Actions of groups of finite Morley rank on abelian groups of rank 2

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Abstract

We classify actions of groups of finite Morley rank on abelian groups of Morley rank 2: there are essentially two, namely the natural actions of $\mathrm{SL}(V)$ and $\mathrm{GL}(V)$ with V a vector space of dimension 2.

1 The result

In [BC05], Borovik and Cherlin asked the following:

Problem 15 of [BC05] *Let G be a connected group of finite Morley rank acting faithfully and definably on an abelian group V of Morley rank 2. Then either G is solvable or V has a structure of a 2-dimensional vector space over an algebraically closed field F and G is one of the groups $\mathrm{SL}_2(F)$ and $\mathrm{GL}_2(F)$ in their natural representations.*

This question appears as a first step towards a more ambitious one; we can but refer the reader to [BC05, §6]. It must also be said that the present article is very undirectly related to the so-called Cherlin-Zilber Algebraicity Conjecture, a quasi-promethean attempt at classifying simple groups of finite Morley rank. It should rather be read as an application of techniques forged to attack the Conjecture. As for Problem 15, we provide a positive answer.

Theorem *Let G be a connected, non-solvable group of finite Morley rank acting definably and faithfully on a connected abelian group V of Morley rank 2. Then there is an algebraically closed field $\mathbb{K}$ of Morley rank 1 such that $V \simeq \mathbb{K}^2$, and G is isomorphic to $\mathrm{GL}_2(\mathbb{K})$ or $\mathrm{SL}_2(\mathbb{K})$ in its natural action.*

The proof we shall give is a geometric as possible, with strong emphasis on the dichotomy between semi-simplicity and unipotence. In the context of groups of finite Morley rank the former notion is conveniently represented by good tori [Che05]; as for the latter, Burdges [Bur04] has developed a suitable 0-unipotence theory which we shall use in its simplest form.

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2 The tools

We collect here some raw meat. The proof of our Theorem takes place in §3.

2.1 Linear actions

Fact 2.1 ([Poi87, Théorème 3.9]) *Let G be a connected group of finite Morley rank acting definably and faithfully on a definable, abelian group A. Assume that G has a definable, connected, normal subgroup $H \trianglelefteq G$ which has an abelian, infinite, normal subgroup $M \trianglelefteq H$, and such that A is H-minimal. Then there is a definable field $\mathbb{K}$ and a definable finite-dimensional $\mathbb{K}$-vector space structure on A such that G embeds into $\mathrm{GL}(A)$. Moreover, M acts on A by scalar multiplication.*

Fact 2.2 (Malcev's Theorem [Poi87, Théorème 3.18]) *Let G be a connected, solvable group of finite Morley rank acting definably and faithfully on a definable, abelian group A. If a definable subgroup $B \leq A$ is G- or G'-minimal, then B is centralized by G'.*

2.2 Around PSL_2

Here is an identification result not entirely explicit in [Poi87, Corollaire 3.28].

Fact 2.3 *Let G be a group of finite Morley rank having a definable subgroup H of corank 1 such that $\bigcap_{g\in G} H^g = 1$. Then G has Morley rank 3 and is isomorphic to $\mathrm{PSL}_2(\mathbb{K})$, where $\mathbb{K}$ is a definable field of Morley rank 1.*

Proof

To see this, it suffices to note that G acts definably, transitively and faithfully on the definable coset space G/H which is strongly minimal. So the Hrushovski analysis [Poi87, Théorème 3.27] yields the conclusions. □

At some point of the proof of our Theorem we shall need to rule out $\mathrm{PSL}_2(\mathbb{K})$ in characteristic not 2. (Of course if $\mathbb{K}$ has characteristic 2, then $\mathrm{PSL}_2(\mathbb{K}) = \mathrm{SL}_2(\mathbb{K})$ will certainly remain.) This is done by the following remark.

Fact 2.4 *Let $\mathbb{K}$ be a field of finite Morley rank of characteristic $\neq 2$. Then there is no definable, faithful, irreducible action of $\mathrm{PSL}_2(\mathbb{K})$ on an abelian group of rank 2.*

Proof

Let $\mathrm{PSL}_2(\mathbb{K})$ act on such an abelian group V, and let us prove a contradiction. We first notice that V is not an elementary abelian 2-group, as otherwise letting $\tau \leq \mathrm{PSL}_2(\mathbb{K})$ be a 2-torus, we find that $V \rtimes \tau = V * \tau$, whence $\tau \leq C_G(V) = 1$. This means that $\mathbb{K}$ has characteristic 2, against our assumption.

Hence V does not have characteristic 2. Let $i \in \mathrm{PSL}_2(\mathbb{K})$ and $w \in C(i) \setminus \{i\}$ be two commuting involutions. As i is not central in $\mathrm{PSL}_2(\mathbb{K})$, we find a decomposition $V = C_V(i) \oplus [V, i]$ and both terms are non-trivial. Notice that w normalizes them; each is either centralized or inverted by w.

If w centralizes $C_V(i)$, then $w \neq 1$ cannot centralize $[V, i]$ too, and must therefore invert it. It follows that i and w have the same action on V, whence $w = i$, a contradiction. Now if w does not centralize $C_V(i)$, then it inverts it. As w is not central in $\mathrm{PSL}_2(\mathbb{K})$, it cannot invert $[V, i]$ too, so it must centralize it. It follows that iw inverts V, so iw must be central in $\mathrm{PSL}_2(\mathbb{K})$, a contradiction again. □

2.3 Around SL_2

We will have to argue that when making G isomorphic to $\mathrm{SL}_2(\mathbb{K})$ via an à la Hrushovski analysis, we actually identify the pair (G, V). In other words, we have to determine small representations of SL_2. This is possible thanks to the following, originally due to Timmesfeld.

Fact 2.5 ([ABC08, Fact II.5.28]) *Let K be a field and let V be a an $\mathrm{SL}_2(\mathbb{K})$-module. Suppose the following:*

1. *$C_V(G) = 0$ and $[G, V] = V$*
2. *$[U, U, V] = 1$, where U is a maximal unipotent subgroup of G.*

Then for some field action on $\langle v^G \rangle$, the vector space $\langle v^G \rangle$ is a natural module for each $v \in C_V(U)^\#$.

Here is the consequence we shall use.

Fact 2.6 *Let $\mathbb{K}$ be a field of Morley rank and let $G \simeq \mathrm{SL}_2(\mathbb{K})$ act definably and faithfully on a connected abelian group V of Morley rank 2. Then V is a natural module for G.*

Proof
We make use of Fact 2.5. It is clear that $C_V(G) = 0$ and $[G, V] = V$. Let U be a maximal unipotent subgroup of G, and $B = N_G^\circ(U)$. V is not B-minimal, as otherwise $B' = 1$. So there is an infinite $V_1 < V$ which is B-invariant. Now by Fact 2.2, U centralizes both V_1 and V/V_1, so $[U, U, V] = 1$. □

By the way, what happens in higher rank? using the same characterization (Fact 2.5), Altınel, Borovik, and Cherlin proved:

Fact 2.7 ([ABC08, Lemma II.5.31]) *Let G be a group of finite Morley rank that is isomorphic to $\mathrm{SL}_2(\mathbb{K})$ as an abstract group with $\mathbb{K}$ an algebraically closed field* of characteristic 2. *Let V be a connected, elementary abelian 2-group on which G acts definably and faithfully. Let $f = \mathrm{rk}\,\mathbb{K}$, and suppose $\mathrm{rk}\,V = 2f$. Then V is a natural module for G.*

And what in characteristic not 2? this is a question we shall discuss another time. Let us merely remark that Fact 2.7 is not about SL_2, but about PSL_2: its proof makes crucial use of Weyl involutions. When the characteristic moves from 2 to odd or 0, these involutions are replaced by elements of order 4, which do not enable the same computations... To be continued, so.

2.4 Further remarks on SL_2

The following result will be the basis of our analysis. It will yield solvability of centralizers (Claim 3.5 below), a good starting point when one wishes to talk about Borel subgroups. We sketch a couple of proofs, hoping one of them will seem of interest to the reader.

Lemma 2.8 *Let G be a connected group of finite Morley rank acting definably and faithfully on a connected abelian group V of Morley rank 2. Let $v_0 \in V^{\#}$. Then $C_G^\circ(v)$ is not isomorphic to SL_2 nor to GL_2.*

Proof

We may assume that there is a subgroup $H \leq C_G^\circ(v_0)$ which is isomorphic to SL_2. By Fact 2.6, we get that H acts on V as SL_2. In particular $H \leq C_G^\circ(v_0)$ is a contradiction. □

The proof we just gave relies on an identification result of the natural SL_2-module. As we have said, we still miss a more general tool when the characteristic differs from 2 and the rank is greater than 2. So the reader might enjoy the following argument, which spares us the concourse of the natural module.

Assume H is isomorphic to $\mathrm{SL}_2(\mathbb{K})$, where $\mathbb{K}$ has characteristic not 2, and V has no involutions. Pick the central involution $i \in Z(H)$. As V is $2^\perp$, we may write the decomposition $V = C_V(i) \oplus [V, i] = C_V^\circ(i) \oplus [V, i]$ by connectedness of V. Notice that $C_G^\circ(i)$ normalizes each term; a quick computation shows that if both $C_V(i)$ and $[V, i]$ have rank 1, then $C_G^\circ(i) \geq H$ is solvable, which is absurd. As $[V, i] \neq 1$, it follows that $C_V(i) = 1$ and $V = [V, i]$, that is i inverts V. This contradicts $i \in C_G^\circ(v_0)$. □

The reader may very well not be pleased at all with the latter argument either. In this case we proceed as follows, regardless of the characteristic.

Alternate proof of Lemma 2.8

Fix an algebraic torus $T \leq H$ and j inverting T (the order of j depends on the characteristic). Let $B \leq H$ be a Borel subgroup containing T, so that B^j is the other such Borel subgroup.

We claim that there is a unique B-minimal subgroup $V_1 < V$. Indeed, if V itself is B-minimal, then by Fact 2.2 $B' = 1$, absurd. So V_1 does exist, and B' centralizes V_1. If there is another B-minimal subgroup, then $B' = 1$ again. (Notice that we make crucial use of $\operatorname{rk} V = 2$ at this point.) It is clear that $B = N_H^\circ(V_1)$, so $V_2 := V_1^j \neq V_1$.

We claim that T acts freely on $V_1^{\#}$ and $V_2^{\#}$. Let $t \in C_T(V_1)$; then t centralizes V_1 and t^j centralizes V_1^j, so t centralizes $V_1 + V_1^j = V$, whence $t = 1$. Hence $C_T(V_1) = 1$, and by the Zilber field theorem, T embeds inside the multiplicative group of a field whose additive group is isomorphic to V_1. The action is thus free on $V_1^{\#}$. The same applies to $V_2^{\#}$.

We claim that $C_T(V/V_1, V/V_2) = 1$. Let $t \in C_T(V/V_1, V/V_2)$. Then fixing $v \in V$, there are $v_1 \in V_1$ and $v_2 \in V_2$ such that $v^t = v + v_1 = v + v_2$, so $v_1 = v_2 \in V_1 \cap V_2$. As this intersection of distinct groups of rank 1 is finite, it follows by connectedness that $[V, t]$ is trivial, and $t = 1$.

Hence we can assume that $K := C_T(V/V_1) < T$. In particular, T/K acts freely on $(V/V_1)^\#$. Hence $v_0 \in V_1$. As $T \le C_G^\circ(v_0)$ acts freely on $V_1^\#$, we find $v_0 = 0$, a contradiction. $\square$

3 The analysis

Notation 3.1 *Let G be a non-solvable connected group of finite Morley rank acting definably and faithfully on a connected abelian group V of Morley rank 2.*

We shall proceed to identifying the action (G, V) to a standard action of $\mathrm{GL}_2(\mathbb{K})$ or $\mathrm{SL}_2(\mathbb{K})$ on $\mathbb{K}^2$. The idea is simple. Our distinction is the following: GL_2 has an infinite center, but SL_2 doesn't. In each case the center is the intersection of all Borel subgroups. So we need a Borel subgroup, which will by definition contain a connected stabilizer. Reverting the last three sentences, we find the outline of a proof.

Before we start, here is a remark on subgroups of G admitting more than one minimal subspace.

Lemma 3.2 *Let $T \le G$ be a definable, connected subgroup that normalizes two distinct minimal subgroups of V. Then T is a good torus of rank ≤ 2.*

Proof

Assume that T normalizes both V_1 and V_2. Clearly T is abelian. Let $K = C_T(V_1)$ and $L = C_T(V_2)$. Then $K \cap L = 1$, so both K and L embed into good tori of rank 1. As T is abelian, it is a good torus too. The rank computation is immediate. $\square$

Recall that an infinite group of finite Morley rank is said *minimal* if it has no proper definable infinite subgroup. If A is such a group, then any definable group of automorphisms of A has rank $\le \mathrm{rk}\, A$ [Poi87, Proposition 3.12].

Claim 3.3 *V is G-minimal but not minimal. It is either torsion-free or an elementary abelian p-group for some prime number p.*

Proof

V can't be minimal, as otherwise G would have rank ≤ 2, and therefore be solvable. If V is not G-minimal then there is a G-invariant $V_1 < V$ of rank 1. In this case, $G/C_G(V_1)$ acts faithfully on the minimal group V_1, and must have rank ≤ 1. Besides, $C_G(V_1)/C_G(V_1, V/V_1)$ acts faithfully on the minimal group V/V_1, and must have rank ≤ 1. Now clearly $[V, C_G(V_1, V/V_1), C_G(V_1, V/V_1)] = 1$ so $C_G(V_1, V/V_1)$ is abelian, and G is an extension of rank ≤ 2 of the latter, hence solvable. This contradiction proves that V is G-minimal.

The second statement now follows from MacIntyre's theorem on abelian groups, together with the remark that V can't have divisible torsion by rigidity of p-tori. $\square$

Notation 3.4 *Let $v_0 \in V^\#$. Let $H = C_G^\circ(v_0) < G$.*

Clearly H has corank at most 2 in G. Our first step is to argue that there is a Borel subgroup of G containing H.

Claim 3.5 *H is solvable.*

Proof

If V is not H-minimal, then there is $V_1 < V$ which is H-minimal. Let $K = C_H(V_1)$ and $L = C_K(V/V_1)$. Notice that $[L,[L,V]] = [L,V_1] = 1$ implies that L is abelian. As K/L acts faithfully on the minimal group V/V_1, it is abelian, so K is solvable. As H/K acts faithfully on the minimal group V_1, it is abelian, so H is solvable.

So we now suppose that V is H-minimal. In particular, $C_V(H)$ is finite and H acts definably and faithfully on the connected abelian group $V/C_V(H)$ of rank 2. As $H < G$, we may apply induction. Then $H \simeq \mathrm{SL}_2(\mathbb{K})$ or $\mathrm{GL}_2(\mathbb{K})$, and we find a contradiction to Lemma 2.8. □

Notation 3.6 *Let $B \geq H$ be a Borel subgroup.*

Claim 3.7 *V is not B-minimal.*

Proof

Otherwise B is abelian by Fact 2.2, so there is an interpretable $\mathbb{L}$ such that $V \simeq \mathbb{L}_+$ and B embeds into $\mathbb{L}^\times$. In particular, $H = C_G^\circ(v_0) \leq B$ must act freely on $V^\#$, a contradiction. □

Notation 3.8 *Let $V_1 < V$ be a B-minimal subgroup.*

We want the Borel subgroup B to behave like in SL_2 or GL_2. The next step is more delicate than the others; we shall prove that B is not morally semi-simple. A very efficient modelisation of semi-simplicity in the context of groups of finite Morley rank is the notion of a *good torus* introduced in [Che05].

Claim 3.9 *B is not a good torus.*

Proof

Assume it is. We shall show that G has rank 3, find a quotient isomorphic to PSL_2, and derive a contradiction.

Step 1: G has rank 3. Let $K = C_H(V_1)$ and $L = C_H(V/V_1)$. Consider the set $X = K \cup L$. If $X \subsetneq H$, then any $h \in H \setminus X$ has a non-trivial image in H/K, which acts freely on $V_1^\#$, and in H/L, which acts freely on $(V/V_1)^\#$. As $H = C_G^\circ(v_0)$, the latter implies $v_0 \in V_1$, and the former a contradiction. It follows $X = H$, and in particular H is equal to K, or to L.

Let $U = C_H^\circ(V_1, V/V_1) = (K \cap L)^\circ$. We claim that $U = 1$. Assume that V has exponent p. Then for each $u \in U$ and $v \in V$, there is $v_1 \in V_1$ with $v^u = v + v_1$. Applying u again and again, we end up with $v^{u^p} = v + pv_1 = v$, whence $u^p \in C_G(v)$, and as this is true regardless of v, we find $u^p = 1$. So U is a p-unipotent subgroup of $H \leq B$; by assumption, $U = 1$. Let us now deal with the characteristic 0 case. Let $u \in U$ be torsion, say $u^n = 1$. For any $v \in V$, there is $v_1 \in V_1$ with $v^u = v + v_1$. Applying u again and again yields this time $v^{u^n} = v = v + nv_1$, so $nv_1 = 0$ and as V is torsion-free, $v_1 = 0$, i.e. $u \in C_G(v)$. It follows that the torsion subgroup of U is trivial; as B is a good torus, $U = 1$ again.

This proves that $\operatorname{rk} K \leq 1$ and $\operatorname{rk} L \leq 1$. As H is either K or L, we deduce $\operatorname{rk} H = 1$. Recall that H has corank ≤ 2 in G; hence as G is non-solvable, it has rank 3.

Step 2: finding a subgroup of rank 2. We now find a subgroup of G of rank 2. If B has rank 1, then $B = H$ is a good torus of rank 1. In particular it contains a p-torus τ for some prime number p, and $B = d(\tau)$. Notice that p can't be the exponent of V, as otherwise τ would centralize V. Pick an element $t \in \tau$ of order p^n. By coprime action [ABC08, Corollary I.9.11], $V = C_V(t) \oplus [V, t]$ and both factors are connected. As $v_0 \in C_V(t)$, we deduce that $V_0 := C_V(t)$ has Morley rank exactly 1. Now if $s \in \tau$ is a root of t of order $p^m > p^n$, then the same applies to s, and in particular $V_0 = C_V(s)$. Thus $V_0 = C_V(\tau) = C_V(B)$.

It follows that if $g \notin N_G(V_0)$ and $v \in V_0 \cap V_0^g$, then $C_G(v) \geq B, B^g$. As a conclusion, V_0 is disjoint from its distinct conjugates, and in particular $\operatorname{rk} V_0^G = \operatorname{rk} G - \operatorname{rk} N_G(V_0) + \operatorname{rk} V_0 \leq 2$ together with G-minimality of V forces $\operatorname{rk} N_G(V_0) = 2$.

Step 3: contradiction. $G/Z(G)$ is a simple group of rank 3 having a definable subgroup of rank 2. By Fact 2.3, $G/Z(G)$ is isomorphic to $\mathrm{PSL}_2(\mathbb{K})$ for some interpretable field $\mathbb{K}$. But now no Borel subgroup of G can be a good torus, against the assumption that B is one. This is a contradiction. $\square$

Now we know that B is not a good torus we reach a more algebraic landscape.

Claim 3.10 *$B = N_G^\circ(V_1)$ and $N_G(B) = N_G(V_1)$.*

Proof

$N_G^\circ(V_1)$ is easily proved solvable. As B is a Borel subgroup, we have $B = N_G^\circ(V_1)$. Also, Lemma 3.2 and Claim 3.9 imply that V_1 is the only B-minimal subgroup of V. Therefore $N_G(B) = N_G(V_1)$. $\square$

A new tool comes into the picture: unipotence. We direct the reader to [DJ07] for terminology and basic results about unipotence notions. As B is not a good torus, it admits a non-trivial unipotence parameter [DJ07, Lemma 2.11].

Notation 3.11 *Let $\tilde{q} \neq (\infty, 0)$ be a maximal unipotence parameter for B and let $U = U_{\tilde{q}}(B)$.*

In general, dealing with Burdges' characteristic 0 unipotence requires care. Here the situation is excellent, insofar as only one unipotence degree is possible.

Claim 3.12 *If V is torsion-free, then $d_\infty(V \rtimes G) \leq 1$. Moreover $\tilde{q} = (p, \infty)$ iff V has exponent p; $\tilde{q} = (\infty, 1)$ iff V is torsion-free. In other words, U and V_1 have the same unipotence parameter. In particular, $U \leq C_G^\circ(V_1)$.*

Proof

If V is torsion-free, then as V is not minimal by Claim 3.3, $d_\infty(V) = 1$. For all the remaining statements, consider the solvable group VU in the light of Burdges' structure theorem for nilpotent groups [DJ07, Fact 2.5]. $\square$

The main tool when using unipotence is Jaligot's Rigidity Lemma which asserts that Borel subgroups tend not to share unipotence. We give here a quite trivial form. The reader may have already met a more subtle version in which some control of the unipotence degree is required; but such control is here provided by Claim 3.12.

Fact 3.13 ([DJ07, Corollary 4.4]) *Let G be a locally$^\circ$ solvable$^\circ$ group of finite Morley rank, $\tilde{p} = (p, r)$ a unipotence parameter with $r > 0$ such that $d_p(G) = r$. Let B be a Borel subgroup of G such that $d_p(B) = r$. Then $U_{\tilde{p}}(B)$ is a Sylow $\tilde{p}$-subgroup of G, and if U_1 is a non-trivial definable $\tilde{p}$-subgroup of B, then $U_{\tilde{p}}(B)$ is the unique Sylow $\tilde{p}$-subgroup of G containing U_1, $N(U_1) \leq N(U_{\tilde{p}}(B)) = N(B)$, and B is the unique Borel subgroup of G containing U_1.*

On the other hand, we have not explained yet what is *local$^\circ$ solvability$^\circ$*, and why G should enjoy such a property. In [DJ07], following Thompson and in spite of another traditional terminology, a group of finite Morley rank G is said *locally$^\circ$ solvable$^\circ$* if whenever $1 < A \leq G$ is a definable, connected, solvable subgroup, then so is $N_G^\circ(A)$. The reader will not be surprised to hear that we do not claim that our group G is locally$^\circ$ solvable$^\circ$: after all, we are trying to identify $\mathrm{GL}_2(\mathbb{K})$! But it turns out that in order to prove and use Fact 3.13, the following weaker condition suffices.

Claim 3.14 *If $1 < U \leq G$ is a $\tilde{q}$-group, then $N_G^\circ(U)$ is solvable.*

Proof
Let $N = N_G^\circ(U)$. If N is not solvable, then V is N-minimal. (This was proved above for H, first paragraph of the proof of Claim 3.5.) In particular, N acts definably and faithfully on $V/C_V(N)$ which is a connected abelian group of rank 2.

If $N < G$ we apply induction: N is isomorphic to either $\mathrm{SL}_2(\mathbb{K})$, or $\mathrm{GL}_2(\mathbb{K})$, or a finite-by-bad group. As N has an infinite, proper normal subgroup, it cannot be isomorphic to $\mathrm{SL}_2(\mathbb{K})$ nor to a finite-by-bad group. Furthermore, if $N \simeq \mathrm{GL}_2(\mathbb{K})$, then U embeds into $\mathbb{K}^\times$ where $\mathbb{K}$ has rank 1, so $U \simeq \mathbb{K}^\times$ is a good torus, a contradiction to $U = U_{\tilde{q}}(B)$.

If $N = G$, then applying Fact 2.1 we find a $\mathbb{K}$-vector space structure on V such that $V \simeq \mathbb{K}^2$ and G embeds into $\mathrm{GL}(V)$, and we argue similarly. □

Claim 3.15 *Up to changing $v_0 \in V^\#$, we may suppose that B has corank 1.*

Proof
If H has corank 1 we are done as $H \leq B$. So assume that H has always corank 2. Then for each $v_0 \in V^\#$, v_0^G is generic, and in particular G is transitive on $V^\#$. If $H < B$ we are done. So suppose $H = B$. Recall from Claim 3.12 that $U \leq C_G^\circ(V_1)$.

Let $g \notin N_G(V_1)$. If there is $v \in (V_1 \cap V_1^g)^\#$, then $C_G^\circ(v) \geq U, U^g$. As G is transitive on $V^\#$, v is conjugate to v_0, and $C_G^\circ(v)$ is solvable by Claim 3.5. Now Fact 3.13 (valid here in view of Claim 3.14) forces $g \in N_G(B) = N_G(V_1)$ by Claim 3.10, a contradiction.

Therefore the subgroups $\{V_1^g\}$ are pairwise disjoint, and $N_G^\circ(V_1)$ must have corank 1. But $N_G^\circ(V_1) = B$ by Claim 3.10. □

We now are ready to recognize the group.

Notation 3.16 *Let $N = \bigcap_{g \in G} B^g$.*

Claim 3.17 *If N is infinite, then there is an interpretable field $\mathbb{K}$ of rank 1 such that $V \simeq \mathbb{K}^2$ and $G = \mathrm{GL}(V)$.*

Proof

Let $g \notin N_G(B) = N_G(V_1) < G$. By Lemma 3.2, $B \cap B^g$ is a good torus; hence $N^\circ \neq 1$ is abelian. Then by Fact 2.1, G embeds into $\mathrm{GL}(V)$ for some $\mathbb{K}$-vector space structure on V. Clearly $\operatorname{rk} \mathbb{K} = 1$ and $V \simeq \mathbb{K}^2$. As G is non-solvable but has an infinite, normal subgroup, $G = \mathrm{GL}(V)$. □

Claim 3.18 *If N is finite, then there is an interpretable field $\mathbb{K}$ of rank 1 such that $V \simeq \mathbb{K}^2$ and $G \simeq \mathrm{SL}(V)$.*

Proof

G/N is a finite, central quotient of G which contains B/N of corank 1. Fact 2.3 yields $G/N \simeq \mathrm{PSL}_2(\mathbb{K})$ for some interpretable field $\mathbb{K}$ of rank 1. As $\operatorname{rk} G = 3$ and G is not solvable, one must have $G' = G$, that is G is perfect: and therefore a perfect central extension of a quasi-simple group. It is known that G is algebraic itself [ABC08, Proposition 3.1 p.136]. It follows $G \simeq \mathrm{PSL}_2$ or SL_2 (arguing for example that G has Lie rank 1). Now Facts 2.4 and 2.6 give the conclusion. □

References

[ABC08] Tuna Altınel, Alexandre Borovik, and Gregory Cherlin. *Simple groups of finite Morley Rank*. Amer. Math. Soc., 2008.

[BC05] Alexandre Borovik and Gregory Cherlin. Permutation groups of finite Morley rank. In *Model Theory and Applications to Algebra and Analysis*. Cambridge University Press, 2005.

[Bur04] Jeffrey Burdges. *Simple Groups of Finite Morley Rank of Odd and Degenerate Type*. PhD thesis, Rutgers University, New Brunswick, New Jersey, 2004.

[Che05] Gregory Cherlin. Good tori in groups of finite Morley rank. *J. Group Theory*, 8(5):613–621, 2005.

[DJ07] Adrien Deloro and Eric Jaligot. Groups of finite Morley rank with solvable local subgroups. Available at arXiv:0802.1394v2 [math.GR]. Submitted, 2007.

[Poi87] Bruno Poizat. *Groupes stables*. Nur al-Mantiq wal-Ma'rifah, Villeurbanne, 1987.