-1$. Let c_{j_0} be such that $\phi_{j_0}(a, c_{j_0})$ and $a \underset{d_1 ... d_{j_0}}{\downarrow} c_{j_0}$. By extension we may assume $ad_1 ... d_n \underset{d_1 ... d_{j_0}}{\downarrow} c_{j_0}$. Since $\eta_{j_0}(d_1 ... d_n, y_{j_0})$ forks over $d_1 ... d_{j_0}$, we know $\neg \eta_{j_0}(d_1 ... d_n, c_{j_0})$. Therefore $\phi_{j_0}(a, y_{j_0}) \wedge \neg \eta_{j_0}(d_1 ... d_n, y_{j_0})$ doesn't fork over $d_1 ... d_{j_0}$ and in particular doesn't fork over $d_1 ... d_n$. Assume now $\Gamma'(d_1)$. Let $a, d_2, ... d_n \in \mathcal{C}$ and assume $\theta(a, d_1, ... d_n)$. It is sufficient to show that for all $0 \leq j \leq n-1$ if $\phi_j(a, y_j)$ forks over $d_1 ... d_j$, then there exists η_j such that $\eta_j(d_1 ... d_n, y_j)$ forks over $d_1 ... d_j$ and $\phi_j(a, y_j) \wedge \neg \eta_j(d_1 ... d_n, y_j)$ forks over $d_1 .... d_n$. Assume otherwise. Fix j, so $\phi_j(a, y_j)$ forks over $d_1 ... d_j$ and $\phi_j(a, y_j) \wedge \neg \eta_j(d_1 ... d_n, y_j)$ doesn't fork over $d_1 .... d_n$ for all η_j such that $\eta_j(d_1 ... d_n, y_j)$ forks over $d_1 ... d_j$. Let

$$\Psi(y_j) \equiv \bigwedge_{\eta_j \in F_j, \mu_j \in E_j} \phi_j(a, y_j) \wedge \neg \eta_j(d_1 ... d_n, y_j) \wedge \neg \mu_j(ad_1 ... d_n, y_j)$$

where

$$F_j = \{\eta_j | \ \eta_j(d_1...d_n, y_j) \text{ forks over } d_1...d_j\},$$

and

$$E_j = \{\mu_j | \ \mu_j(ad_1...d_n, y_j) \text{ forks over } d_1...d_n\}.$$

By our assumption and compactness, $\Psi(y_j)$ is consistent. Let $c_j \models \Psi(y_j)$. Then $\phi_j(a, c_j)$, $d_1...d_n \underset{d_1...d_j}{\downarrow} c_j$, and $ad_1...d_n \underset{d_1...d_n}{\downarrow} c_j$. By transitivity, $ad_1...d_n \underset{d_1...d_j}{\downarrow} c_j$. A contradiction to the assumption that $\phi_j(a, y_j)$ forks over $d_1...d_j$. The proof of Subclaim 8.5 is complete.

Since the extension property is first-order for T, the relation Λ_0 defined by $\Lambda_0(d_1, ...d_n) \equiv \forall a \Lambda(a, d_1, ...d_n)$ is type-definable. Now, clearly for all d_1, $\Gamma'(d_1)$ iff

$$\bigwedge_{\{\eta_j\}_{j=0}^{n-1} \in L} \forall d_2...d_n(\neg\Lambda_0(d_1, ..., d_n) \to \bigvee_{j=0}^{n-1} \eta_j(d_1...d_n, y_j) \text{ dnfo } d_1...d_j).$$

Now, if $n = 1$ then this is clearly τ^f-closed. If $n > 1$, then we finish by the induction hypothesis.

Corollary 8.6 *Assume the extension property is first-order in T. Let $m \leq l < \omega$ and let $d_1^*, ...d_m^* \in \mathcal{C}$. Let $\theta \in L$ and $\phi_i \in L$ for $i \leq l$. Let V be defined by*

$$V(a, d_1, ..., d_l) \textit{ iff } [\theta(a, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(a, y_i) \textit{ forks over } d_1 d_2...d_i)].$$

Then the set U defined by

$$U(d_{m+1}) \textit{ iff } \exists a \exists d_{m+2}...d_l \ V(a, d_1^*, ...d_m^*, d_{m+1}, ...d_l)$$

is a τ^f-open set over $d_1^*...d_m^*$.

Proof: By Fact 2.7, there are formulas $\{\psi_j(\tilde{x}, w_j) \in L(d_1^*...d_m^*)\}_{j \in J}$ such that

$$\forall a \ [\bigwedge_{i=0}^{m} (\phi_i(a, y_i) \text{ forks over } d_1^* d_2^*...d_i^*) \text{ iff } \bigvee_{j \in J} (\psi_j(a, w_j) \text{ forks over } d_1^* d_2^*...d_m^*)].$$

Therefore by Lemma 8.4 (since by Lemma 3.2, the extension property is first-order over $\bar{d}^*$ as well) U is a union over $j \in J$ of τ^f-open sets over $d_1^* d_2^* ... d_m^*$.

Theorem 8.7 *Assume the extension property is first-order in T. Then*
1) Let $\mathcal{U}$ be an unbounded $\tilde{\tau}^f$-set over $\emptyset$. Then there exists an unbounded τ^f-open set $\mathcal{U}^*$ over some finite set A^* such that $\mathcal{U}^* \subseteq \mathcal{U}$. In fact, if $V(x, z_1, ..., z_l)$ is a pre-$\tilde{\tau}^f$-set relation such that $\mathcal{U} = \{a | \exists d_1 ... d_l V(a, d_1, ..., d_l)\}$, and $(d_1^*, ..., d_m^*)$ is any maximal sequence (with respect to extension) such that $\exists d_{m+1} ... d_l V(\mathcal{C}, d_1^*, ..., d_m^*, d_{m+1}, ..., d_l)$ is unbounded, then

$$\mathcal{U}^* = \exists d_{m+1} ... d_l V(\mathcal{C}, d_1^*, ..., d_m^*, d_{m+1}, ..., d_l)$$

is a τ^f-open set over $d_1^* ... d_m^*$.
2) Let $\mathcal{U}$ be an unbounded $\tilde{\tau}^f_{st}$-set over $\emptyset$. Then there exists an unbounded τ^f_∞-open set $\mathcal{U}^*$ over some finite set A^* such that $\mathcal{U}^* \subseteq \mathcal{U}$. In fact, if $V(x, z_1, ..., z_l)$ is a pre-$\tilde{\tau}^f_{st}$-set relation such that $\mathcal{U} = \{a | \exists d_1 ... d_l V(a, d_1, ..., d_l)\}$, and $(d_1^*, ..., d_m^*)$ is any maximal sequence (with respect to extension) such that $\exists d_{m+1} ... d_l V(\mathcal{C}, d_1^*, ..., d_m^*, d_{m+1}, ..., d_l)$ is non-algebraic, then

$$\mathcal{U}^* = \exists d_{m+1} ... d_l V(\mathcal{C}, d_1^*, ..., d_m^*, d_{m+1}, ..., d_l)$$

is a basic τ^f_∞-open set over $d_1^* ... d_m^*$.

Proof: By Remark 8.3, (2) is an immediate corollary of (1). It suffices, of course, to prove the second part of (1). T is PCFT by Corollary 3.5. Let $\bar{d}^* = d_1^* ... d_m^*$. First, if $m = l$ then the assertion follows immediately by Fact 2.7. So, we may assume $m < l$. By maximality of $\bar{d}^*$, we know $\exists d_{m+2} ... d_l V(\mathcal{C}, d_1^*, ..., d_m^*, d'_{m+1}, d_{m+2}, ... d_l)$ is bounded (equivalently, a union of algebraic sets over $\bar{d}^*$) for every d'_{m+1}. Thus for every $a \in \mathcal{U}^*$, there exist $\chi_a(x, \bar{z}^*, z) \in L$, $k = k(\chi_a) < \omega$ and $d'_{m+1}(a) \in \mathcal{C}$, such that $\forall z \forall \bar{z}^* \exists^{=k} x \chi_a(x, \bar{z}^*, z)$ (*1) and $V(a, d_1^*, ..., d_m^*, d'_{m+1}(a), d_{m+2}, ... d_l)$ for some $d_{m+2}, ... d_l \in \mathcal{C}$ and $\chi_a(x, \bar{d}^*, d'_{m+1}(a))$ isolates the type $tp(a/\bar{d}^*, d'_{m+1}(a))$. Let $\Xi = \{\chi_a\}_{a \in \mathcal{U}^*}$. For $\chi \in \Xi$, let $k = k(\chi)$ and let U_χ be the $\bar{d}^*$-invariant set defined by $U_\chi(d_{m+1})$ iff

$$\exists \text{ distinct } a_1 .... a_k [\bigwedge_{j=1}^{k} \chi(a_j, \bar{d}^*, d_{m+1}) \wedge \bigwedge_{j=1}^{k} \exists d_{m+2} ... d_l V(a_j, \bar{d}^*, d_{m+1}, d_{m+2}, ... d_l)]$$

Subclaim 8.8 *U_χ is a τ^f-open set over $\bar{d}^*$.*

Proof: Let V be given by:

$$V(a, d_1, ..., d_l) \text{ iff } \exists \tilde{a}\ [\theta(a, \tilde{a}, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 d_2 ... d_i)].$$

for some $\theta, \phi_i \in L$. Since T is PCFT, it is sufficient to show that there exists a τ^f-open set $W = W(x, z_{m+1}, \bar{d}^*)$ over $\bar{d}^*$ such that if U'_χ is defined by

$$U'_\chi(d_{m+1}) \text{ iff } \exists \text{ distinct } a_1....a_k[\bigwedge_{j=1}^{k} \chi(a_j, \bar{d}^*, d_{m+1}) \wedge \bigwedge_{j=1}^{k} W(a_j, d_{m+1}, \bar{d}^*)]$$

then $U'_\chi = U_\chi$. To show this let W be defined by: $W(a, d_{m+1}, \bar{d}^*)$ iff

$$\exists \tilde{a} \exists d'_{m+2}...d'_l[\theta(a, \tilde{a}, d^*_1, d^*_2, ...d^*_m, d_{m+1}, d'_{m+2}, ...d'_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d'_1 d'_2 ... d'_i)]$$

where d'_i is defined in the following way: for $1 \leq i \leq m$, d'_i denotes d^*_i, and d'_{m+1} denotes $d_{m+1}a$ (and the rest are quantified variables). First note that for all a, d_{m+1} with $a \in acl(d_{m+1}, \bar{d}^*)$, $W(a, d_{m+1}, \bar{d}^*)$ iff $\exists d_{m+2}...d_l V(a, \bar{d}^*, d_{m+1}, d_{m+2}, ...d_l)$. Thus by (*1), $U'_\chi = U_\chi$. By Corollary 8.6, W is a τ^f-open set over $\bar{d}^*$. So, the proof of Subclaim 8.8 is complete.

Now, for each $\chi \in \Xi$ define $Y_\chi(x) \equiv \exists d_{m+1}(\chi(x, \bar{d}^*, d_{m+1}) \wedge U_\chi(d_{m+1}))$. Since T is PCFT, Subclaim 8.8 implies Y_χ is a τ^f-open set over $\bar{d}^*$. Note that by the definition of U_χ and (*1), $Y_\chi \subseteq \mathcal{U}^*$ for all $\chi \in \Xi$. Now, if $a \in \mathcal{U}^*$, then by the choice of $d'_{m+1}(a)$, χ_a and $k = k(\chi_a)$, we have $\chi_a(a, \bar{d}^*, d'_{m+1}(a)) \wedge U_{\chi_a}(d'_{m+1}(a))$. Thus $a \in Y_{\chi_a}$. Hence $\mathcal{U}^* = \bigcup_{\chi \in \Xi} Y_\chi$, and so $\mathcal{U}^*$ is a τ^f-open set over $\bar{d}^*$. The proof of Theorem 8.7 is complete.

9 Main Result

Definition 9.1 *For $a \in \mathcal{C}$ and $A \subseteq \mathcal{C}$ the SU_{se}-rank is defined by induction on α: $SU_{se}(a/A) \geq \alpha + 1$ if there exist $B_1 \supseteq B_0 \supseteq A$ such that $a \not\downarrow^{s}_{B_0} B_1$ and $SU_{se}(a/B_1) \geq \alpha$.*

Remark 9.2 *Note that $SU_{se}(a/B) \leq SU_{se}(a/A)$ for all $a \in \mathcal{C}$ and $A \subseteq B \subseteq \mathcal{C}$ (this is the reason for introducing SU_{se}). Also, clearly $SU_s(a/A) \leq SU_{se}(a/A) \leq SU(a/A)$ for all a, A. Clearly $SU_{se}(a/A) = 0$ iff $SU_s(a/A) = 0$ iff $a \in acl(A)$ for all a, A.*

Theorem 9.3 *Let T be a countable imaginary simple unidimensional theory. Then T is supersimple.*

Proof: By adding countably many constants we may assume there exists $p_0 \in S(\emptyset)$ of SU-rank 1 (each of the assumptions is preserved, as well as the corollary). By Lemma 7.5, it will be sufficient to show there exists an unbounded τ^f_∞-open set of bounded finite SU_s-rank over some finite set. Fix a non-algebraic sort s. Since T is unidimensional and imaginary, by Fact 2.1 for every $a \in \mathcal{C}^s \backslash acl(\emptyset)$ there exists $a' \in dcl(a) \backslash acl(\emptyset)$ such that $tp(a')$ is p_0-internal and thus has finite SU-rank. To finish the proof it will be sufficient to show that in any countable simple theory T in which the extension property is first-order, the non-existence of an unbounded τ^f_∞-open set of bounded finite SU_{se}-rank over a finite set implies $\exists a^* \in \mathcal{C}^s \backslash acl(\emptyset)$ such that for every $\emptyset$-definable function f, either $f(a^*) \in acl(\emptyset)$ or $SU_{se}(f(a^*)) \geq \omega$. To show this, assume the above assumptions on T. For every $\emptyset$-definable function f and $n < \omega$, let

$$S_{f,n} = \{a \in \mathcal{C}^s |\ 0 < SU_{se}(f(a)) < n\}.$$

Subclaim 9.4 *For every non-empty $\tilde{\tau}^f_{st}$-set $\mathcal{U} \subseteq \mathcal{C}^s$ (with $\mathcal{U} \cap acl(\emptyset) = \emptyset$) for all $\emptyset$-definable function f, and $n < \omega$, there exists a non-empty $\tilde{\tau}^f_{st}$-set $\mathcal{U}^* \subseteq \mathcal{U} \cap (\mathcal{C}^s \backslash S_{f,n})$.*

Assuming Subclaim 9.4 is true, let $((f_i, n_i)|i < \omega)$ be an enumeration of all such pairs (f, n). By induction, let $\mathcal{U}_0 = \mathcal{C}^s \backslash acl(\emptyset)$, and let $\mathcal{U}_{i+1} \subseteq \mathcal{U}_i \cap (\mathcal{C} \backslash S_{f_i,n_i})$ be a non-empty $\tilde{\tau}^f_{st}$-set. Since each $\mathcal{U}_i$ is type-definable, by compactness $\bigcap_{i<\omega} \mathcal{U}_i \neq \emptyset$. So, any $a^* \in \bigcap_{i<\omega} \mathcal{U}_i$ will work.

Proof of Subclaim 9.4: Let $\mathcal{U}$, (f, n) be as in Subclaim 9.4. Now, if there exists $a \in \mathcal{U}$ such that $f(a) \in acl(\emptyset)$, let $\chi(x) \in L$ be algebraic such that $\chi(f(a))$. By letting $\mathcal{U}^* = \{a \in \mathcal{U} |\ \models \chi(f(a))\}$ we are done. Hence we may assume $f(a) \notin acl(\emptyset)$ for every $a \in \mathcal{U}$. Let $V(x, z_1, ...z_n)$ be a pre-$\tilde{\tau}^f_{st}$-set relation such that

$$\mathcal{U} = \{a|\ \exists d_1, d_2, ...d_l\ V(a, d_1, ..., d_l)\}.$$

where V is defined by:

$$V(a, d_1, ..., d_l) \text{ iff } \exists \tilde{a}\ [\theta(a, \tilde{a}, d_1, d_2, ..., d_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 d_2 ... d_i)]$$

for some $\theta(x, \tilde{x}, z_1, z_2, ..., z_l) \in L$ and stable $\phi_i(\tilde{x}, y_i) \in L$ for $0 \leq i \leq l$. Now, let V_f be defined by: for all $b, d_1, ..., d_l \in \mathcal{C}$,

$$V_f(b, d_1, ..., d_l) \text{ iff } \exists a(b = f(a) \wedge V(a, d_1, ..., d_l)).$$

Then, clearly V_f is a pre-$\tilde{\tau}^f_{st}$-set relation. Let

$$\mathcal{U}_f = \{b |\ \exists d_1, d_2, ...d_l\ V_f(b, d_1, ..., d_l)\}.$$

Let $\bar{d}^* = (d^*_1, ..., d^*_m)$ be a maximal sequence, with respect to extension, $(m \leq l)$ such that

$$\tilde{V}_f(v) \equiv \exists d_{m+1}, d_{m+2}, ...d_l V_f(v, d^*_1, ...d^*_m, d_{m+1}, ...d_l)$$

is non-algebraic, or equivalently unbounded (since $\mathcal{U} \neq \emptyset$ and we assume $f(a) \notin acl(\emptyset)$ for all $a \in \mathcal{U}$, the empty sequence satisfies this property). By Theorem 8.7, $\tilde{V}_f(\mathcal{C})$ is a basic τ^f_∞-open set over $\bar{d}^*$. By our assumption $\tilde{V}_f(\mathcal{C})$ is not of bounded finite SU_{se}-rank. Thus there are a^* and $d^*_{m+1}, ...d^*_l$ such that $V(a^*, \bar{d}^*, d^*_{m+1}, ..., d^*_l)$ and $SU_{se}(f(a^*)/\bar{d}^*) \geq n$. Let $E = \langle (c^*_i, e^*_i) | 1 \leq i \leq n \rangle$ be such that $f(a^*) \underset{\bar{d}^* c^*_1 e^*_1 ... c^*_i}{\not\downarrow^{s}} e^*_i$ for all $1 \leq i \leq n$ (*1). Note that since both dcl and forking have finite character, we may assume that c^*_i, e^*_i are finite tuples. Let $\tilde{a}^*$ be such that:

$$\theta(a^*, \tilde{a}^*, d^*_1, d^*_2, ..., d^*_l) \wedge \bigwedge_{i=0}^{l} (\phi_i(\tilde{a}^*, y_i) \text{ forks over } d^*_1 d^*_2 ... d^*_i) \quad (*2).$$

Now, by maximality of $\bar{d}^*$, $f(a^*) \in acl(\bar{d}^* d^*_{m+1})$. By taking a non-forking extension of $tp(E/acl(\bar{d}^* d^*_{m+1}))$ over $acl(d^*_1 ... d^*_l a^* \tilde{a}^*)$ we may assume that $a^* \tilde{a}^* d^*_1 ... d^*_l \underset{\bar{d}^* d^*_{m+1}}{\downarrow} E$ and (*1) and (*2) still hold. Thus $a^* \tilde{a}^* \underset{d^*_1 ... d^*_i}{\downarrow} d^*_1 ... d^*_i E$ for all $m+1 \leq i \leq l$. Hence by (*2), we conclude $\phi_i(\tilde{a}^*, y_i)$ forks over $d^*_1 d^*_2 ... d^*_i E$ for all $m+1 \leq i \leq l$. By (*1) and symmetry of $\downarrow^{s}$ (Lemma 6.7), there are stable $\psi_i(x_i, w_i) \in L$ and $\emptyset$-definable functions g_i for $1 \leq i \leq n$ such that if $a_i^* = g_i(f(a^*), \bar{d}^* c_1^* e_1^* ... c_i^*)$, then $\psi_i(a_i^*, e_i^*)$ and $\psi_i(a_i^*, w_i)$ forks over $\bar{d}^* c_1^* e_1^* ... c_i^*$. Now, let us define a relation V^* in the following way:

$V^*(a, d_1, ...d_m, c_1, e_1, ...c_n, e_n, d_{m+1}, ...d_l)$ iff $\exists \tilde{a}, \tilde{a}' = \tilde{a}_1' ... \tilde{a}_n' (\theta^* \wedge V_0 \wedge V_1 \wedge V_2)$

where, θ^* is defined by: $\theta^*(a, \tilde{a}, \tilde{a}', d_1, ...d_m, c_1, e_1, ...c_n, e_n, d_{m+1}, ...d_l) \equiv$

$$\theta(a, \tilde{a}, d_1, ...d_l) \wedge \bigwedge_{i=1}^{n} \psi_i(\tilde{a}_i', e_i) \wedge \ (\tilde{a}_i' = g_i(f(a), d_1, ...d_m, c_1, e_1, ...c_i))$$

V_0 is defined by:

$$V_0(\tilde{a}, d_1, ...d_m) \text{ iff } \bigwedge_{i=0}^{m} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 d_2 ... d_i)$$

V_1 is defined by:

$$V_1(\tilde{a}', d_1, ...d_m, c_1, e_1, ...c_n, e_n) \text{ iff } \bigwedge_{i=1}^{n} (\psi_i(\tilde{a}_i', w_i) \text{ forks over } d_1 d_2 ... d_m c_1 e_1 ... c_i),$$

and V_2 is defined by:

$$V_2(\tilde{a}, d_1, ..d_m, c_1, e_1, ..c_n, e_n, d_{m+1}, ..d_l) \text{ iff } \bigwedge_{i=m+1}^{l} (\phi_i(\tilde{a}, y_i) \text{ forks over } d_1 d_2 .. d_m c_1 e_1 .. c_n e_n d_{m+1} .. d_i).$$

Note that V^* is a pre-$\tilde{\tau}_{st}^f$-set relation. Thus

$\mathcal{U}^* = \{a | \exists d_1, ..d_m, c_1, e_1, ..c_n, e_n, d_{m+1}, ..d_l \, V^*(a, d_1, ..d_m, c_1, e_1, ..c_n, e_n, d_{m+1}, ..d_l)\}$

is a $\tilde{\tau}_{st}^f$-set. By the construction of $a^*, d_1^*, ..d_m^*, c_1^*, e_1^*, ..c_n^*, e_n^*, d_{m+1}^*, ..d_l^*$, $\mathcal{U}^* \neq \emptyset$. By the definition of $\mathcal{U}^*$, $\mathcal{U}^* \subseteq \mathcal{U} \cap (\mathcal{C}^s \backslash S_{f,n})$ (note that if $a \in \mathcal{U}^*$, then there are $d_1, ..., d_m \in \mathcal{C}$ such that $SU_{se}(f(a)/d_1...d_m) \geq n$ and thus by Remark 9.2, $SU_{se}(f(a)) \geq n$). So, the proof of Subclaim 9.4 is complete, and thus so is the proof of the theorem.

Recall that a theory T has the *wnfcp(=weak non finite cover property)* if for each L-formula $\phi(x, y)$, the D_ϕ-rank is finite and definable (the D_ϕ-rank of a formula $\psi(x, a)$ is defined by: $D_\phi(\psi(x, a)) \geq 0$ if $\psi(x, a)$ is consistent; $D_\phi(\psi(x, a)) \geq \alpha + 1$ if for some b, $D_\phi(\psi(x, a) \wedge \phi(x, b)) \geq \alpha$ and $\phi(x, b)$ divides over a; and for limit δ, $D_\phi(\psi(x, a)) \geq \delta$ if it is $\geq \alpha$ for all $\alpha < \delta$).

Corollary 9.5 *Let T be a countable imaginary simple unidimensional theory. Then T is low and thus has the wnfcp.*

Proof: By Fact 2.4, T has bounded finite SU-rank in any given sort. Thus the global D-rank in any given sort has a finite bound. Now, let $\phi(x,y) \in L$. Then $\phi(x,y)$ is low in x iff $Sup\{D(x=x,\phi(x,y),k) | \, k<\omega\} < \omega$. So, clearly every $\phi(x,y)$ is low in x. Thus T is low. By Corollary 3.8 the extension property is first-order in any unidimensional theory. We conclude T has the wnfcp (see [BPV], Corollary 4.6).

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