Analytic Zariski structures, predimensions and non-elementary stability

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The notion of an analytic Zariski structure was introduced in [1] by the author and N.Peatfield in a form slightly different from the one presented here. Analytic Zariski generalises the previously known notion of a Zariski structure (see [2] for one-dimensional case and [3], [4] for the general definition) mainly by dropping the requirement of Noetherianity and weakening the assumptions on the projections of closed sets. This leads to interesting new fenomena, in particular, the family of closed-in-open subsets forms a hierarchy which starts with *analytic sets* and continues by induction to more complex ones, called in [5] *generalised analytic sets* (defined classically on the complex numbers and in the context of rigid analytic geometry).

In [1] we assumed that the Zariski structure is compact (or compactifiable), here we drop this assumption (some interesting structures are not compactifiable in the strict sense).

The class of analytic Zariski structures is much broader and geometrically more rich than the class of Noetherian Zariski structures. The main examples come from two sources:

(i) structures which are constructed in terms of complex analytic functions and relations;

(ii) "new stable structures" introduced by Hrushovski's construction; in many cases these objects exhibit properties similar to those of class (i).

Although there are concrete examples for both (i) and (ii), in many cases we can only conjecturally identify a particular structure as an analytic Zariski one. In particular despite some attempts the conjecture that $\mathbb{C}_{\exp}$ is analytic Zariski is still open (even assuming this is the same as pseudoexponentiation).

Note that the results of [1] are valid in the present context, in particular, if M is compact and $\mathcal{C}^0$ is the subfamily of closed sets $S \subseteq M^n$, all n, which are analytic in M^n, then $(M, \mathcal{C}^0)$ is a Noetherian Zariski structure. Hence, by [3], $(M, \mathcal{C}^0)$ has elimination of quantifiers and is of finite Morley rank.

The aim of this paper is to carry out a model-theoretic analysis of the whole $\mathrm{M} = (M, \mathcal{C})$. We do it in the spirit of the theory of *abstarct elementary classes.* We start by introducing a suitable countable fragment of the family of basic Zariski relations and a correspondent substructure of constants over which all the further analysis is carried out. Then we proceed to the analysis of the notion of dimension, already present in Zariski setting. We define a more delicate notions of the *predimension* and *dimension* of a tuple in M. In fact by doing this we reinterprete dimensions which are present in every analytic structure in terms familiar to many from Hrushovski's construction, thus establishing once again conceptual links between classes (i) and (ii).

Our main results are proved under assumption that M is one-dimensional (as an analytic Zariski structure) and irreducible. No assumption on presmoothness is needed. We prove for such an M, in the terminology closely related to [6] and [7]:

1. M is quasiminimal with regards to a closure operator cl associated with the predimension;

2. M is homogeneous over countable submodels (see [6]) in the $\exists$-definable expansion of the language;

3. there are at most countably many $L_{\infty,\omega}$-types over any countable submodel realised in M (ω-stability).

It is tempting to conjecture that submodels of M form an excellent quasiminimal class. The condition for this is the amalgamation over independent submodels. This property is very difficult to prove in concrete cases (needs a lot of concrete algebra) but on the other hand we do not know counterexamples to this conjecture even in the present very general setting. We prove that if the class is indeed excellent then we can uniquely lift M to any uncountable cardinality.

I would like to remark that M.Gavrilovich in his DPhil thesis [8] proved 2 and 3 above for a very interesting class of actual complex analytic structures which includes universal covers of Abelian varieties. His method also produces a natural countable language for each case and exhibits deep relations of the model-theoretic conditions assumed below with some facts and conjectures of complex geometry.

Finally I want to mention that some natural questions in this context are widely open. In particular, we have no classification for presmooth analytic Zariski groups (with the graph of multiplication analytic).

I want to express my thanks to Assaf Hasson who saw a very early version of this work and made many useful comments.

1 Analytic Zariski structures

Let M $= (M, \mathcal{C})$ a structure with primitives $\mathcal{C}$. We assume that M^n, for all $n \geq 1$, is endowed with a topology and relations of $\mathcal{C}$ define the closed subsets of the M^n's. More precisely we assume for the primitives of the language:

(L)

1. arbitrary intersections of closed sets are closed;
2. finite unions of closed sets are closed;
3. the domain of the structure is closed;
4. the graph of equality is closed;
5. any singleton of the domain is closed;
6. Cartesian products of closed sets are closed;
7. the image of a closed $S \subseteq M^n$ under a permutation of coordinates is closed;
8. for $a \in M^k$ and S a closed subset of $M^{k+\ell}$ defined by a predicate $S(x^\frown y)$, $x \in M^k$, $y \in M^\ell$, the fibre over a, $S(a, M) := \{y \in M^\ell : a^\frown y \in S\}$, is closed.

L6 needs some clarification. If $S_1 \subseteq M^n$ and $S_2 \subseteq M^m$ are closed the assumption states that $S_1 \times S_2$ canonically identified with a subset of M^{n+m} is closed in the latter.

Remark 1.1 Note that it follows that projections $\mathrm{pr}_{i_1,\ldots,i_m} : M^n \to M^m$ are continuous in the sense that the inverse image of a closed set under a projection is closed. Indeed, $\mathrm{pr}^{-1} S = S \times M^{n-m}$.

We write $X \subseteq_{op} V$ to say that X is open in V and $X \subseteq_{\rm cl} V$ to say it is closed.

We say that $X \subseteq M^n$ is **$\mathcal{C}$-constructible** (usually omitting $\mathcal{C}$) if X is a finite union of some sets S, such that $S \subseteq_{\rm cl} U \subseteq_{op} M^n$.

A subset $P \subseteq M^n$ will be called **projective** if P is a union of finitely many sets of the form $\mathrm{pr}\, S$, for some $S \subseteq_{\rm cl} U \subseteq_{op} M^{n+k}$, $\mathrm{pr} : M^{n+k} \to M^n$.

Note that any set S such that $S \subseteq_{\rm cl} U \subseteq_{op} M^{n+k}$, is constructible, a projection of a constructible set is projective and that any constructible set is projective.

Dimension.

To any nonempty projective S a non-negative integer called **the dimension of** S, $\dim S$, is attached.

We assume:

(SI) (**strong irreducibility**) for an irreducible set $S \subseteq_{\rm cl} U \subseteq_{op} M^n$ (that is S is not a union of two proper closed subsets) and its closed subset $S' \subseteq_{\rm cl} S$,

$$\dim S' = \dim S \Rightarrow S' = S;$$

(DP) (**dimension of points**) for a nonempty projective S, $\dim S = 0$ if and only if S is at most countable.

(CU) (**countable unions**) If $S = \bigcup_{i\in\mathbb{N}} S_i$, all projective, then $\dim S = \max_{i\in\mathbb{N}} \dim S_i$;

(WP) (**weak properness**) given irreducible $S \subseteq_{\rm cl} U \subseteq_{op} M^n$ and $F \subseteq_{\rm cl} V \subseteq_{op} M^{n+k}$ with the projection $\mathrm{pr} : M^{n+k} \to M^n$ such that $\mathrm{pr}\, F \subseteq S$ and $\dim \mathrm{pr}\, F = \dim S$, there exists $D \subseteq_{op} S$ such that $D \subseteq \mathrm{pr}\, F$.

Remark 1.2 (CU) in the presence of (DCC) implies the *essential uncountability property* (EU) usually assumed for Noetherian Zariski structures.

We postulate further, for S constructible irreducible:

(AF) $\dim \mathrm{pr}\, S = \dim S - \min_{u\in \mathrm{pr}\, S} \dim(\mathrm{pr}^{-1}(u) \cap S)$;

(FC) The set $\{a \in \operatorname{pr} S : \dim(\operatorname{pr}^{-1}(a) \cap S) \geq k\}$ is of the form $T \cap \operatorname{pr} S$ for some constructible T, and there exists an open set V such that $V \cap \operatorname{pr} S \neq \emptyset$ and

$$\min_{a \in \operatorname{pr} S} \dim(\operatorname{pr}^{-1}(a) \cap S) = \dim(\operatorname{pr}^{-1}(v) \cap S), \text{ for any } v \in V \cap \operatorname{pr}(S).$$

The following helps to understand the dimension of projective sets.

Lemma 1.3 *Let $P = \operatorname{pr} S \subseteq M^n$, for S irreducible constructible, and $U \subseteq_{op} M^n$ with $P \cap U \neq \emptyset$. Then*

$$\dim P \cap U = \dim P.$$

Proof. We can write $P \cap U = \operatorname{pr} S' = P'$, where $S' = S \cap \operatorname{pr}^{-1} U$ constructible irreducible, $\dim S' = \dim S$ by (SI). By (FC), there is $V \subseteq_{op} M^n$ such that for all $c \in V \cap P$,

$$\dim \operatorname{pr}^{-1}(c) \cap S = \min_{a \in P} \dim \operatorname{pr}^{-1}(a) \cap S = \dim S - \dim P.$$

Note that $\operatorname{pr}^{-1} U \cap \operatorname{pr}^{-1} V \cap S \neq \emptyset$, since S ir irreducible. Taking $s \in \operatorname{pr}^{-1} U \cap \operatorname{pr}^{-1} V \cap S$ and $c = \operatorname{pr} s$ we get, using (AF) for S',

$$\dim \operatorname{pr}^{-1}(c) \cap S' = \dim \operatorname{pr}^{-1}(c) \cap S = \min_{a \in P'} \dim \operatorname{pr}^{-1}(a) \cap S = \dim S - \dim P'.$$

So, $\dim P' = \dim P$. □

Analytic subsets.

Definition 1.4 A subset S, $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$, is called **analytic in** U if for every $a \in S$ there is an open $V_a \subseteq_{op} U$ such that $S \cap V_a$ is the union of finitely many closed in V_a irreducible subsets.

We postulate the following properties

(INT) (**Intersections**) If $S_1, S_2 \subseteq_{an} U$ are irreducible then $S_1 \cap S_2$ is analytic in U;

(CMP) (**Components**) If $S \subseteq_{an} U$ and $a \in S$ then there is $S_a \subseteq_{an} U$, a finite union of irreducible analytic subsets of U, and some $S'_a \subseteq_{an} U$ such that $a \in S_a \setminus S'_a$ and $S = S_a \cup S'_a$;

Each of the irreducible subsets of S_a above is called an **irreducible component of S containing** a.

(CC) (**Countability of the number of components**) Any $S \subseteq_{an} U$ is a union of at most countably many irreducible components.

Remark 1.5 For S analytic and $a \in \mathrm{pr}\, S$, the fibre $S(a, M)$ is analytic.

Lemma 1.6 *If $S \subseteq_{an} U$ is irreducible, V open, then $S \cap V$ is an irreducible analytic subset of V and, if non-empty,* $\dim S \cap V = \dim S$.

Proof. Immediate.□

Lemma 1.7 *(i) $\emptyset$, any singleton and U are analytic in U;*

(ii) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cup S_2$ is analytic in U;

(iii) If $S_1 \subseteq_{an} U_1$ and $S_2 \subseteq_{an} U_2$, then $S_1 \times S_2$ is analytic in $U_1 \times U_2$;

(iv) If $S \subseteq_{an} U$ and $V \subseteq U$ is open then $S \cap V \subseteq_{an} V$;

(v) If $S_1, S_2 \subseteq_{an} U$ then $S_1 \cap S_2$ is analytic in U.

Proof. Immediate. □

Definition 1.8 Given a subset $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ we define the notion of the **analytic rank** of S in U, $\mathrm{ark}_U(S)$, which is a natural number satisfying

1. $\mathrm{ark}_U(S) = 0$ iff $S = \emptyset$;
2. $\mathrm{ark}_U(S) \leq k + 1$ iff there is a set $S' \subseteq_{\mathrm{cl}} S$ such that $\mathrm{ark}_U(S') \leq k$ and with the set $S^0 = S \setminus S'$ being analytic in $U \setminus S'$.

Obviously, any nonempty analytic subset of U has analytic rank 1.

Example 1.9 In [5] we have discussed the following notion of generalised analytic subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and, more generally, of $[\mathbf{P}^1(K)]^n$ for K algebraically closed complete valued field.

Let $F \subseteq \mathbb{C}^2$ be a graph of an entire analytic function and $\bar{F}$ its closure in $[\mathbf{P}^1(\mathbb{C})]^2$. It follows from Picar's Theorem that $\bar{F} = F \cup \{\infty\} \times \mathbf{P}^1(\mathbb{C})$, in particular $\bar{F}$ has analytic rank 2.

Generalised analytic sets are defined as the subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ for all n, obtained from classical (algebraic) Zariski closed subsets of $[\mathbf{P}^1(\mathbb{C})]^n$ and $\bar{F}$ by applying the positive operations: Cartesian products, finite intersections, unions and projections.

It has been proven in [5] (by a simple induction on the number of operation) that *any generalised analytic set is of finite analytic rank.*

The next assumptions guarantees that the class of analytic subsets explicitly determines the class of closed subsets in M.

(AS) [**Analytic stratification**] For any $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$, $\mathrm{ark}_U S$ is defined and finite.

We also are going to consider the property

(PS) [**Presmoothness**] If $S_1, S_2 \subseteq_{an} U \subseteq_{op} M^n$ both S_1, S_2 irreducible, then for any irreducible component S_0 of $S_1 \cap S_2$

$$\dim S_0 \geq \dim S_1 + \dim S_2 - \dim U.$$

Definition 1.10 A topological structure satisfying axioms (L)-(AS) above will be called an **analytic Zariski structure**. An analytic Zariski structure will be called **presmooth** if it has the presmoothness property (PS).

2 Model theory of analytic Zariski structures

Definition 2.1 Let M_0 be a nonempty subset of M and $\mathcal{C}_0$ a subfamily of $\mathcal{C}$. We will say that $(M_0, \mathcal{C}_0)$ is a **core substructure** if

1. if $\{\langle x_1, \ldots, x_n \rangle\} \in \mathcal{C}_0$ (a singleton) then $x_1, \ldots, x_n \in M_0$;

2. finite intersections of $\mathcal{C}_0$-closed sets are in $\mathcal{C}_0$;

3. $\mathcal{C}_0$ satisfies (L1)-(L7), and (L8) with $a \in M_0^k$;

4. $\mathcal{C}_0$ satisfies (WP), (AF), (FC) and (AS);

5. for any $\mathcal{C}_0$-constructible $S \subseteq_{an} U \subseteq_{op} M^n$, every irreducible component S_i of S is $\mathcal{C}_0$-constructible;

6. for any nonempty $\mathcal{C}_0$-constructible $U \subseteq M$, $\quad U \cap M_0 \neq \emptyset$.

Lemma 2.2 *Given any countable $N \subseteq M$ and countable $C \subseteq \mathcal{C}$ there exist countable $M_0 \supseteq N$ and $\mathcal{C}_0 \supseteq C$ such that $(M_0, \mathcal{C}_0)$ is a core substructure.*

Proof. Standard. □

We fix below a core substructure $(M_0, \mathcal{C}_0)$ with M_0 and $\mathcal{C}_0$ countable.

Definition 2.3 For finite $X \subseteq M$ we define the $\mathcal{C}_0$-predimension

$$\delta(X) = \min\{\dim S : \ \vec{X} \in S,\ S \subseteq_{an} U \subseteq_{op} M^n,\ S \text{ is } \mathcal{C}_0\text{-constructible}\}$$

and dimension

$$\partial(X) = \min\{\delta(XY) : \quad \text{finite } Y \subset M\}.$$

For $X \subseteq M$ finite, we say that X is **self-sufficient** and write $X \leq M$, if $\partial(X) = \delta(X)$.

For infinite $A \subseteq M$ we say $A \leq M$ if for any finite $X \subseteq A$ there is a finite $X \subseteq X' \subseteq A$ such that $X' \leq M$.

We work now under

Assumption $\dim M = 1$ and M is irreducible.

Note that we then have

$$0 \leq \delta(Xy) \leq \delta(X) + 1, \text{ for any } y \in M,$$

since $\vec{X}y \in S \times M$.

Lemma 2.4 *Given $F \subseteq_{an} U \subseteq_{op} M^k$, $\dim F > 0$, there is $i \le k$ such that for $\mathrm{pr}_i : (x_1, \ldots, x_k) \mapsto x_i$,*

$$\dim \mathrm{pr}_i F > 0.$$

Proof. Use (AF) and induction on k.□

Proposition 2.5 *Let $P = \mathrm{pr}\, S$, for some $\mathcal{C}_0$-constructible $S \subseteq_{an} U \subseteq_{op} M^{n+k}$, $\mathrm{pr} : M^{n+k} \to M^n$. Then*

$$\dim P = \max\{\partial(x) : \ x \in P(M)\}. \qquad (1)$$

Moreover, this formula is true when $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^{n+k}$.

Proof. We use induction on $\dim S$.

We first note that by induction on $\mathrm{ark}_U S$, if (1) holds for all analytic S of dimension less or equal to k then it holds for all closed S of dimension less or equal to k.

The statement is obvious for $\dim S = 0$ and so we assume that $\dim S > 0$ and for all analytic S' of lower dimension the statement is true.

By (CU) and (CMP) we may assume that S is irreducible. Then by (AF)

$$\dim P = \dim S - \dim S(c, M) \qquad (2)$$

for any $c \in P(M) \cap V(M)$ (such that $S(c, M)$ is of minimal dimension) for some open $\mathcal{C}_0$-constructible V.

Claim 1. It is enough to prove the statement of the proposition for the projective set $P \cap V'$, for some $\mathcal{C}_0$-open $V' \subseteq_{op} M^n$.

Indeed,

$$P \cap V' = \mathrm{pr}(S \cap \mathrm{pr}^{-1} V'), \quad S \cap \mathrm{pr}^{-1} V' \subseteq_{\mathrm{cl}} \mathrm{pr}^{-1} V' \cap U \subseteq_{op} M^{n+k}.$$

And $P \setminus V' = \mathrm{pr}(S \cap T)$, $T = \mathrm{pr}^{-1}(M^n \setminus V') \in \mathcal{C}_0$. So, $P \setminus V'$ is the projection of a proper analytic subset, of lower dimension. By induction, for $x \in P \setminus V'$, $\partial(x) \le \dim P \setminus V' \le \dim P$ and hence, using 1.3,

$$\dim P \cap V' = \max\{\partial(x) : \ x \in P \cap V'\} \Rightarrow \dim P = \max\{\partial(x) : \ x \in P\}.$$

Claim 2. The statement of the proposition holds if $\dim S(c, M) = 0$ in (2).

Proof. Given $x \in P$ choose a tuple $y \in M^k$ such that $S(x^\frown y)$ holds. Then $\delta(x^\frown y) \le \dim S$. So we have $\partial(x) \le \delta(x^\frown y) \le \dim S = \dim P$.

It remains to notice that there exists $x \in P$ such that $\partial(x) \ge \dim P$.

Consider the $\mathcal{C}_0$-type

$$x \in P \ \&\{x \notin R : \ \dim R \cap P < \dim P \text{ and } R \text{ is projective}\}.$$

This is realised in M, since otherwise $P = \bigcup_R (P \cap R)$ which would contradict (CU) because (M_0, C_0) is countable.

For such an x let y be a tuple in M such that $\delta(x^\frown y) = \partial(x)$. By definition there exist $S' \subseteq_{an} U' \subseteq_{op} M^m$ such that $\dim S' = \delta(x^\frown y)$. Let $P' = \operatorname{pr} S'$, the projection into M^n. By our choice of x, $\dim P' \ge \dim P$. But $\dim S' \ge \dim P'$. Hence, $\partial(x) \ge \dim P$. Claim proved.

Claim 3. There is a $\mathcal{C}_0$-constructible $R \subseteq_{an} S$ such that all the fibers $R(c, M)$ of the projection map $R \to \operatorname{pr} R$ are 0-dimensional and $\dim \operatorname{pr} R = \dim P$.

Proof. We have by construction $S(c, M) \subseteq M^k$. Assuming $\dim S(c, M) > 0$ on every open subset we show that there is a $b \in M_0$ such that (up to the order of coordinates) $\dim S(c, M) \cap \{b\} \times M^{k-1} < \dim S(c, M)$, for all $c \in P \cap V' \ne \emptyset$, for some open $V' \subseteq V$ and $\dim \operatorname{pr} S(c, M) \cap \{b\} \times M^{k-1} = \dim P$. By induction on $\dim S$ this will prove the claim.

To find such a b choose $a \in P \cap V$ and note that by 2.4, up to the order of coordinates, $\dim \operatorname{pr}_1 S(a, M) > 0$, where $\operatorname{pr}_1 : M^k \to M$ is the projection on the first coordinate.

Consider the projection $\operatorname{pr}_{M^n,1} : M^{n+k} \to M^{n+1}$ and the set $\operatorname{pr}_{M^n,1} S$. By (AF) we have

$$\dim \operatorname{pr}_{M^n,1} S = \dim P + \dim \operatorname{pr}_1 S(a, M) = \dim P + 1.$$

Using (AF) again for the projection $\operatorname{pr}^1 : M^{n+1} \to M$ with the fibers $M^n \times \{b\}$, we get, for all b in some open subset of M,

$$1 \ge \dim \operatorname{pr}^1 \operatorname{pr}_{M^n,1} S = \dim \operatorname{pr}_{M^n,1} S - \dim[\operatorname{pr}_{M^n,1} S] \cap [M^n \times \{b\}] =$$

$$= \dim P + 1 - \dim[\operatorname{pr}_{M^n,1} S] \cap [M^n \times \{b\}].$$

Hence $\dim[\operatorname{pr}_{M^n,1} S] \cap [M^n \times \{b\}] \ge \dim P$, for all such b, which means that the projection of the set $S_b = S \cap (M^n \times \{b\} \times M^{k-1})$ on M^n is of dimension $\dim P$, which finishes the proof if $b \in M_0$. But $\dim S_b = \dim S - 1$ for all

$b \in M \cap V'$, some $\mathcal{C}_0$-open V', so for any $b \in M_0 \cap V'$. The latter is not empty since $(M_0, \mathcal{C}_0)$ is a core substructure. This proves the claim.

Claim 4. Given R satisfying Claim 3,

$$P \setminus \operatorname{pr} R \subseteq \operatorname{pr} S', \text{ for some } S' \subseteq_{\mathrm{cl}} S, \ \dim S' < \dim S.$$

Proof. Consider the cartesian power

$$M^{n+2k} = \{x^\frown y^\frown z : x \in M^n, \ y \in M^k, \ z \in M^k\}$$

and its $\mathcal{C}_0$-constructible subset

$$R\&S := \{x^\frown y^\frown z : \quad x^\frown z \in R \ \& \ x^\frown y \in S\}.$$

Clearly $R\&S \subseteq_{an} W \subseteq_{op} M^{n+2k}$, for an appropriate $\mathcal{C}_0$-constructible W.

Now notice that the fibers of the projection $\operatorname{pr}_{xy} : x^\frown y^\frown z \mapsto x^\frown y$ over $\operatorname{pr}_{xy} R\&S$ are 0-dimensional and so, for some irreducible component $(R\&S)^0$ of the analytic set $R\&S$, $\dim \operatorname{pr}_{xy}(R\&S)^0 = \dim S$. Since $\operatorname{pr}_{xy} R\&S \subseteq S$ and S irreducible, we get by (WP) $D \subseteq \operatorname{pr}_{xy} R\&S$ for some $D \subseteq_{op} S$. Clearly

$$\operatorname{pr} R = \operatorname{pr} \operatorname{pr}_{xy} R\&S \supseteq \operatorname{pr} D$$

and $S' = S \setminus D$ satisfies the requirement of the claim.

Now we complete the proof of the proposition: By Claims 2 and 3

$$\dim P = \max_{x \in \operatorname{pr} R} \partial(x).$$

By induction on $\dim S$, using Claim 4, for all $x \in P \setminus \operatorname{pr} R$,

$$\partial(x) \leq \dim \operatorname{pr} S' \leq \dim P.$$

The statement of the proposition follows. □

In what follows a $\mathcal{C}_0$-substructure of M is a $\mathcal{C}_0$-structure on a subset $N \supseteq M_0$. Recall that $\mathcal{C}_0$ is purely relational.

Recall the following well-known

Fact Given $a, a' \in M^n$ the $L_{\infty,\omega}(\mathcal{C}_0)$-types of the two n-tuples in M are equal if and only if they are *back and forth equivalent* that is there is a nonempty set I of isomorphisms of $\mathcal{C}_0$-substructures of M such that $a \in \mathrm{Dom}\, f_0$ and $a' \in \mathrm{Range}\, f_0$, for some $f_0 \in I$, and

(forth) for every $f \in I$ and $b \in M$ there is a $g \in I$ such that $f \subseteq g$ and $b \in \mathrm{Dom}\, g$;

(back) For every $f \in I$ and $b' \in M$ there is a $g \in I$ such that $f \subseteq g$ and $b' \in \mathrm{Range}\, g$.

Definition 2.6 For $a \in M^n$, the **projective type of** a **over** M is

$$\{P(x):\ a \in P,\ P \text{ is a projective set over } \mathcal{C}_0\} \cup$$

$$\cup \{\neg P(x):\ a \notin P,\ P \text{ is a projective set over } \mathcal{C}_0\}.$$

Lemma 2.7 *Suppose $X \le M$, $X' \le M$ and the (first-order) quantifier-free $\mathcal{C}_0$-type of X is equal to that of X'. Then the $L_{\infty,\omega}(\mathcal{C}_0)$-types of X and X' are equal.*

Proof. We are going to construct a back-and-forth system for X and X'.

Let $S_X \subseteq_{an} V \subseteq_{op} M^n$, S_X irreducible, all $\mathcal{C}_0$-constructible, and such that $X \in S_X(M)$ and $\dim S_X = \delta(X)$.

Claim 1. The quantifier-free $\mathcal{C}_0$-type of X (and X') is determined by formulas equivalent to $S_X \cap V'$, for V' open such that $X \in V'(M)$.

Proof. Use the stratification of closed sets (AS) to choose $\mathcal{C}_0$-constructible $S \subseteq_{\mathrm{cl}} U \subseteq_{op} M^n$ such that $X \in S$ and $\mathrm{ark}_U S$ is minimal. Obviously then $\mathrm{ark}_U S = 0$, that is $S \subseteq_{an} U \subseteq_{op} M^n$. Now S can be decomposed into irreducible components, so we may choose S to be irreducible. Among all such S choose one which is of minimal possible dimension. Obviously $\dim S = \dim S_X$, that is we may assume that $S = S_X$. Now clearly any constructible set $S' \subseteq_{\mathrm{cl}} U' \subseteq_{op} M^n$ containing X must satisfy $\dim S' \cap S_X \ge \dim S_X$, and this condition is also sufficient for $X \in S'$.

Let y be an element of M. We want to find a finite Y containing y and an Y' such that the quantifier-free type of XY is equal to that of $X'Y'$ and both are self-sufficient in M. This, of course, extends the partial isomorphism $X \to X'$ to $XY \to X'Y'$ and will prove the lemma.

We choose Y to be a minimal set containing y and such that $\delta(XY)$ is also minimal, that is

$$1 + \delta(X) \ge \delta(Xy) \ge \delta(XY) = \partial(XY)$$

and $XY \leq M$.

We have two cases: $\delta(XY) = \partial(X) + 1$ and $\delta(XY) = \partial(X)$. In the first case $Y = \{y\}$. By Claim 1 the quantifier-free $\mathcal{C}_0$-type r_{Xy} of Xy is determined by the formulas of the form $(S_X \times M) \setminus T$, $T \subseteq_{cl} M^{n+1}$, $T \in \mathcal{C}_0$, $\dim T < \dim(S_X \times M)$.

Consider

$$r_{Xy}(X', M) = \{z \in M : X'z \in (S_X \times M) \setminus T,\ \dim T < \dim S_X,\ \text{all } T\}.$$

We claim that $r_{Xy}(X', M) \neq \emptyset$. Indeed, otherwise M is the union of countably many sets of the form $T(X', M)$. But the fibers $T(X', M)$ of T are of dimension 0 (since otherwise $\dim T = \dim S_X + 1$, contradicting the definition of the T). This is impossible, by (CU).

Now we choose $y' \in r_{Xy}(X', M)$ and this is as required.

In the second case, by definition, there is an irreducible $R \subseteq_{an} U \subseteq_{op} M^{n+k}$, $n = |X|, k = |Y|$, such that $XY \in R(M)$ and $\dim R = \delta(XY) = \partial(X)$. We may assume $U \subseteq V \times M^k$.

Let $P = \operatorname{pr} R$, the projection into M^n. Then $\dim P \leq \dim R$. But also $\dim P \geq \partial(X)$, by 2.5. Hence, $\dim R = \dim P$. On the other hand, $P \subseteq S_X$ and $\dim S_X = \delta(X) = \dim P$. By axiom (WP) we have $S_X \cap V' \subseteq P$ for some $\mathcal{C}_0$-constructible open V'.

Hence $X' \in S_X \cap V' \subseteq P(M)$, for P the projection of an irreducible analytic set R in the $\mathcal{C}_0$-type of XY. By Claim 1 the quantifier-free $\mathcal{C}_0$-type of XY is of the form

$$r_{XY} = \{R \setminus T : T \subseteq_{\text{cl}} R,\ \dim T < \dim R\}.$$

Consider

$$r_{XY}(X', M) = \{Z \in M^k : X'Z \in R \setminus T,\ T \subseteq_{\text{cl}} R,\ \dim T < \dim R\}.$$

We claim again that $r_{XY}(X', M) \neq \emptyset$. Otherwise the set $R(X', M) = \{X'Z : R(X'Z)\}$ is the union of countably many subsets of the form $T(X', M)$. But $\dim T(X', M) < \dim R(X', M)$ as above, by (AF).

Again, an $Y' \in r_{XY}(X', M)$ is as required.$\square$

Corollary 2.8 *There is at most countably many $L_{\infty,\omega}(\mathcal{C}_0)$-types of tuples $X \leq M$.*

Indeed, any such type is determined uniquely by the choice of a $\mathcal{C}_0$-constructible $S_X \subseteq_{an} U \subseteq_{op} M^n$ such that $\dim S_X = \partial(X)$.

Lemma 2.9 *Suppose, for finite $X, X' \subseteq M$, the projective $\mathcal{C}_0$-types of X and X' coincide. Then the $L_{\infty,\omega}(\mathcal{C}_0)$-types of the tuples are equal.*

Proof. Choose finite Y such that $\partial(X) = \delta(XY)$. Then $XY \leq M$. Let $XY \in S \subseteq_{an} U \subseteq_{op} M^n$ be $\mathcal{C}_0$-constructible and such that $\dim S$ is minimal possible, that is $\dim S = \delta(XY)$. We may assume that S is irreducible. Notice that for every proper closed $\mathcal{C}_0$-constructible $T \subseteq_{\rm cl} U$, $XY \notin T$ by dimension considerations.

By assumptions of the lemma $X'Y' \in S$, for some Y' in M. We also have $X'Y' \notin T$, for any T as above, since otherwise a projective formula would imply that $XY'' \in T$ for some Y'', contradicting that $\partial(X) > \dim T$.

We also have $\delta(X'Y') = \dim S$. But for no finite Z' it is possible that $\delta(X'Z') < \dim S$, for then again a projective formula will imply that $\delta(XZ) < \dim S$, for some Z.

It follows that $X'Y' \leq M$ and the quantifier-free types of XY and $X'Y'$ coincide, hence the $L_{\infty,\omega}(\mathcal{C}_0)$-types are equal, by 2.7.$\Box$

Definition 2.10 Set, for finite $X \subseteq M$,

$$\mathrm{cl}_{\mathcal{C}_0}(X) = \{y \in M : \ \partial(Xy) = \partial(X)\}.$$

We fix $\mathcal{C}_0$ and omit the subscript below.

Lemma 2.11 *$b \in \mathrm{cl}(A)$, for $\vec{A} \in M^n$, if and only if $b \in P(\vec{A}, M)$ for some projective $P \subseteq M^{n+1}$ such that $P(\vec{A}, M)$ is at most countable. In particular, $\mathrm{cl}(A)$ is countable for any finite A.*

Proof. Let $d = \partial(A) = \delta(AV)$, and $\delta(AV)$ is minimal for all possible finite $V \subseteq M$. So by definition $d = \dim S_0$, some analytic irreducible S_0 such that $\vec{AV} \in S_0$ and S_0 of minimal dimension. This corresponds to a $\mathcal{C}_0$-definable relation $S_0(x, v)$, where x, v strings of variables of length n, m

First assume that b belongs to a countable $P(\vec{A}, M)$. By definition

$$P(x, y) \equiv \exists w \, S(x, y, w),$$

for some analytic $S \subseteq M^{n+1+k}$, x, y, w strings of variables of length $n, 1$ and k and the fiber $S(\vec{A}, b, M^k)$ is nonempty. We also assume that P and S are of minimal dimension, answering this description. By (FC), (AS) and minimality we may choose S so that $\dim S(\vec{A}, b, M^k)$ is minimal among all the fibers $S(\vec{A}', b', M^k)$.

Consider the analytic set $S^\sharp \subseteq M^{n+m+1+k}$ given by $S_0(x, v) \,\&\, S(x, y, w)$. By (AF), considering the projection of the set on (x, v)-coordinates,

$$\dim S^\sharp \leq \dim S_0 + \dim S(\vec{A}, M, M^k),$$

since $S(\vec{A}, M, M^k)$ is a fiber of the projection. Now we note that by countability $\dim S(\vec{A}, M, M^k) = \dim S(\vec{A}, b, M^k)$, so

$$\dim S^\sharp \leq \dim S_0 + \dim S(\vec{A}, b, M^k).$$

Now the projection $\mathrm{pr}_w S^\sharp$ along w (corresponding to $\exists w\, S^\sharp$) has fibers of the form $S(\vec{X}, y, M^k)$, so by (AF)

$$\dim \mathrm{pr}_w S^\sharp \leq \dim S_0 = d.$$

Projecting further along v we get $\dim \mathrm{pr}_v \mathrm{pr}_w S^\sharp \leq d$, but $\vec{A}b \in \mathrm{pr}_v \mathrm{pr}_w S^\sharp$ so by Proposition 2.5 $\partial(\vec{A}b) \leq d$. The inverse inequality holds by definition, so the equality holds. This proves that $b \in \mathrm{cl}(A)$.

Now, for the converse, we assume that $b \in \mathrm{cl}(A)$. So, $\partial(\vec{A}b) = \partial(\vec{A}) = d$. By definition there is a projective set P containing $\vec{A}b$, defined by the formula $\exists w\, S(x, y, w)$ for some analytic S, $\dim S = d$. Now $\vec{A}$ belongs to the projective set $\mathrm{pr}_y P$ (defined by the formula $\exists y \exists w\, S(x, y, w)$) so by Proposition 2.5 $d \leq \dim \mathrm{pr}_y P$, but $\dim \mathrm{pr}_y P \leq \dim P \leq \dim S = d$. Hence all the dimensions are equal and so, the dimension of the generic fiber is 0, but as above We may assume without loss of generality that all fibers are of minimal dimension, so

$$\dim S(\vec{A}, M, M^k) = 0.$$

Hence, b belongs to a 0-dimensional set $\exists w\, S(\vec{A}, y, w)$, which is projective and countable. □

Lemma 2.12 *(i)*

$$\mathrm{cl}(\emptyset) = \mathrm{cl}(M_0) = M_0.$$

(ii) Given finite $X \subseteq M$, $y, z \in M$,

$$z \in \mathrm{cl}(X, y) \setminus \mathrm{cl}(X) \Rightarrow y \in \mathrm{cl}(X, z).$$

(iii)

$$\mathrm{cl}(\mathrm{cl}(X)) = \mathrm{cl}(X).$$

Proof. (i) Clearly $M_0 \subseteq \mathrm{cl}(\emptyset)$, by definition.

We need to show the converse, that is if $\partial(y) = 0$, for $y \in M$, then $y \in M_0$. By definition $\partial(y) = \partial(\emptyset) = \min\{\delta(Y) : y \in Y \subset M\} = 0$. So, $y \in Y$, $\vec{Y} \in S \subseteq_{an} U \subseteq_{op} M^n$, $\dim S = 0$. The irreducible components of S are points (singletons), so $\{\vec{Y}\}$ is one and must be in $\mathcal{C}_0$, since $(M_0, \mathcal{C}_0)$ is a core substructure. By 2.1.1, $y \in M_0$.

(ii) Assuming the left-hand side $\partial(Xyz) = \partial(Xy) > \partial(X)$, $\partial(Xz) > \partial(X)$. By the definition of ∂ then,

$$\partial(Xy) = \partial(X) + 1 = \partial(Xz),$$

so $\partial(Xzy) = \partial(Xz)$, $y \in \mathrm{cl}(Xz)$.

(iii) Immediate by 2.11. □

Below, if not stated otherwise, we use the language $\mathcal{C}_0^{\exists}$ the primitives of which correspond to relations $\exists$-definable in M. Also, we call a **submodel of** M any $\mathcal{C}_0^{\exists}$-substructure closed under cl.

Theorem 2.13 *(i) Every $L_{\infty,\omega}(\mathcal{C}_0)$-type realised in* M *is equivalent to a projective type, that is a type consisting of existential (first-order) formulas and the negations of existential formulas.*

(ii) There are only countably many $L_{\infty,\omega}(\mathcal{C}_0)$-types realised in M.

(iii) $(M, \mathcal{C}_0^{\exists})$ is quasiminimal ω-homogeneous over countable submodels.

Proof. (i) Immediate from 2.9.

(ii) By 2.8 there are only countably many types of finite tuples $Z \leq M$. Let $N \subseteq M_0$ be a countable subset of M such that any finite $Z \leq M$ is $L_{\infty,\omega}(\mathcal{C}_0)$-equivalent to some tuple in N. Every finite tuple $X \subset M$ can be extended to $XY \leq M$, so there is a $L_{\infty,\omega}(\mathcal{C}_0)$-monomorphism $XY \to N$. This monomorphism identifies the $L_{\infty,\omega}(\mathcal{C}_0)$-type of X with one of a tuple in N, hence there are no more than countably many such types.

(iii) Lemma 2.12 proves that cl defines a pregeometry on M. Letting $G := \mathrm{cl}_{\mathcal{C}_0}(A)$ we we need to check, according to [6], that (a)for any X and X', both cl-independent over G, of the same size n, a bijection $\phi : X \to X'$ is a G-monomorphism, and (b) given any G-monomorphism $\phi : X \to X'$ and $y \in M$ we can extend ϕ so that $y \in \mathrm{Dom}\,\phi$.

Consider first (a). Note that $GX \leq M$ and $GX' \leq M$ and so the types of X and X' over G are $\mathcal{C}_0$-quantifier-free. But there is no proper $\mathcal{C}_0$-closed subset $S \subseteq_{\mathrm{cl}} M^n$ such that $\vec{X} \in S$ or $\vec{X'} \in S$. Hence the types are equal.

For (b) just use the fact that the G-monomorphism by our definition preserves $\exists$-formulas, so by 2.9 complete $L_{\infty,\omega}(\mathcal{C}_0)$-types of X and X' coincide, so ϕ can be extended. $\Box$

Now we want to define an *abstract elementary class* $\mathcal{A}$ associated with M. Set

$$\mathcal{A}_0(\mathrm{M}) = \{ \text{ countable } \mathcal{C}_0^{\exists}\text{-structures N} : \ \mathrm{N} \cong \mathrm{N}' \subseteq \mathrm{M}, \ \mathrm{cl}(N') = N' \}$$

and define embedding $\mathrm{N}_1 \preccurlyeq \mathrm{N}_2$ in the class as an $\mathcal{C}_0$-embedding $f : \mathrm{N}_1 \to \mathrm{N}_2$ such that there are isomorphisms $g_i : \mathrm{N}_i \to \mathrm{N}_i'$, $\mathrm{N}_1' \subseteq \mathrm{N}_2' \subseteq \mathrm{M}$, all embeddings commuting and $\mathrm{cl}(N_i') = N_i'$.

Recall the construction from [9] (of a special set and quasi-minimal excellence) corresponding to Shelah's notion of an independent system and excellence.

Let $B \subset M$ be a countable cl-independent set and $B_1, \ldots, B_k \subset B$, $\bigcup B_i = B$. Consider the structures $N_i = \mathrm{cl}(B_i)$, these obviously belong to $\mathcal{A}_0(\mathrm{M})$. A system of structures N_i of the form $\mathrm{cl}(B_i)$ as above will be called an **independent system** in $\mathcal{A}_0(\mathrm{M})$.

We say that $\mathcal{A}_0(\mathrm{M})$ is **excellent** if for any independent system $\{N_i : i \leq k\}$ and a finite $X \subset C = \mathrm{cl}(\bigcup_i N_i)$ the $L_{\infty,\omega}(\mathcal{C}_0)$-type of X over C is equivalent to a subtype over a finite $C' \subset C$.

Now define $\mathcal{A}(\mathrm{M})$ to be the class of $\mathcal{C}_0$-structures H with $\mathrm{cl}_{\mathcal{C}_0}$ defined with respect to H and satisfying:

(i) $\mathcal{A}_0(\mathrm{H}) \subseteq \mathcal{A}_0(\mathrm{M})$ as classes with embeddings

and

(ii) for every finite $X \subseteq H$ there is $\mathrm{N} \in \mathcal{A}_0(\mathrm{H})$, such that $X \subseteq N$.

Note that in any $\mathrm{H} \in \mathcal{A}(\mathrm{M})$, given a finite $X \subseteq H$, the set $\mathrm{cl}(X)$ is countable. That is $\mathcal{A}(\mathrm{M})$ *satisfies the countable closure property.* Indeed, by definition there exists a countable $\mathrm{N} \in \mathcal{A}_0(\mathrm{H})$ with $X \subseteq N$. Since $\mathrm{cl}_{\mathrm{H}}(N) = N$, also $\mathrm{cl}_{\mathrm{H}}(X) \subseteq N$.

Given $\mathrm{H}_1 \subseteq \mathrm{H}_2$, $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{A}(\mathrm{M})$, we define $\mathrm{H}_1 \preccurlyeq \mathrm{H}_2$ to hold in the class, if for every finite $X \subseteq H_1$, $\mathrm{cl}(X)$ is the same in H_1 and H_2. More generallly, for $\mathrm{H}_1, \mathrm{H}_2 \in \mathcal{A}(\mathrm{M})$ we define $\mathrm{H}_1 \preccurlyeq_f \mathrm{H}_2$ to be an embedding f such that there are isomorphisms $\mathrm{H}_1 \cong \mathrm{H}_1'$, $\mathrm{H}_2 \cong \mathrm{H}_2'$ such that $\mathrm{H}_1' \subseteq \mathrm{H}_2'$, all embeddings commute, and $\mathrm{H}_1 \preccurlyeq \mathrm{H}_2$.

Lemma 2.14 $\mathcal{A}(\mathrm{M})$ *is closed under the unions of ascending* $\preccurlyeq$*-chains.*

Proof. Immediate from the fact that for infinite $Y \subseteq M$,

$$\mathrm{cl}(Y) = \bigcup \{\mathrm{cl}(X) : \ X \subseteq_{\text{finite}} Y\}.$$

□

Theorem 2.15 *Given an analytic Zariski structure* M *and a countable core substructure* $(M_0, \mathcal{C}_0)$, *assume that* $\mathcal{A}_0(\mathrm{M})$ *is excellent. Then the class* $\mathcal{A}(\mathrm{M})$ *contains structures of any infinite cardinality and is categorical in uncountable cardinals.*

Proof. This is essentially the same proof that of Theorem 2 of [9], which is based on Theorem 1 of the same paper.

The starting point of the proof in [9] establishes the existence in the class of an H_0 of the form $H_0 = \mathrm{cl}(A_0)$, for A_0 an infinite countable cl-independent set. Under the present assumptions this is immediate since such H_0 exists already in $\mathcal{A}_0(\mathrm{M})$.

The rest of the proof just repeats that of Theorem 2 of [9]. One constructs an ascending chain of structures $\mathrm{H}_\alpha \in \mathcal{A}(\mathrm{M})$ of the form $\mathrm{cl}(A_\alpha)$, for some cl-independent sets A_α, up to a given cardinal κ, so that $A_{\alpha+1} \setminus A_\alpha = \{a_\alpha\}$. $\mathrm{H}_{\alpha+1}$ is defined uniquely up to isomorphism by a bijection $\psi_\alpha : A_\alpha \to A_{\alpha+1}$ and the fact proved in Theorem 1 of [9]: under assumptions of quasi-minimal excellence ψ_α can be extended to an isomorphism of $\mathrm{cl}(A_\alpha) \to \mathrm{cl}(A_{\alpha+1})$. □

Problem Given an analytic Zariski structure M is there always a big enough countable core substructure $(M_0, \mathcal{C}_0)$ such that the class $\mathcal{A}_0(\mathrm{M})$ is excellent?

The example in the next section, of the cover of the algebraic torus $K^{\times}$, for K an algebraically closed field, is excellent for a natural choice of $\mathcal{C}_0$ and some M_0. The proof even in this basic example is difficult. In other, more general cases, the answer is not known or known partially and is closely linked to difficult problems in arithmetic algebraic geometry, see [11].

We introduce one more condition which holds for many known analytic Zariski structures. We say that **the language of** M **is essentially countable**, (CL) for short, if there is a countable $\mathcal{C}_{\text{base}} \subset \mathcal{C}$ such that every $S \in \mathcal{C}$ is of the form $S = P(a, M)$, for some $P \in \mathcal{C}_{\text{base}}$ and $a \in M^k$.

Proposition 2.16 *Under assumptions of Theorem 2.15, if also the language of* M *is essentially countable and assuming without loss of generality* $\mathcal{C}_{\text{base}} \subseteq \mathcal{C}_0$, *any uncountable* $\mathrm{H} \in \mathcal{A}(\mathrm{M})$ *is an analytic Zariski structure in the language* $\mathcal{C}_0$ *with parameters in* H. *Also* H *is presmooth if* M *is.*

Proof. We define $\mathcal{C}(\mathrm{H})$ to consist of the subsets of H^n of the form $P(a, H)$ for $P \in \mathcal{C}_0$ of arity $k + n$, $a \in H^k$. The assumption (L) is obviously satisfied.

Now note that the constructible and projective sets in $\mathcal{C}(\mathrm{H})$ are also of the form $P(a, H)$ for some $\mathcal{C}_0$-constructible or $\mathcal{C}_0$-projective P.

Define $\dim P(a, H) = d$ if $\dim P(b, M) = d$ for some $b \in M^k$ such that The $\mathcal{C}_0^{\exists}$-quantifier-free types of a and b are equal. This is well-defined by (FC) and the fact that the any $\mathcal{C}_0^{\exists}$-quantifier-free type realised in H is also realised in M. Moreover, we have the following.

Claim. The set of $\mathcal{C}_0^{\exists}$-quantifier-free types realised in H is equal to that realised in M.

Indeed, this is immediate from the definition of the class $\mathcal{A}(\mathrm{M})$, stability of $\mathcal{A}(\mathrm{M})$ and the fact that the class is categorical in uncountable cardinalities.

The definition of dimension immediately implies (DP), (CU),(AF) and (FC) for H.

(SI): if $P_1(a_1, H) \subseteq_{\text{cl}} P_0(a_0, H)$, $\dim P_1(a_1, H) = \dim P_0(a_0, H)$ and the two sets are not equal, then the same holds for $P_1(b_1, M)$ and $P_0(b_0, M)$ for equivalent b_0, b_1 in M. Then, $P_0(b_0, M)$ is reducible, tha is for some proper $P_2(b_2, M) \subseteq_{\text{cl}} P_0(b_0, M)$, $P_0(b_0, M) = P_1(b_1, M) \cup P_2(b_2, M)$. Now, by homogeneity we can choose a_2 in H such that $P_0(a_0, H) = P_1(a_1, H) \cup P_2(a_2, H)$, a reducible representation.

This also shows that the notion of irreducibility is preserved by equivalent substituition of parameters. Then the same is true for the notion of analytic subset. Hence (INT), (CMP),(CC) and (PS) follow. For the same reason (AS) holds. Next we notice that the axioms (WP) follows by the homogeneity argument. □

3 Some examples

We consider the **universal cover of** $\mathbb{C}^\times$ as a topological structure and show that this is analytic Zariski.

This is a structure with the universe V identified with the set of complex numbers $\mathbb{C}$ and we are going to use the additive structure on it. We also consider the usual exponentiation map

$$\exp : V \to \mathbb{C}^\times$$

and want to take into our language and topology the usual Zariski topology (of algebraic varieties) on $(\mathbb{C}^\times)^n$ as well as exp as a continuous map.

A model-theoretic analysis of this structure was carried out in [9], [10], [11] and in the DPhil thesis [12] of Lucy Smith. The latter work, based on [9], provides the description of the topology $\mathcal{C}$ on V and proves that $(V, \mathcal{C})$ is analytic Zariski. (It then addresses the issue of possible *compacifications* of the structure).

Note that the whole analysis below up to Corollary 3.9 uses only the first order theory of the structure $(V, \mathcal{C})$, so one can wonder what changes if we replace $\mathbb{C}$ and exp with its abstract analogues. The answer to this question is known in the form of the categoricity theorem proved in [9] and [6] (see some corrections in [10]): *if* $\mathrm{ex} : U \to K^\times$ *is a group homomorphism,* U *a divisible torsion-free group,* K *an algebraically closed field of characteristic* 0 *and cardinality continuum and* $\ker(\mathrm{ex})$ *is cyclic, then the structure is isomorphic to the structure* $(V, \mathcal{C})$ *on the complex numbers.* More generally, any two covers of 1-tori over algebraically closed fields of characteristic 0 of the same uncountable cardinality and with cyclic kernels are isomorphic.

We follow [12] pp.17-25 with modifications and omission of some technical details.

The base of the PQF-topology (positive quantifier-free) on V and its cartesian powers V^n is, in short, the family of subsets of V^n defined by PQF-formulae.

Definition 3.1 A PQF-closed set is defined as a finite union of sets of the form

$$L \cap m \cdot \ln W \tag{3}$$

where $W \subseteq (\mathbb{C}^\times)^n$ an algebraic subvariety and L is a **$\mathbb{Q}$-linear subspace** of V^n, that is defined by equations of the form $m_1 x_1 + \ldots + m_n x_n = a$, $m_i \in \mathbb{Z},\ a \in V$.

Slightly rephrasing the quantifier-elimination statement proved in [11] Corollary 2 of section 3, we have

Lemma 3.2 *(i) Projection of a PQF-closed set is PQF-constructible, that is a boolean combination of PQF-closed sets.*

(ii) The image of a constructible set under exponentiation is a Zariski-constructible (algebraic) subset of $(\mathbb{C}^\times)^n$. The image of the set of the form (3) is Zariski closed.

The PQF$_\omega$-topology is given by closed basic sets of the form

$$\cup_{a \in I}(S + a)$$

where S is of the form (3) and I a subset of $(\ker \exp)^n$.

We define $\mathcal{C}$ to be the family of all PQF$_\omega$-closed sets.

Corollary 3.3 $\mathcal{C}$ *satisfies* (L).

We assign **dimension** to a closed set of the form (3)

$$\dim L \cap m \cdot \ln W := \dim \exp\left(L \cap m \cdot \ln W\right).$$

using the fact that the object on the right hand side is an algebraic variety. We extend this to an arbitrary closed set assuming (CU), that is that the dimension of a countable union is the maximum dimension of its members.This immediately gives (DP). Using 3.2 we also get (WP).

For a variety $W \subseteq (\mathbb{C}^\times)^n$ consider the system of its roots

$$W^{\frac{1}{m}} = \{\langle x_1, \ldots, x_n\rangle \in (\mathbb{C}^\times)^n : \langle x_1^m, \ldots, x_n^m\rangle \in W\}.$$

Let $d_W(m)$ be the number of irreducible components of $W^{\frac{1}{m}}$. We say that the sequence $W^{\frac{1}{m}}$, $m \in \mathbb{N}$, **stops branching** if the sequence $d_W(m)$ is eventually constant.

Obviously, in case W is a singleton, $W = \{\langle w_1, \ldots, w_n \rangle\} \subseteq (\mathbb{C}^\times)^n$, the sequence $W^{\frac{1}{m}}$ does not stop branching as $d_W(m) = m^n$. This is the simplest case when W is contained in a coset of a torus, namely given by the equations $\bigwedge_i x_i = 1$. Similarly, if W is contained in a coset of an irreducible torus given by k independent equations of the form

$$x_1^{\ell_{i1}} \cdot \ldots \cdot x_n^{\ell_{in}} = 1$$

then $d_W(m) = m^k$ so does not stop branching.

Fact ([9], Theorem 2, case $n = 1$ and its Corollary) The sequence $W^{\frac{1}{m}}$ stops branching if and only if W is not contained in a coset of a proper subtorus of $(\mathbb{C}^\times)^k$.

Lemma 3.4 *Any irreducible closed subset of V^n is of the form (3), for W not contained in a coset of a proper torus, $m \in \mathbb{Z}$.*

In case W is contained in a coset of a proper torus T, note that $T = \exp L$, for some L a $\mathbb{Q}$-linear subspace of V^n. Also there is an obvious $\mathbb{Q}$-linear isomorphism $\sigma_L : L \to V^k$, $k = \dim L$, which induces a biregular isomorphism $\sigma_T : T \to (\mathbb{C}^\times)^k$. Now $\sigma_T(W) \subseteq (\mathbb{C}^\times)^k$ is not contained in a coset of a proper torus and so $\sigma_T(W)^{\frac{1}{m}}$ stops branching.

Note that L is defined up to the shift by $a \in (\ker \exp)^n$.

Proposition 3.5 *Let $W \subseteq (\mathbb{C}^\times)^n$ be an irreducible subvariety, $T = \exp L$ the minimal coset of a torus containing W and m is the level where $\sigma_T(W^{\frac{1}{m}})$ stops branching. Let $\sigma_T(W_i^{\frac{1}{m}})$ be an irreducible component of $\sigma_T(W^{\frac{1}{m}})$. Then*

$$L \cap m\sigma_T^{-1}\sigma_T(W_i^{\frac{1}{m}}) \tag{4}$$

is an irreducible component of $\ln W$. Moreover, any irreducible component of $\ln W$ has this form for some choice of L, $\exp L = T$.

Remark 3.6 (i) The irreducible components of the form (4) for distinct choices of L do not intersect.

(ii) There are finitely many irreducible components of the form (4) for a fixed L and W.

Remark 3.7 Proposition 3.5 eventually provides a description of the irreducible decomposition for any set of the form (3), so for any closed set. Indeed, the irreducible components of the set $L \cap m \cdot \ln W$ are among irreducible components of $\ln X$, for the algebraic variety $X = \exp(L \cap m \cdot \ln W)$.

Corollary 3.8 *Any closed subset of V^n is analytic in V^n.*

It is easy now to check that (SI), (INT), (CMP), (CC), (AS) and (PS) are satisfied.

Corollary 3.9 *The structure $(V, \mathcal{C})$ is analytic Zariski one-dimensional and presmooth.*

An inquisitive reader will notice that the analysis above treats only *formal* notion of analyticity on the cover $\mathbb{C}$ of $\mathbb{C}^\times$ but does not address the classical one. In particular, *is the formal analytic decomposition as described by 3.5 the same as the actual complex analytic one?* In a private communication F.Campana answered this question in positive, using a cohomological argument. M.Gavrilovich proved this and much more general statement in his thesis (see [8], III.1.2) by a similar argument.

Now we look into yet another version of a cover structure which is proven to be analytic Zariski, a **cover of the one-dimensional algebraic torus over an algebraically closed field of a positive characteristic.**

Let $(V, +)$ be a divisible torsion free abelian group and K an algebraically closed field of a positive characteristic p. We assume that V and K are both of the same uncountable cardinality. Under these assumptions it is easy to construct a surjective homomorphism

$$\mathrm{ex} : V \to K^\times.$$

The kernel of such a homomorphism must be a subgroup which is p-divisible but not q-divisible for each q coprime with p. One can easily construct ex so that

$$\ker \mathrm{ex} \cong \mathbb{Z}[\frac{1}{p}],$$

the additive group (which is also a ring) of rationals of the form $\frac{m}{p^n}$, $m, n \in \mathbb{Z}$, $n \geq 0$. In fact in this case it is convenient to view V and $\ker \mathrm{ex}$ as $\mathbb{Z}[\frac{1}{p}]$-modules.

In this new situation Lemma 3.2 is still true, with obvious alterations, and we can use the definition 3.1 to introduce a topology and the family $\mathcal{C}$ as above. The above Fact (right before 3.1) for $K^{\times}$ is proved in [10]. Hence the corresponding versions of 3.5 - 3.9 follow.

Finally we mention a wide class of structures on $\mathbb{C}$ considered by M.Gavrilovich in [8]. These are universal covers U of smooth complex algebraic varieties $\mathbb{A} = \mathbb{A}(\mathbb{C})$, not necessarily of dimension one. Gavrilovich defines, quite cleverly, a family $\mathcal{C}$ of closed (analytic) subsets on V^n closely related to etale coverings of $\mathbb{A}$. In case $\mathbb{A}$ is $\mathbb{C}^{\times}$, his $\mathcal{C}$ coincides with the one discussed above, but already for elliptic curves it can be richer. Gavrilovich then proves that this uncountable family can be defined in terms of a countable subfamily $\mathcal{C}_0$.

He calls a complex variety $\mathbb{A}$ *Shafarevich,* if V is holomorphically convex and the fundamental groups of $\mathbb{A}^n$ are subgroup separable (properties studied elsewhere). It is known that Abelian varieties satisfy these properties. Summarising the results of chapter III of [8] we get the following.

Theorem 3.10 (M.Gavrilovich) *Assume $\mathbb{A}$ is Shafarevich. Then, for its universal cover V, the structure $(V, \mathcal{C})$ is analytic Zariski.*

References

[1] N.Peatfield and B.Zilber, *Analytic Zariski structures and the Hrushovski construction,* Annals of Pure and Applied Logic, Vol 132/2-3 (2004) pp 127-180

[2] E.Hrushovski and B.Zilber, *Zariski Geometries*, Journal of AMS, 9(1996), 1-56

[3] B.Zilber, *Model Theory and Algebraic Geometry.* In **Seminarberichte Humboldt Universitat zu Berlin**, Nr 93-1, Berlin 1993, 202-222

[4] B.Zilber,**Zariski Geometries** forthcoming book, CUP

[5] B.Zilber, *Generalized Analytic Sets* Algebra i Logika, Novosibirsk, v.36, N 4 (1997), 361 - 380 (translation on the author's web-page and published by Kluwer as 'Algebra and Logic')

[6] B.Zilber, *A categoricity theorem for quasi-minimal excellent classes*. In: **Logic and its Applications** eds. Blass and Zhang, Cont.Maths, v.380, 2005, pp.297-306

[7] J.Baldwin, **Categoricity** authors web-page

[8] M.Gavrilovich, **Model Theory of the Universal Covering Spaces of Complex Algebraic Varieties**, DPhil Thesis, Oxford 2006, http://misha.uploads.net.ru/misha-thesis.pdf

[9] B.Zilber, *Covers of the multiplicative group of an algebraically closed field of characteristic zero* J. London Math. Soc. (2), 74(1):41–58, 2006

[10] M.Bays and B.Zilber, *Covers of Multiplicative Groups of Algebraically Closed Fields of Arbitrary Characteristic* April 2007, arXive math.AC/0401301

[11] B.Zilber, *Model theory, geometry and arithmetic of the universal cover of a semi-abelian variety* In: **Model Theory and Applications**, ed. L.Belair and el. (Proc. of Conf. in Ravello 2000) Quaderni di Matematica, v.11, Series edited by Seconda Univli, Caserta, 2002

[12] L.Smith, **Toric Varieties as Analytic Zariski Structures**, DPhil Thesis, Oxford 2008