A NOTE ON DOLICH'S PAPER

SERGEI STARCHENKO

Let T be an o-minimal theory extending the theory of real closed fields. We fix a large saturated model $\mathfrak{C}$ of T. Our goal is to give a '"more conceptual proof" of the following theorem of Dolich (see Definition 1.3 for the notion of "goodness").

Theorem 0.1 ([2, Theorem 3.5]). *For a formula $\varphi(\bar{x}, \bar{y})$, $\bar{a} \in \mathfrak{C}$ and $A \subset \mathfrak{C}$ the following conditions are equivalent.*
(1) $\varphi(\bar{x}, \bar{a})$ is good over A.
(2) $\varphi(\bar{x}, \bar{a})$ does not fork over A.

In the paper [4] Y. Peterzil and A. Pillay considered a special case when the set $\varphi(\mathfrak{C}, \bar{a})$ is closed and bounded and derived the following result from Dolich's paper.

Theorem 0.2 ([4, Theorem 6.5]). *Let $M \prec \mathfrak{C}$, $\varphi(\bar{x}, \bar{y})$ be a formula, and $\bar{a} \in \mathfrak{C}$. Assume the set $\varphi(\mathfrak{C}, \bar{a})$ is closed and bounded in $\mathfrak{C}$. Then $\phi(\bar{x}, \bar{a})$ does not fork over M if and only if $\varphi(\mathfrak{C}, \bar{a})$ has a point in M.*

The proof of the above theorem, derived from ideas of A. Dolich, is more direct then the original proof of Theorem 0.1.

Unfortunately, Theorem 0.2 fails in the case when $\varphi(\mathfrak{C}, \bar{a})$ is not closed and bounded.

Our proof of Theorem 0.1 is similar to the proof of Y. Peterzil and A. Pillay, and is also more direct then the original proof of A. Dolich. We also derive an appropriate generalization of Theorem 0.2

Theorem 0.3 (see Corollary 4.3). *Let $M \prec \mathfrak{C}$, $\varphi(\bar{x}, \bar{y})$ be a formula, and $\bar{a} \in \mathfrak{C}$. The $\phi(\bar{x}, \bar{a})$ does not fork over M if and only if $\varphi(\mathfrak{C}, \bar{a})$ has a point in $M\langle\delta\rangle$ for any (some) $\delta \in \mathfrak{C}$ satisfying $\delta > M\langle\bar{a}\rangle$.*

As a corollary we have a characterisation of forking for types.

Corollary 0.4. *Let $M \prec N \prec \mathfrak{C}$ and $p \in S(N)$. The type p does not fork over M if and only if for some $\delta \in \mathfrak{C}$ with $\delta > N$ the type p has an extension $q \in S(N\langle\delta\rangle)$ such that q is finitely realizable in $M\langle\delta\rangle$.*

Remark 0.5. We don't really need that T expands the theory of real closed field. We only need that T has definable Skolem functions, and any two definable 1-types are inter-algebraic.

In the case when $\varphi(\bar{x}, \bar{a})$ defines a bounded set, we only need to consider bounded definable types, and in this case it is sufficient to assume that T expands an ordered group.

Date: August 14, 2008.

1. Preliminaries

Since T has definable Skolem functions for any $A \subseteq \mathfrak{C}$ we have that $\mathrm{dcl}(A)$ is the universe of an elementary substructure of $\mathfrak{C}$.

1.1. **Notations.** We use $A \subset \mathfrak{C}$ and $M \prec C$ to denote that A and M are **small** subset and elementary substructure of $\mathfrak{C}$, respectively.

For $M \prec \mathfrak{C}$ and $A \subseteq \mathfrak{C}$ we will use $M\langle A\rangle$ to denote $\mathrm{dcl}(M \cup A)$.

If $A, B \subseteq \mathfrak{C}$ then we will write $A < B$ if $a < b$ for all $a \in A, b \in B$.

For a set $A \subseteq \mathfrak{C}$ and a type q over A (possibly not complete) we will denote by $\mathrm{dcl}_q(A)$ the set $\mathrm{dcl}(A) \cap q(\mathfrak{C})$.

By a global type we mean a **complete** type over $\mathfrak{C}$.

1.2. **The notion of "goodness".** First we recall the definition of a definable type.

Definition 1.1. Let $N \preccurlyeq \mathfrak{C}$, $A \subseteq N$ and $p(\bar{x})$ a type over N. The type p *is definable over* A if for every formula $\varphi(\bar{x}, \bar{y})$ there is a formula $\psi(\bar{y})$ over A such that for every $\bar{b} \in N$ we have $N \models \psi(\bar{b})$ if and only if $\varphi(\bar{x}, \bar{b}) \in p(\bar{x})$.

If $A = N$ then we just say that p is definable.

It is very easy to see that if $A \subseteq N$ and $p(\bar{x}) \in S(N)$ is definable over A then p has a unique global extension definable over A.

We introduce strongly definable types.

Definition 1.2. Let $N \preccurlyeq \mathfrak{C}$, $A \subseteq N$, and $p(\bar{x}) \in S_n(N)$. We say that the type p *is strongly definable over* A if for a realization $(a_1, \dots, a_n)$ of p we have that the type $\mathrm{tp}(a_i/N\langle a_1, \dots, a_{i-1}\rangle)$ is definable over $A \cup \{a_1, \dots, a_{i-1}\}$ for all $i = 1, \dots, n$.

It is easy to see that if a type $p \in S(N)$ is strongly definable over A then it is definable over A, however converse is not true in general.

We are ready to unwrap Dolich's definition of goodness.

Definition 1.3. For $\bar{a} \in \mathfrak{C}$ and a formula $\varphi(\bar{x}, \bar{y})$ we say that the formula $\varphi(\bar{x}, \bar{a})$ is *good over a set* $A \subset \mathfrak{C}$ if either $\mathrm{length}(\bar{x}) = 0$ and $\mathfrak{C} \models \varphi(\bar{x}, \bar{a})$, or $\mathrm{length}(\bar{x}) > 0$ and there is a type $p(\bar{x}) \in S(A\bar{a})$ such that $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$ and $p(\bar{x})$ is strongly definable over A.

1.3. **Splitting.** Recall that if $A \subseteq B \subsetneq \mathfrak{C}$ and $p(\bar{x}) \in S(B)$ then p *does not split over* A if for any formula $\psi(\bar{x}, \bar{y})$ and $\bar{b}, \bar{b}' \in B$ with $\mathrm{tp}(\bar{b}/A) = \mathrm{tp}(\bar{b}'/A)$ we have $\psi(\bar{x}, \bar{b}) \in p$ if and only if $\psi(\bar{x}, \bar{b}') \in p$.

It is easy to see that if p is a global type and $A \subset \mathfrak{C}$ then p does not split over A if and only if p is invariant under all automorphisms of $\mathfrak{C}$ fixing A. In this case we also say that p is A-invariant.

In the following fact we list some basic properties of splitting.

Fact 1.4.
1. *If $A \subseteq N \preccurlyeq \mathfrak{C}$ and $p(\bar{x}) \in S(N)$ is definable over A then p has unique global extension q that is definable over A and this type q does not-split over A.*
2. *If $M \preccurlyeq N \prec \mathfrak{C}$, $p \in S(N)$, and p does not split over M then p has a global M-invariant extension. Moreover, if N is $|M|^+$-saturated then such global M-invariant extension is unique.*
3. *Let $p \in S(\mathfrak{C})$. If p is definable and does not split over $M \prec \mathfrak{C}$ then p is definable over M.*

The following claim is also easy.

Claim 1.5. *Let $A \subseteq N \preccurlyeq \mathfrak{C}$, and $p(\bar{x}) \in S_n(N)$ strongly definable over A. If q is the global extension of p definable over A then q is strongly definable over A.*

Definition 1.6. For global types $p(\bar{x}), q(\bar{y})$ and a set $A \subset \mathfrak{C}$ we say that p and q *commute over A* if

(a) both p and q do not split over A, and
(b) for any $A \subseteq B \subset \mathfrak{C}$, a realization $\bar{a}$ of $p{\restriction} B$ and any realization $\bar{b}$ of $q{\restriction} B\bar{a}$ we have $\bar{a} \models p{\restriction} B\bar{b}$.

Remark. In notations of Hrushovski-Pillay (see [3]) , part (b) means $p \ltimes q = p \rtimes q$.

Remark 1.7. In the above definition, since non-splitting is equivalent to be A-invariant, "for any realization" can be replaced by "for some realization", and commuting is a symmetric relation.

2. Some properties of types

2.1. **More on definable types.** The following fact follows from definability of types in o-minimal theories.

Fact 2.1. *Let $M \prec \mathfrak{C}$, $p(x) \in S_1(M)$ a non-algebraic type, and $c \models p$. The following are equivalent.*

(a) $p(x)$ is definable.
(b) M is not co-final in $M\langle c\rangle$.

Let $\Delta(x)$ be the global 1-type over $\mathfrak{C}$ with $c < x \in \Delta(x)$ for every $c \in \mathfrak{C}$, i.e. $\Delta(x)$ is the type of infinitely large positive element. Obviously, Δ is definable over $\varnothing$.

Corollary 2.2. *Let $M \prec \mathfrak{C}$ and $\delta \models \Delta(x){\restriction} M$. A 1-type $p(x) \in S_1(M)$ is definable if and only if p has a realization in $M\langle\delta\rangle$.*

Proof. Easy. ☐

Construction 1. By induction on n we construct global types $\Delta_n(\bar{x}) \in S_n(\mathfrak{C})$ all strongly definable over the empty set. We take $\Delta_1(x) = \Delta(x)$. To construct Δ_{n+1}, let $\bar{a}$ be a realization of $\Delta_n \restriction \mathrm{dcl}(\varnothing)$, and δ a realization of $\Delta \restriction \mathrm{dcl}(\bar{a})$. Then the type $\mathrm{tp}(\bar{a}\delta/\mathrm{dcl}(\varnothing))$ is strongly definable and we take Δ_{n+1} to be its unique global extension definable over $\varnothing$.

Remark. In notations of Hrushovski-Pillay, $\Delta_{n+1} = \Delta_n \rtimes \Delta$.

Obviously, for a model $M \prec \mathfrak{C}$ and $(\delta_1, \ldots, \delta_n) \models \Delta_n \restriction M$ we have $\delta_{i+1} > M\langle\delta_1, \ldots, \delta_i\rangle$ for all $i = 1, \ldots, n$.

Claim 2.3. *Let $M \prec \mathfrak{C}$ and $p(\bar{x}) \in S_n(M)$. The type p is strongly definable if and only if there is a realization $\bar{a} = (a_1, \ldots, a_n)$ of p and a realization $(\delta_1, \ldots, \delta_n)$ of $\Delta_n{\restriction} M$ such that $a_i \in M\langle\delta_1, \ldots, \delta_i\rangle$ for all $i = 1, \ldots, n$.*

Proof. Follows from Corollary 2.2. ☐

We need the following claim and its consequences.

Claim 2.4. *Let $M \preccurlyeq N \prec \mathfrak{C}$ with N finitely generated over M. Then there is $t \in N$ such that $M\langle t\rangle$ is co-finial in N.*

Proof. We prove it by induction on $\dim(N/M)$. The claim is easy if $\dim(N/M) \le 1$. Assume $\dim(N/M) = n+1$ with $n \ge 1$. By [5, Lemma], there is $M \prec M_1 \prec N$ such that M_1 is co-final in N and $\dim(M_1/M) = n$, Applying induction hypothesis, we can find $t \in M_1$ such that $M\langle t\rangle$ is co-final in M_1, hence in N. $\square$

Corollary 2.5. *If $M \prec N$ and $\delta \models \Delta\restriction N$ then $M\langle\delta\rangle$ is co-final in $N\langle\delta\rangle$.*

Proof. It is sufficient to consider the case when N is finitely generated over M. Then, by Claim 2.4, there is $t \in N\langle\delta\rangle$ such that $M\langle t\rangle$ is co-final in $N\langle\delta\rangle$. Thus there $t' \in M\langle t\rangle$ with $t' \ge \delta > N$.

Since $t' \in M\langle t\rangle \setminus M$, we have $M\langle t\rangle = M\langle t'\rangle$, and, for the same reason, $N\langle\delta\rangle = N\langle t'\rangle$. Thus $M\langle t'\rangle$ is co-final in $N\langle t'\rangle$. Since $t' > N$, t' realizes $\Delta\restriction N$, and there is an automorphism of $\mathfrak{C}$ fixing N that maps δ to t'. $\square$

Corollary 2.6. *Let $A \subseteq \mathfrak{C}$, $c \in \mathfrak{C}$ such that $c \notin \operatorname{dcl}(A)$, and $\delta \models \Delta\restriction Ac$. Then $\operatorname{tp}(c/\operatorname{dcl}(A\delta))$ is not definable.*

Proof. For $M = \operatorname{dcl}(A)$ and $M_1 = \operatorname{dcl}(A\delta)$, by the previous Corollary, we have that M_1 is co-final in $M_1\langle c\rangle$, hence $\operatorname{tp}(c/M_1\langle\delta\rangle)$ is not definable. $\square$

Corollary 2.7. *Let $A \subseteq \mathfrak{C}$, $c \in \mathfrak{C}$ such that $c \notin \operatorname{dcl}(A)$, $k \in \mathbb{N}$, and $\bar\delta \models \Delta_k\restriction Ac$. Then $\operatorname{tp}(c/\operatorname{dcl}(A\bar\delta))$ is not definable.*

Proof. Follows from Corollary 2.6 by induction on k. $\square$

Corollary 2.8. *Let $M \prec \mathfrak{C}$ and $\bar a \in \mathfrak{C}$. Assume $\operatorname{tp}(\bar a/M)$ is not algebraic. Then there are $c_1, \dots, c_m \in \mathfrak{C}$ independent over M such that $\bar a$ and $\bar c = (c_1, \dots, c_m)$ are interdefinable over M, and for $i = 2, \dots, m$ the type $\operatorname{tp}(c_i/M\langle c_1, \dots, c_{i-1}\rangle)$ is not definable.*

Proof. Let $N = M\langle\bar a\rangle$. If $\dim(N/M) < 2$, then we can take $\bar c = c_1$ such that $N = M\langle c_1\rangle$.

Assume $\dim(N/M) \ge 2$. By Claim 2.4, we can choose $c_1, \dots c_m \in N$ independent over M such that $M\langle c_1\rangle$ is co-final in N and $N = M\langle c_1, \dots, c_m\rangle$. Then, for $i = 2, \dots, m$, $M\langle c_1, \dots, c_{i-1}\rangle$ is co-final in $M\langle c_1, \dots, c_i\rangle$, hence $\operatorname{tp}(c_i/M\langle c_1, \dots, c_{i-1}\rangle)$ is not definable. $\square$

2.2. **Special Extensions.** In this section we consider "nice" extensions introduced by Dolich.

Let $A \subseteq B \subset \mathfrak{C}$, $p(x) \in S_1(A)$ a non-algebraic type, and $a \models p$. Then, by o-minimality, the type $\operatorname{tp}(a/B)$ does not split over A if and only if either $a > \operatorname{dcl}_p(B)$ or $a < \operatorname{dcl}_p(B)$.

Thus, if $A \subset \mathfrak{C}$ and $p \in S_1(A)$ is non-algebraic then p has exactly two A-invariant global extensions, that we will denote by p^0 and p^1, where $p^0(x) = p(x) \cup \{x < c\colon c \in p(\mathfrak{C})\}$; and similarly, $p^1(x) = p(x) \cup \{x > c\colon c \in p(\mathfrak{C})\}$.

For a convenience, in the case when $p \in S_1(A)$ is algebraic, we take $p_0 = p_1$ to be the unique global extension of p.

Notice that if $A \subseteq B \subset C$, $p \in S_1(A)$ is not algebraic, and $a \models p$ then $a \models p^0\restriction B$ if and only if $a < \operatorname{dcl}_p(B)$, and $a \models p^1\restriction B$ if and only if $a > \operatorname{dcl}_p(B)$.

Construction 2. For a set $A \subset \mathfrak{C}$, an n-type $p(x_1, \dots, x_n) \in S_n(A)$, and $\eta = (\eta(1), \dots, \eta(n)) \in 2^n$, by induction on n, we construct a global extension $p^\eta(\bar{x})$ of $p(\bar{x})$ that does not split over A.

We have constructed such extensions for $n = 1$.

Assume we have such extensions for n, and let $p(x_1, \dots, x_{n+1}) \in S_{n+1}(A)$. We first choose an $|M|^+$-saturated elementary extension N of M. Let $q(x_1, \dots, x_n) \in S_n(A)$ be the restriction of p to the first n variables. Let $(a_1, \dots, a_n)$ be the realization of $q^{\eta \restriction n} \restriction N$. Let $r(x_{n+1}) \in S_1(A \cup \{a_1, \dots, a_n\})$ be the type $p(a_1, \dots, a_n, x_{n+1})$, and a_{n+1} be a realization of $r^{\eta(n+1)} \restriction N\langle a_1, \dots, a_n \rangle$. It is easy to see that the type $\mathrm{tp}(a_1, \dots, a_{n+1}/N)$ does not split over A (since $\mathrm{tp}(a_1, \dots, a_n/N)$ does not split over A and $\mathrm{tp}(a_{n+1}/N\langle a_1, \dots, a_n \rangle)$ does not split over $A \cup \langle a_1, \dots, a_n \}$), and, since N is $|M^+|$-saturated, this type has unique global extension p^η that does not split over A.

Remark. In notations of Hrushovski-Pillay for $p(x_1, \dots, x_{n+1}) \in S_{n+1}(A)$ and $\eta \in 2^{n+1}$ the type p^η is $q^{\eta \restriction n} \rtimes r^{\eta(n+1)}$, where $q = \mathrm{tp}(a_1, \dots, a_n/A)$ and $r = \mathrm{tp}(a_{n+1}/A \cup \{a_1, \dots, a_n\})$ for some $(a_1, \dots, a_{n+1})$ realizing p.

It is not hard to see that if $A \subset \mathfrak{C}$, $p \in S_n(A)$, $\eta \in 2^n$, $B \supset A$ then $\bar{a} = (a_1, \dots, a_n)$ realizes $p^\eta \restriction B$, if and only if $\bar{a}$ realizes p, and, for $i = 1, \dots, n$ and $p_i(x) = \mathrm{tp}(a_i/A \cup \{a_1, \dots, a_{i-1}\})$ we have $a_i < \mathrm{dcl}_{p_i}(B \cup \{a_1, \dots, a_{i-1}\})$ if $\eta(i) = 0$; and $a_i > \mathrm{dcl}_{p_i}(B \cup \{a_1, \dots, a_{i-1}\})$ if $\eta(i) = 0$.

2.3. **On relations between 1-types.** Let $A \subset \mathfrak{C}$ and $p, q \in S_1(A)$. It follows from o-minimality that either $p(x) \cup q(y)$ is a complete type or there is an A-definable function f such that f maps $p(\mathfrak{C})$ onto $q(\mathfrak{C})$, and in the latter case the function f must be either order-preserving on $p(\mathfrak{C})$ or order reversing on $p(\mathfrak{C})$. We will write $f(p) = q$, if f is an A-definable function from $p(\mathfrak{C})$ onto $q(\mathfrak{C})$.

Claim 2.9 ([2])**.** *Let $A \subset \mathfrak{C}$, $p, q \in S_1(\mathfrak{C})$, and f, g A-definable functions such that $f(p) = q$ and $g(p) = q$. Then either both f and g are order-preserving on $p(\mathfrak{C})$ or they are both order-reversing.*

Corollary 2.10. *Let $A \subset \mathfrak{C}$ and $p, q \in S_1(A)$ with p non-algebraic. Then either*

(a) for any $B \supset A$ and $a_i \models p^i \restriction B$, $i = 0, 1$, we have $\mathrm{dcl}_q(Aa_0) < \mathrm{dcl}_q(B)$ and $\mathrm{dcl}_q(Aa_1) > \mathrm{dcl}_q(B)$; or

(b) for any $B \supset A$ and $a_i \models p^i \restriction B$, $i = 0, 1$, we have $\mathrm{dcl}_q(Aa_0) > \mathrm{dcl}_q(B)$ and $\mathrm{dcl}_q(Aa_1) < \mathrm{dcl}_q(B)$.

Proof. Easy. □

Corollary 2.11. *Let $A \subset \mathfrak{C}$ and $a, b \in \mathfrak{C}$. Then $\mathrm{tp}(a/\mathrm{dcl}(Ab))$ does not split over A if and only if $\mathrm{tp}(b/\mathrm{dcl}(Aa))$ does not split over A.*

Proof. Easy □

Remark. Unfortunately the above claim holds only for 1-types.

Claim 2.12. *Let $M \prec \mathfrak{C}$ and let $p \in S_1(M)$ be a non-definable type. Then $p(x) \cup \Delta(y) \restriction M$ is a complete type over M.*

Proof. Let $a \models p$ and $\delta \models \Delta \restriction M$ It is sufficient to show that $\Delta \restriction M$ has unique extension over $M\langle a \rangle$, namely $\Delta \restriction M\langle a \rangle$. Let $N = M\langle a \rangle$. Since p is not definable, M is co-final in N and $\delta > N$, since $\delta > M$. Thus $\delta \models \Delta \restriction N$. □

Corollary 2.13. *Let $M \prec \mathfrak{C}$ and let $p(x) \in S_1(M)$ be a non-definable type. Then, for any $k \in \mathbb{N}$, $p(x) \cup \Delta_k(\bar{y})\restriction M$ is a complete type over M.*

Proof. Follows, by induction on k, from Claim 2.12 and Corollary 2.6. □

Claim 2.14. *Let $M \prec \mathfrak{C}$ and $r(x) \in S_1(\mathfrak{C})$ a global 1-type that does not split over M. If r is not definable then r and Δ commute over M.*

Proof. Let $a \models r\restriction M$ and $\delta \models \Delta\restriction M\langle a\rangle$. We need to show that $a \models r\restriction M\langle\delta\rangle$. Let $p(x) = r(x)\restriction M$.

Case 1: The type $p(x)$ is not definable. Then, by Claim 2.12, $p(x) \cup \Delta(y)\restriction M$ is a complete type, hence $a \models r\restriction M\langle\delta\rangle$.

Case 2: The type $p(x)$ is definable. In this case $p(x)$ has a global extension $r'(x)$ such that $r'(x)$ is definable and does not split over M. Obviously, $r' \neq r$.

Since $\mathrm{tp}(\delta/M\langle a\rangle)$ does not split over M, by Corollary 2.12, $\mathrm{tp}(a/M\langle\delta\rangle)$ does not split over M, and it has a global extension $r_1(x) \in S_1(\mathfrak{C})$ such that r_1 that does not split over M. Since every non-algebraic 1-type has exactly two non-splitting global extensions, it is sufficient to show that $r_1 \neq r'$. By Corollary 2.6, the type $\mathrm{tp}(a/M\langle\delta\rangle)$ is not definable, hence all its global $M\langle\delta\rangle$-invariant extensions are not definable. Hence r_1 is not definable and $r_1 \neq r'$. □

Corollary 2.15. *Let $M \prec \mathfrak{C}$ and $r(x) \in S_1(\mathfrak{C})$ a global 1-type that does not split over M. If r is not definable then r and Δ_k commute over M for any $k \in \mathbb{N}$*

Proof. Follows from the previous claim by induction on k. □

3. Proof of Theorem 0.1

In this section we prove Theorem 0.1. At some points during the proof we will make some choices and assumptions that we will carry on throughout the proof. We will itemize some of them.

- We fix $A, \bar{a}$, and $\varphi(\bar{x}, \bar{y})$ as in Theorem 0.1.
- Taking $M_0 = \mathrm{dcl}(A)$ if needed we assume $A = M_0 \prec \mathfrak{C}$.

By a result of H. Adler (see [1]), since T has NIP, a global type does not fork over M_0 if and only if it does not split over M_0. Thus, since $\varphi(\bar{x}, \bar{a})$ does not fork over M_0 if and only if there is a global type p containing φ such that p does not fork over M_0, Theorem 0.1 can be restated in the following way.

Theorem 3.1 (Restatement of Theorem 0.1)**.** *The following conditions are equivalent.*

(1) $\varphi(\bar{x}, \bar{a})$ is good over M_0.
(2) Thee is a global M_0-invariant type containing $\varphi(\bar{x}, \bar{a})$.

The direction (1) $\Longrightarrow$ (2) is easy: If $\varphi(\bar{x}, \bar{a})$ is good over M_0 then there is a global type $p(\bar{x})$ containing $\varphi(\bar{x}, \bar{a})$ such that p is strongly definable over M_0. Obviously $p(\bar{x})$ is M_0-invariant.

The direction (2) $\Longrightarrow$ (1) will follow from the following proposition.

Proposition 3.2. *If $\varphi(\bar{x}, \bar{a})$ is not good over M_0 then there are $\bar{a}^1, \ldots, \bar{a}^s$ all realizing the type $\mathrm{tp}(\bar{a}/M_0)$ such that*

$$\mathfrak{C} \models \neg\,\exists \bar{x}\big[\,\varphi(\bar{x}, \bar{a}) \bigwedge_{i=1}^{s} \varphi(\bar{x}, \bar{a}^i)\,\big]$$

We prove Proposition 3.2 by induction on the length of $\bar{x}$.

If the length of $\bar{x}$ is zero, then the proposition follows from the definition.

The following claim provides the induction step.

Claim 3.3. *If $\varphi(x_1, \dots, x_n, x_{n+1}, \bar{a})$ is not good over M_0 then there are $\bar{a}^1, \dots, \bar{a}^k$ all realizing the type $\mathrm{tp}(\bar{a}/M_0)$ such that the formula*

$$\exists x_{n+1} \big[\varphi(x_1, \dots, x_n, x_{n+1}, \bar{a}) \bigwedge_{i=1}^{k} \varphi(x_1, \dots, x_n, x_{n+1}, \bar{a}^i) \big]$$

is not good over M_0.

Since $\varphi(\bar{x}, \bar{a})$ is not good over M_0, $\bar{a} \notin M_0$. As the first step, using Corollary 2.8, we can find $\bar{c} = (c_1, \dots, c_m)$ such that

- $\bar{a}$ and $\bar{c}$ are interdefinable over M_0;
- $c_1, \dots, c_m$ are independent over M_0, and
- $\mathrm{tp}(c_i/M_0\langle c_1, \dots, c_{i-1}\rangle)$ is not definable for $i = 2, \dots, m$.

- Let $r(\bar{z}) = \mathrm{tp}(\bar{c}/M_0)$;
- and for each $\eta \in 2^m$ we chose $\bar{c}^\eta$ realizing $r^\eta{\restriction} M_0\langle\bar{c}\rangle$.

Since $\bar{a}$ and $\bar{c}$ are interdefinable over M_0,

- there is a formula $\psi(\bar{x}, \bar{z})$ over M_0 such that $\mathfrak{C} \models \varphi(\bar{x}, \bar{a}) \leftrightarrow \psi(\bar{x}, \bar{c})$.

Thus Claim 3.3 follows from the following claim.

Claim 3.4. *If $\psi(\bar{x}, \bar{c})$ is not good over M_0, then the formula*

$$\exists x_{n+1} \big[\psi(x_1, \dots, x_{n+1}, \bar{c}) \bigwedge_{\eta \in 2^m} \psi(x_1, \dots, x_{n+1}, \bar{c}^\eta) \big]$$

is not good over M_0.

We proof Claim 3.4 by a contradiction. We assume that

- $\psi(\bar{x}, \bar{c})$ is not good over M_0;
- but $\exists x_{n+1} \big[\psi(x_1, \dots, x_{n+1}, \bar{c}) \bigwedge_{\eta \in 2^m} \psi(x_1, \dots, x_{n+1}, \bar{c}^\eta) \big]$ is good over M_0.

- Let $C = \bar{c} \cup_{\eta \in 2^m} \bar{c}^\eta$. We choose $\bar{\delta} = (\delta_1, \dots, \delta_n) \models \Delta_n{\restriction} M_0\langle C\rangle$,

and let

- $M = M_0\langle\bar{\delta}\rangle$.

To proceed as in [4], we will need one more lemma.

Lemma 3.5. *Let $q(x) \in S_1(M)$ be a non-algebraic 1-type. Then there is $\eta \in 2^m$ such that $r^\eta(\bar{z})$ commutes with Δ_n over M_0 and one of the following holds.*

(A) For any $B \supseteq M$ and $\bar{d} \models r^\eta{\restriction} B$, we have $\mathrm{dcl}_q(M\langle\bar{d}\rangle) < \mathrm{dcl}_q(B)$.

(B) For any $B \supseteq M$ and $\bar{d} \models r^\eta{\restriction} B$, we have $\mathrm{dcl}_q(M\langle\bar{d}\rangle) > \mathrm{dcl}_q(B)$.

Proof. For $j = 1, \dots, m$, let $r_j(z_1, \dots, z_j) = \mathrm{tp}(c_1, \dots, c_j/M_0)$, i.e. r_j is the restriction of r to the first j variables.

By induction on $j = 1, \dots, m$ we will construct $\eta_j \in 2^j$ such that $r_j^{\eta_j}$ commutes with Δ_n over M_0, and either (A) or (B) holds for $r_j^{\eta_j}$ in place of r^η.

Case $j = 1$**.** Since r_1 has exactly two M_0-invariant global extensions, and only one of them can be definable, there is $i \in \{0, 1\}$ such that r_1^i is not definable. By Corollary 2.15, r_1^i and Δ_n commute.

By Corollary 2.10, applied to $A = M$, $p = r^i_1{\restriction}M$, and q, either (A) or (B) holds for r^i_1.

Induction Step: $j+1$**.** Assume $j > 0$ and we have η_j such that $r_j^{\eta_j}$ commutes with Δ_n over M_0 and either (A) or (B) holds for $r_j^{\eta_j}$. For simplicity we assume that (A) holds for $r_j^{\eta_j}$, i.e. for any $B \supseteq M$ and $\bar d^j \models r_j^{\eta_j}{\restriction}B$, we have $\mathrm{dcl}_q(M\langle \bar d^j\rangle) < \mathrm{dcl}_q(B)$.

Let $\bar d^j \models r_j^{\eta_j}{\restriction}M$. Since $\bar d^j \models r_j$ we can find $e \in \mathfrak{C}$ such that $(\bar d_j, e) \models r_{j+1}$. Since $j > 0$, by our choice of $\bar c$ the type $\mathrm{tp}(e/M_0\langle \bar d^j\rangle)$ is not definable. Since $r_j^{\eta_j}$ commutes with Δ_n over M_0, and $\bar d^j \models r_j^{\eta_j}{\restriction}M_0\langle\bar\delta\rangle$, we have $\bar\delta \models \Delta_n{\restriction}M_0\langle \bar d_j\rangle$, hence, by Corollary 2.13, the type $p(z) = \mathrm{tp}(e/M_0\langle \bar d^j\rangle)$ is a complete type over $M\langle \bar d^j\rangle$.

Let $q_1(x) = q(x) \cup \{x > a \colon a \in \mathrm{dcl}_q(M\langle \bar d^j\rangle)\}$. Obviously, q_1 is a complete type over $M\langle \bar d^j\rangle$. Applying Corollary 2.10 to $M\langle \bar d^j\rangle$, q_1 and $p(z)$, we can find $i \in \{0,1\}$ such that for any $B \supseteq M\langle \bar d^j\rangle$ and $a \models p^i{\restriction}B$ we have $\mathrm{dcl}_{q_1}(M\langle \bar d^j a\rangle) < \mathrm{dcl}_{q_1}(B)$. We take $\eta_{j+1} = (\eta_j, i)$.

First we show that $r_{j+1}^{\eta_{j+1}}$ commutes with Δ_n over M_0. Let $N \succ M_0$, $\bar\delta' \models \Delta_n{\restriction}N$, and $\bar b = (b_1,\dots,b_{j+1}) \models r_{j+1}^{\eta_{j+1}}{\restriction}N\langle\bar\delta'\rangle$. Since $r_{j+1}^{\eta_{j+1}}$ is an M_0-invariant type, we can apply an M_0-automorphism of $\mathfrak{C}$ and assume that $(b_1,\dots,b_j) = \bar d^j$, and $b_{j+1} = e$. Since $r_j^{\eta_j}$ commutes with Δ_n, we have $\bar\delta' \models \Delta_n{\restriction}N\langle \bar d^j\rangle$. Since $\mathrm{tp}(e/M_0\langle \bar d^j\rangle)$ is not definable every its global $M_0\langle \bar d_j\rangle$-invariant extension commutes with Δ_n over $M_0\langle \bar d'\rangle$, hence $\bar\delta' \models \Delta_n{\restriction}N\langle \bar d^j e\rangle$. Thus $r_{j+1}^{\eta_{j+1}}$ commutes with Δ_n over M_0.

Secondly, let $B \supseteq M$ and $\bar d^{j+1} \models r_{j+1}^{\eta_{j+1}}{\restriction}B$. We will show that $\mathrm{dcl}_q(M\langle \bar d^{j+1}\rangle) < \mathrm{dcl}_q(B)$. Assume not, then there are $b \in \mathrm{dcl}_q(B)$ and $u \in \mathrm{dcl}_q(M\langle \bar d^{j+1}\rangle)$ such that $b < u$. Applying an appropriate automorphism of $\mathfrak{C}$ if needed we may assume $\bar d^{j+1} = \bar d^j\hat{\ }e$. By the choice of η_j we have $b > \mathrm{dcl}_q(M\langle \bar d_j\rangle)$, and, since $u > b$, $u > \mathrm{dcl}_q(M\langle \bar d_j\rangle)$. Thus $b, u \models q_1$ with $b < u$. It contradicts the choice of i.

Lemma 3.5 is proved. $\square$

We continue the proof of Claim 3.4.

Since $\exists x_{n+1}\big[\,\psi(x_1,\dots,x_{n+1},\bar c) \bigwedge_{\eta\in 2^m} \psi(x_1,\dots,x_{n+1},\bar c^\eta)\,\big]$ is good over M_0, there is

- $\bar\xi = (\xi_1,\dots,\xi_n)$ such that $\xi_i \in M_0\langle\delta_1,\dots,\delta_i\rangle$, for $i = 1,\dots,n$,

and

$$(\star)\qquad \mathfrak{C} \models \exists x_{n+1}\big[\,\psi(\xi_1,\dots,\xi_n,x_{n+1},\bar c) \bigwedge_{\eta\in 2^m} \psi(\xi_1,\dots,\xi_n,x_{n+1},\bar c^\eta)\,\big].$$

Let $\delta \models \Delta{\restriction}M$. Since $\psi(x_1,\dots,x_n,\bar c)$ is not good over M_0, we have that

- the formula $\theta(x_n) = \psi(\xi_1,\dots,\xi_{n-1},x_n,\bar c)$ has no realization in $M\langle\delta\rangle$.

We follow the proof of [4, Lemma 6.4] almost word by word, replacing their Proposition 6.3 with Lemma 3.5, so we will be brief.

Let $I_{\bar c} = \theta(\mathfrak{C})$. (Notice $I_{\bar c}$ is definable over $M\langle\bar c\rangle$.) We know that $I_{\bar c}$ has no point in $M\langle\delta\rangle$. It is easy to then that the topological closure $\mathrm{cl}(I_{\bar c})$ is disjoint from $M \cup \{\pm\infty\}$ (for example, if $m \in M \cap \mathrm{cl}(I_{\bar c})$ then either $m+1/\delta \in I_{\bar c}$ or $m-1/\delta \in I_{\bar c}$).

Thus $I_{\bar c}$ is a finite union of intervals $I^1_{\bar c} < \ldots < I^k_{\bar c}$ whose endpoints are in $M\langle\bar c\rangle$, and with each $I^i_{\bar c}$ contained in $q_i(\mathfrak{C})$ for some non-algebraic $q_i(x) \in S_1(M)$.

For each $i = 1,\dots,k$ we choose η_i as in Lemma 3.5. We write r^i for r^{η_i} and $\bar c_i$ for $\bar c_{\eta_i}$.

For each $i = 1, \dots, k$ we have that $\bar{c}_i$ and $\bar{c}$ realize the same type over M_0. Since $\bar{\delta} \models \Delta_n {\restriction} M_0\langle C\rangle$, we also have that $\bar{c}_i\hat{\ }\bar{\delta}$ and $\bar{c}\hat{\ }\bar{\delta}$ realize the same type over M_0. Thus $\bar{c}_i$ and $\bar{c}$ are conjugate over M and we will denote by $I^j_{\bar{c}_i}$ for the corresponding copy of $I^j_{\bar{c}}$.

Since r^i and Δ_n commute over M_0, $c_i \models r^i {\restriction} M_0\langle\bar{c}\rangle$, and $\bar{\delta} \models \Delta {\restriction} M_0\langle\bar{c}\bar{c}_i\rangle$, we have $c_i \models r^i {\restriction} M\langle\bar{c}\rangle$. Arguing as in [4, Lemma 6.4], and using Claim 3.5 with $B = M\langle\bar{c}\rangle$, we obtain that $\cap_{i=1}^k I_{\bar{c}_i} = \varnothing$. Contradiction with $(\star)$.

4. A restatement of Theorem 0.1

Theorem 4.1. *For $M \prec \mathfrak{C}$, a formula $\varphi(x_1, \dots, x_n, \bar{y})$, and $\bar{a} \in \mathfrak{C}$ the following conditions are equivalent.*

(1) The formula $\varphi(\bar{x}, \bar{a})$ is good over M.
(2) The formula $\varphi(\bar{x}, \bar{a})$ has a realization in $M\langle\bar{\delta}\rangle$ for any $\bar{\delta} \models \Delta_n {\restriction} M\langle\bar{a}\rangle$.
(3) The formula $\varphi(\bar{x}, \bar{a})$ has a realization in $M\langle\delta\rangle$ for any $\delta \models \Delta {\restriction} M\langle\bar{a}\rangle$.

Proof. Obviously (1) implies (2), and (3) implies (1).

For (2)$\Longrightarrow$(3), it is sufficient to show the following claim.

Claim 4.2. *Let M and $\varphi(\bar{x}, \bar{a})$ be as above, $m > 0$, and $(\delta_1 \dots, \delta_{m+1}) \models \Delta_{m+1} {\restriction} M\langle\bar{a}\rangle$. If $\varphi(\bar{x}, \bar{a})$ has a realization in $M\langle\delta_1, \dots, \delta_{m+1}\rangle$, then it has a realization in $M\langle\delta_1, \dots, \delta_m\rangle$.*

Proof. Let $\bar{\xi}(\xi_1, \dots, \xi_n)$ be a realization of $\varphi(\bar{x}, \bar{a})$ in $M\langle\delta_1, \dots, \delta_{n+1}\rangle$. Thus there are M-definable functions $h_1(z_1, \dots, z_{m+1}), \dots, h_n(z_1, \dots, z_{m+1})$ such that $\xi_i = h_i(\delta_1, \dots, \delta_{m+1})$ and

$$\mathfrak{C} \models \varphi(\bar{h}(\delta_1, \dots, \delta_{m+1}), \bar{a}).$$

Hence $\varphi(\bar{h}(\delta_1, \dots, \delta_m, y, \bar{a})) \in \Delta(y) {\restriction} M\langle\bar{a}, \delta_1, \dots, \delta_m\rangle$, and there is $\alpha \in M\langle\bar{a}, \delta_1, \dots, \delta_m\rangle$ such that $\mathfrak{C} \models \forall y\ [y > \alpha \to \varphi(\bar{h}(\delta_1, \dots, \delta_m, y), \bar{a})]$. Since, by Corollary 2.5, $M\langle\delta_m\rangle$ is co-final in $M\langle\bar{a}, \delta_1, \dots, \delta_m\rangle$, we may assume $\alpha \in M\langle\delta_m\rangle$.

Then $\mathfrak{C} \models \varphi(\bar{h}(\delta_1, \dots, \delta_m, \alpha{+}1))$, hence $\varphi(\bar{x}, \bar{a})$ has a realization in $M\langle\delta_1, \dots, \delta_m\rangle$. □

This proofs Theorem 4.1. □

Corollary 4.3. *For $M \prec \mathfrak{C}$, a formula $\varphi(\bar{x}, \bar{y})$, and $\bar{a} \in C$. The following conditions are equivalent.*

(1) $\varphi(\bar{x}, \bar{a})$ does not fork over M.
(2) $\varphi(\bar{x}, \bar{a})$ has a realization in $M\langle\delta\rangle$ for any $\delta \models \Delta {\restriction} M\langle\bar{a}\rangle$.

Remark 4.4. Corollary 4.3 implies Theorem 0.2. Indeed, assume $\varphi(\mathfrak{C}, \bar{a})$ is closed and bounded. If $\varphi(\bar{x}, \bar{a})$ has a realization in $M\langle\delta\rangle$, for some $\delta \models \Delta {\restriction} M\langle\bar{a}\rangle$, then there is an M-definable map $h\colon \mathfrak{C} \to \mathfrak{C}^n$ such that $\varphi(h(t), \bar{a}) \in \varphi(\mathfrak{C}, \bar{a})$ for all sufficiently large $t \in \mathfrak{C}$. Then $\lim_{t\to+\infty} h(t)$ is a realization of $\varphi(\bar{x}, \bar{a})$ in M.

References

[1] Hans Adler, *Introduction to theories without the indepenence property.* Preprint.
[2] Alfred Dolich, *Forking and independence in o-minimal theories*, J. Symbolic Logic **69** (2004), no. 1, 215–240. MR **MR2039358 (2005g:**03050)
[3] Ehud Hrushovski and Anand Pillay, *On nip and invariant measures.* Preprint.
[4] Ya'acov Peterzil and Anand Pillay, *Generic sets in definably compact groups*, Fund. Math. **193** (2007), no. 2, 153–170. MR **MR2282713 (2008g:**03057)

[5] Marcus Tressl, *Valuation theoretic content of the Marker-Steinhorn theorem*, J. Symbolic Logic **69** (2004), no. 1, 91–93. MR **MR2039348 (2005a:**03076)

Department of Mathematics, University of Notre Dame, Notre Dame, IN 46556
E-mail address: `Starchenko.1@nd.edu`