RED FIELDS

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ABSTRACT. We apply Hrushovski-Fraïssé's amalgamation procedure to obtain a theory of fields of prime characteristic of Morley rank 2 equipped with a definable additive subgroup of rank 1.

1. Introduction

In the early 90's E. Hrushovski came up with an interesting application of Fraïssé's construction to obtain certain structures whose rank was described by a priori exhibited predimension function δ. Unfortunately, this procedure has remained an obscure area of model theory to the general audience, in spite of its relevant applications. Hrushovski's amalgamation method is – in principle – a two-step process: first, one obtains a Fraïssé generic model from a certain class so that the above model exhibits some of the properties we are interested in, though of infinite rank. Secondly, one makes it *collapse*, i.e. we impose certain restrictions on the class we are amalgamating in, related to algebraizing certain types which have small δ-*dimension* but large rank in the aforementioned model. This relies on some combinatorial arguments in order to determine on advance the maximal number of realizations allowed for a given type within our class.

In [11] a theory of fields of rank $\omega \cdot 2$ with a definable additive subgroup of rank ω was constructed by applying the first part of the amalgamation method as aforementioned described. It was asked by by B. Poizat in [9] after Corollary 3.3 whether a similar structure of finite rank could exist. It is known that such a field cannot be of characteristic 0, because they have no definable non-trivial additive subgroups. In this paper, we will collapse the above structure and answer the previous question positively. Moreover, an explicit description of the axioms of the generic model obtained is given. The *red* additive subgroup has Morley rank 1 and the whole structure Morley rank 2.

The main source of this work is E. Hrushovski's fusion of two strongly minimal sets [6]. Using Hrushovski's paper B. Poizat [10] and J. Baldwin and K. Holland [1] produced a field with black points of rank 2. In [2] we gave an exposition of [6], simplifying some of the arguments, which allowed us in [3] to give a new construction of a field with black points of rank 2 with a simpler axiomatization. The methods used there will be developed further in this paper in order to obtain a field of rank 2 with a *red* additive subgroup. Similar ideas yield a fusion over a vector space over a finite field of two strongly minimal sets with DMP in [4].

1991 *Mathematics Subject Classification.* Primary: 03C65; Secondary:03C50.

Key words and phrases. Model Theory, Fields of finite Morley Rank.

A. Martin-Pizarro conducted research supported by a Marie Curie research grant MEIF-CT-2005-009541 and a DFG-Forschungsstipendium MA3310/1-1.

The authors would like to thank the Newton Institute (Cambridge, UK) for the hospitality provided during the semester *Model Theory and its applications to Algebra and Analysis*, in which the three authors took part. Moreover, the second author would like to thank the **D**eutsche **F**orschungs**g**emeinschaft for its support during Spring Semester 2005.

2. Group sets

The following Lemma was proven in 1990 by the third author [12].

Lemma 2.1. *Let G be a stable abelian group, and a, b and c in G be pairwise independent over B with $a+b+c=0$. Then:*

(1) *Their strong types over B have the same stabilizer U. Furthermore, U is connected.*
(2) *a, b and c are generic elements of $\operatorname{acl}(B)$–definable cosets of U.*

Moreover, if G is totally transcendental, it follows that a, b and c have the same Morley rank over B, namely $\operatorname{MR}(U)$, and U is definable over $\operatorname{acl}(B)$.

Note 2.2. Suppose that $a \in U$. Then $-b$ and c have the same type over B, since they are generic in the same coset of U.

As in Section 2 in [3] we will encode types using *codes*. These codes will enable us to define the class of structures we are concerned with and later use them to give an axiomatization of the Fraïssé generic model.

We work inside $\mathbb{U}$, a universal algebraically closed field of prime characteristic p and all formulae are L–formulas, where L denotes the language of rings. All sets in this section will be subsets of some cartesian power $\mathbb{U}^n$, unless specified. We use $\deg_{\mathrm{M}}$ to denote the Morley degree.

Definition 2.3. Let X be a definable subset of $\mathbb{U}^n$ with $\deg_{\mathrm{M}}(X)=1$. We say that X is a *group set* (resp. *torsor set*) if its generic type is the generic type of a definable subgroup G (resp. coset of a subgroup) of $(\mathbb{U}^n,+)$. If X is not a torsor set, then X is called *groupless*.

Notation 2.4. Two definable sets X and Y are *equivalent*, if $\operatorname{MR}(X)=\operatorname{MR}(Y)$ and $\operatorname{MR}\big(X \vartriangle Y\big) < \operatorname{MR}\big(X\big)$. We write $X \sim Y$.

Given $A \subset \mathbb{U}$, we denote by $\langle A\rangle$ the subspace generated by A (considering $\mathbb{U}$ as an $\mathbb{F}_p$-vector space in the natural way).

Lemma 2.5 (Hasson, Hils [5])**.** *With notation as above, the following holds:*

(1) *Given $Y \sim X$, if X is a group set, so is Y.*
(2) *Given H in $\mathrm{GL_n}(\mathbb{F}_p)$, if X is a group set, so is $H(X)=\{H\vec{x}\,|\,\vec{x}\in X\}$.*
(3) *The set $\{\,\vec{b}\,|\,\varphi(\mathbb{U},\vec{b})$ is a group-set $\}$ is definable for every formula $\varphi(\vec{x},\vec{y})$.*

Similarly for torsor sets.

Proof. The first two statements are clear.

For the third point, note that algebraically closed fields have the DMP. So $\deg_{\mathrm{M}}\varphi(\vec{x},\vec{b})=1$ is an elementary property of $\vec{b}$. We claim that $\varphi(\mathbb{U},\vec{b})$ is a group set iff "There exist two generic $\vec{b}$–independent realizations $\vec{a}_1$ and $\vec{a}_2$ of $\varphi(\vec{x},\vec{b})$ whose sum $\vec{a}_1+\vec{a}_2$ lies in $\varphi(\vec{x},\vec{b})$." This is easily seen to be an elementary property of b. One direction of the equivalence is clear. To prove the other direction let $\vec{a}_0$, $\vec{a}_1$ have the property above and let k be the Morley rank of $\varphi(\vec{x},\vec{b})$. We have $\mathrm{MR}(\vec{a}_0+\vec{a}_1/\vec{b}\vec{a}_0) = \mathrm{MR}(\vec{a}_1/\vec{b}\vec{a}_0) = k$ and therefore $\mathrm{MR}(\vec{a}_0+\vec{a}_1/\vec{b}) \geq k$. $\models \varphi(\vec{a}_0+\vec{a}_1,\vec{b})$ implies that $\mathrm{MR}(\vec{a}_0+\vec{a}_1/\vec{b}) = k$, so $\vec{a}_0$, $\vec{a}_1$, $\vec{a}_0+\vec{a}_1$ are pairwise independent over $\vec{b}$. And they all have the same type over $\vec{b}$. Hence by Lemma 2.1 $\mathrm{tp}(\vec{a}_0/\vec{b})$ is a group type.

Note also that X is a torsor set if and only if for some x the set $X-x$ is a group set. □

Definition 2.6. Given a group set X, its *invariant group* is the set $\mathrm{Inv}(X) = \{H \in \mathrm{GL_n}(\mathbb{F}_p) \mid H(X) \sim X\}$.

Note that "$H \in \mathrm{Inv}(\varphi(\vec{x},\vec{b}))$" is an elementary property of $\vec{b}$.

Lemma 2.7. *Let X be a B-definable set of Morley degree 1, and $\vec{e}_0$ and $\vec{e}_1$ two generic B-independent elements. If $\vec{e}_0 - H\vec{e}_1 \mathop{\smile\!\!\!\!\!\!\;|}_B \vec{e}_0$ for some H in $\mathrm{GL_n}(\mathbb{F}_p)$, then X is a torsor set. Moreover, if X is a group set, then H is in $\mathrm{Inv}(X)$.*

Proof. From

$$\mathrm{MR}(H\vec{e}_1/B,\vec{e}_0-H\vec{e}_1) = \mathrm{MR}(\vec{e}_0/B,\vec{e}_0-H\vec{e}_1) = \mathrm{MR}(\vec{e}_0/B) \geq \mathrm{MR}(H\vec{e}_1/B)$$

we obtain that $\{\vec{e}_0, H\vec{e}_1, \vec{e}_0-H\vec{e}_1\}$ is a pairwise B-independent triple. By 2.1 X is an torsor set. If G is the group that corresponds to X, then $\vec{e}_0$ and $H\vec{e}_1$ are generic elements of some cosets of G. That implies that $H(G)=G$. □

3. Prealgebraic Sets

Let us fix some notation. For any subset A of $\mathbb{U}$ let $\langle A\rangle$ be the $\mathbb{F}_p$–vector space generated by A, and $\dim(A)$ the dimension of $\langle A\rangle$. The relative dimension $\dim(A/B)$ is the dimension of $\langle A,B\rangle/\langle B\rangle$. We write $\mathrm{trdeg}(A/B)$ for the transcendence degree of A over B.

Definition 3.1. Let $n=2k>0$ and X be a $\vec{b}$–definable subset of $\mathbb{U}^n$, of Morley rank k and Morley degree 1. We call X *prealgebraic* (over the parameters $\vec{b}$) if for all generic $\vec{a}\in X$, the following holds:

a) $\dim(\vec{a}) = n$.
b) Given a non-trivial subspace U of $\langle\vec{a}\rangle$, then

$$2\cdot \mathrm{trdeg}(U/\vec{b}) > \dim(U). \tag{3.1}$$

Equivalently,

$$2\cdot \mathrm{trdeg}(\vec{a}/U\vec{b})) < n - \dim(U). \tag{3.2}$$

Lemma 3.2. *If X is prealgebraic over $\vec{b}$ and $\vec{a}$ in X is generic, then* $\dim(\vec{a}/\vec{b}) = n$.

Proof. Clearly, $\langle\vec{a}\rangle \cap \mathrm{acl}(\vec{b}) = 0$ by Equation (3.1). □

Lemma 3.3. *Prealgebraicity is preserved under equivalence*[1]*, translation and the action of* $\mathrm{GL_n}(\mathbb{F}_p)$.

From the first assertion it follows that prealgebraicity does not depend on the choice of parameters. We use this to prove the second statement.

[1]in the sense of 2.4

Proof. Suppose X is prealgebraic defined over $\vec{b}$ and $X \sim Y$ with Y defined over $\vec{b}'$. Choose a generic element $\vec{a}$ of $X \cap Y$. For a subspace U of $\langle \vec{a} \rangle$, we have that $U \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{b}} \vec{b}'$ and $U \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{b}'} \vec{b}$. So $\mathrm{MR}(U/\vec{b}) = \mathrm{MR}(U/\vec{b}')$ and Y is prealgebraic.

Let now $\vec{m}$ be in $\mathbb{U}^n$. We want to show that $X - \vec{m}$ is again prealgebraic. Choose $\vec{a}$ in X, generic over $\vec{m}, \vec{b}$. Hence, $\vec{a} - \vec{m}$ is generic in $X - \vec{m}$ over $\vec{b}$, $\vec{m}$. Given x in $\langle \vec{a} \rangle \cap \mathrm{acl}(\vec{b}\vec{m})$, it follows from $\vec{a} \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{b}} \vec{m}$ that x lies in $\mathrm{acl}(\vec{b})$. By 3.2 $x = 0$. In particular,

$$\langle \vec{a} \rangle \cap \mathrm{acl}(\vec{m}) = 0$$

Therefore, the coordinates of $\vec{a} - \vec{m}$ are linearly independent. Let $U \subset \langle \vec{a} - \vec{m} \rangle$ be a non-trivial subspace. There is some subspace $V \subset \mathbb{F}_p^n$ with

$$U = \{ v \cdot (\vec{a} - \vec{m}) \mid v \in V \}$$

where $\cdot$ denotes the scalar product. By the above, $\dim U = \dim V$ and we have

$$2 \cdot \mathrm{trdeg}(U/b, m) = 2 \cdot \mathrm{trdeg}(Va/b, m) = 2 \cdot \mathrm{trdeg}(Va/b) > \dim(Va) = \dim(U).$$

The case of a transformation via an invertible matrix is clear. □

4. Codes

Definition 4.1. The formula $\varphi(\vec{x}, \vec{y})$ *encodes* X if there is some $\vec{b}$ in $\mathbb{U}$ such that $X \sim \varphi(\vec{x}, \vec{b})$.

Definition 4.2. A *code* α is a tuple consisting of the following objects: natural numbers n_α and k_α and a formula $\varphi_\alpha(\vec{x}, \vec{y})$ with the following properties:

(a) $\mathrm{length}(\vec{x}) = n_\alpha = 2k_\alpha$.
(b) The set $\varphi_\alpha(\vec{x}, \vec{b})$ is either empty or has Morley rank k_α and Morley degree 1.
(c) Let $\vec{a}$ be a realization of $\varphi_\alpha(\vec{x}, \vec{b})$. Then $\dim(\vec{a}) = n_\alpha$ and Equation (3.2) in Definition 3.1 holds for all non-trivial subspaces of $\langle a \rangle$.
(d) $\varphi_\alpha(\vec{x}, \vec{b}) \sim \varphi_\alpha(\vec{x}, \vec{b}') \implies b = b'$.
(e) If some non-empty $\varphi_\alpha(\vec{x}, \vec{b})$ is groupless, then all $\varphi_\alpha(\vec{x}, \vec{b}')$ are. (We call hence α a *groupless* or *coset* code accordingly.)
(f) $\varphi_\alpha(\vec{x} + \vec{m}, \vec{b})$ is encoded by φ_α for all $\vec{m}$.
(g) For all H in $\mathrm{GL}_{\mathrm{n}_\alpha}(\mathbb{F}_p)$, the set $\varphi(H\vec{x}, \vec{b})$ is encoded by φ_α.

It follows from (d) that $\vec{b}$ is a canonical basis of the type of Morley rank k_α determined by $\varphi_\alpha(\vec{x}, \vec{b})$. Moreover, either $\varphi_\alpha(\vec{x}, \vec{b})$ is empty or prealgebraic by (c).

We first show that these codes enable us to encode prealgebraic sets.

Lemma 4.3. *Every prealgebraic set $X \subset \mathbb{U}^n$ is encoded by some code α as above.*

Proof. A straight-forward generalization of the argument exhibited in Lemma 2.3 [3] following [6] shows how to find such a formula $\varphi_0(\vec{x}, \vec{b})$ with properties (a)-(d) for a given set X as above.[2]

By case 3 in Lemma 2.5, we can strengthen φ_0 so that (e) holds. Let us call such a formula $\varphi_0(\vec{x}, \vec{b})$ a *good* formula.

Let $T(\vec{x})$ denote an affine transformation (compositum of an element of $\mathrm{GL}_{\mathrm{n}}(\mathbb{F}_p)$ and a translation by some tuple $\vec{m}$). By Lemma 3.3 $T(X)$ is again a set as in the

[2] For (c) we use that the dimension of $\langle a \rangle$ can be elementarily expressed, since $\mathbb{F}_p$ is *finite*.

statement, hence it can be encoded by a good formula. Therefore, by compactness there are finitely many $\varphi_1, \ldots, \varphi_r$ that encode all possible $T(X)$'s. Moreover, we may assume that either all (*or none*) encode groupless sets by 2.5. Choose now $w_1, \ldots, w_r$ different $\mathbb{F}_p$-tuples with the same length. Define:

$$\begin{aligned} \theta_i^1(\vec{b}) &= \text{“ No } \varphi_j \ (j < i) \text{ encodes } \varphi_i(\vec{x}, \vec{b})\text{”} \\ \theta_i^2(\vec{b}) &= \text{“ } \varphi_{\alpha_i}(\vec{x}, \vec{b}) \text{ is equivalent to some } \varphi_0(H\vec{x} + \vec{m}', \vec{b}') \text{ ”} \\ \varphi_i'(\vec{x}, \vec{y}) &= \varphi_i(\vec{x}, \vec{y}) \wedge \theta_i^1(\vec{y}) \wedge \theta_i^2(\vec{y}) \end{aligned}$$

Finally, let

$$\varphi_\alpha(\vec{x}, \vec{y}_1, \vec{y}) = \bigvee_{i=1}^{r} (\varphi_i'(\vec{x}, \vec{y}) \wedge \vec{y}_1 = w_i)$$

It is clear that φ_α has properties (a)-(e). To show that (f) and (g) hold, let $\vec{b}$, $\vec{m}$ and H be given. By construction, $\varphi_\alpha(\vec{x}, \vec{b})$ is equivalent to some $\varphi_0(H'\vec{x} + \vec{m}', \vec{b}')$. Hence,

$$\varphi_\alpha(H\vec{x} + \vec{m}, \vec{b}) \sim \varphi_0\left((H'H)\vec{x} + (H'\vec{m} + \vec{m}'), \vec{b}'\right)$$

The above is again encoded by α by construction. □

Now, we can choose our set of representatives:

Theorem 4.4. *There is a set $\mathcal{C}$ of codes such that any definable prealgebraic set X of Morley degree 1 is encoded by a unique α in $\mathcal{C}$.*

Proof. Let α_i be a list of all codes. Again, define:

$$\begin{aligned} \theta_i(\vec{b}) &= \text{“ No } \alpha_j \ (j < i) \text{ encodes } \varphi_{\alpha_i}(\vec{x}, \vec{b})\text{”} \\ \varphi_{\alpha_i'}(\vec{x}, \vec{y}) &= \varphi_\alpha(\vec{x}, \vec{y}) \wedge \theta_i(\vec{y}) \end{aligned}$$

We need only show that the $\varphi_{\alpha_i'}$ still satisfy (f) and (g). By construction, $\varphi_{\alpha_i'}(H\vec{x} + \vec{m}, \vec{b})$ is encoded by α_i. We need only show that no α_j with $j < i$ encodes it. Suppose that

$$\varphi_{\alpha_i}(H\vec{x} + \vec{m}, \vec{b}) \sim \varphi_{\alpha_j}(\vec{x}, \vec{b}')$$

Then

$$\varphi_{\alpha_i}(\vec{x}, \vec{b}) \sim \varphi_{\alpha_j}(H^{-1}\vec{x} - H^{-1}\vec{m}, \vec{b}') \sim \varphi_{\alpha_j}(\vec{x}, \vec{b}'')$$

for some $\vec{b}''$, which contradicts our definition of φ'_α. Finally set

$$\mathcal{C} = \{ \alpha_i' \mid i = 0, 1, \ldots \}.$$

□

For the rest of the paper we fix a set C of codes as given by the above theorem. We call the codes in C *good* codes.

We now choose for each α in $\mathcal{C}$ a natural number m_α such for each Morley sequence $\vec{e_i}$ of $\varphi_\alpha(\vec{x}, \vec{b})$ of length m_α and all $\vec{b}'$ with $|\vec{b}'| = |\vec{b}|$ we have $\vec{e_i} \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{b}} \vec{b}'$ for some i. We can always find $m_\alpha \leq |\vec{b}| + 1$.

Theorem 4.5. *For each $\alpha \in \mathcal{C}$ and $\lambda \geq m_\alpha$ there is a formula $\Psi_\alpha(\vec{x}_0, \ldots, \vec{x}_\lambda)$ with following properties:*

(a) *For any initial segment* $\{\vec{e}_0, \ldots, \vec{e}_\lambda, f\}$ *of a Morley sequence of* $\varphi_\alpha(\vec{x}, \vec{b})$,

$$\Psi_\alpha(\vec{e}_0 - f, \ldots, \vec{e}_\lambda - f)$$

holds.

(b) *For each realization* $(\vec{e}_0, \ldots, \vec{e}_\lambda)$ *of* Ψ_α *there is a unique* $\vec{b}$ *with* $\models \varphi_\alpha(\vec{e}_i, \vec{b})$ *for* $0 \leq i \leq \lambda$. *Moreover,* $\vec{b} \in \mathrm{dcl}(\vec{e}_{i_1}, \ldots, \vec{e}_{i_{m_\alpha}})$ *for any* $i_1 < \cdots < i_{m_\alpha}$. *(We call* $\vec{b}$ *the canonical parameter of the sequence* $\vec{e}_0, \ldots, \vec{e}_\lambda$*).*

(c) *Each realization of* Ψ_α *is* $\mathbb{F}_p$*–linear independent.*

(d) *If* $\models \Psi_\alpha(\vec{e}_0, \ldots, \vec{e}_\lambda)$, *then for* $i \in \{0, \ldots, \lambda\}$:

$$\models \Psi_\alpha(\vec{e}_0 - \vec{e}_i, \ldots, \vec{e}_{i-1} - \vec{e}_i, -\vec{e}_i, \vec{e}_{i+1} - \vec{e}_i, \ldots, \vec{e}_\lambda - \vec{e}_i)$$

(e) *Given a realization* $(\vec{e}_0, \ldots, \vec{e}_\lambda)$ *of* Ψ_α *with canonical parameter* $\vec{b}$ *as in* (b), *we have the following:*

Suppose α *is groupless:*

1) *If* $\vec{e}_i$ *generic in* $\varphi(\vec{x}, \vec{b})$, *then*

$$\vec{e}_i - H\vec{e}_j \not\mathop{\smile}_{\vec{b}} \vec{e}_i$$

for all H *in* $\mathrm{GL}_{\mathrm{n}_\alpha}(\mathbb{F}_p)$ *and* $j \neq i$.

Suppose α *is an coset code, then:*

2) $\varphi_\alpha(\vec{x}, \vec{b})$ *is a group-set.*
3) $\Psi_\alpha(e_0, \ldots, e_{i-1}, e_i - e_j, e_{i+1}, \ldots, e_\lambda)$ *for* $j \neq i$.[3]
4) $\Psi_\alpha(e_0, \ldots, e_{i-1}, He_i, e_{i+1}, \ldots, e_\lambda)$ *for all* H *in* $\mathrm{Inv}(\varphi_\alpha(\vec{x}, \vec{b}))$.[3]
5) *Moreover, if* $\vec{e}_i$ *is generic in* $\varphi_\alpha(\vec{x}, \vec{b})$, *then*

$$\vec{e}_i - H\vec{e}_j \not\mathop{\smile}_{\vec{b}} \vec{e}_i$$

for all $j \neq i$ *and* H *in* $\mathrm{GL}_{\mathrm{n}_\alpha}(\mathbb{F}_p) \setminus \mathrm{Inv}(\varphi_\alpha(\vec{x}, \vec{b}))$.

In Section 7 we will assign a fixed large $\lambda = \mu(\alpha)$ to every α.

Proof. Consider the following partial type

$$\Sigma(\vec{e}_0, \ldots, \vec{e}_\lambda) = \text{“ there is some } \vec{b}' \text{ and some Morley sequence } \vec{a}_0, \ldots, \vec{a}_\lambda, \vec{f} \text{ of } \varphi_\alpha(\vec{x}, \vec{b}') \text{ with } \vec{e}_i = \vec{a}_i - \vec{f}\text{”.}$$

Claim. Σ *has properties* (a)–(e).

Proof of the claim. By definition, Σ has property (a). Given a realization $\vec{e}_0, \ldots, \vec{e}_\lambda$ of Σ, there are some $\vec{b}'$ and $\vec{a}_0, \ldots, \vec{a}_\lambda, \vec{f}$ as above. Hence, $\{\vec{e}_i\}_{0 \leq i \leq \lambda}$ is a Morley sequence of $\varphi_\alpha(\vec{x} + \vec{f}, \vec{b}')$. Then $\varphi_\alpha(\vec{x} + \vec{f}, \vec{b}') \sim \varphi_\alpha(\vec{x}, \vec{b})$ for some $\vec{b}$ by (f) in Definition 4.2. Since $\vec{b}$ is the canonical base of the type determined by $\varphi_\alpha(\vec{x}, \vec{b})$, the sequence $\{\vec{e}_i\}_{0 \leq i \leq \lambda}$ is a Morley sequence for $\varphi_\alpha(\vec{x}, \vec{b})$. Given another $\vec{b}^*$ which satisfies $\varphi(\vec{e}_i, \vec{y})$ for m_α many i's, there is some i such that

$$\mathbb{U} \models \varphi(\vec{e}_i, \vec{b}^*)$$

[3]By (b) and $\lambda \geq m_\alpha$ this new sequence has also canonical parameter $\vec{b}$.

and

$$\vec{e_i} \underset{\vec{b}}{\perp} \vec{b}^*$$

by the choice of m_α. It follows that $\varphi_\alpha(\vec{x}, \vec{b}^*) \sim \varphi_\alpha(\vec{x}, \vec{b})$ and by (d) in Definition 4.2, $\vec{b}^* = \vec{b}$. Hence, (b) holds for Σ.

The linear independence in (c) follows from Lemma 3.2. We get

$$\mathrm{acl}(\vec{b}, \vec{e}_0, \dots, \vec{e}_{i-1}) \cap \langle \vec{e}_i \rangle = 0.$$

Given $\vec{a}_0, \dots, \vec{a}_\lambda, \vec{f}$ as above, the sequence $\vec{a}_0 \dots, \vec{a}_{i-1}, \vec{f}, \vec{a}_{i+1}, \dots, \vec{a}_\lambda, \vec{a}_i$ is again Morley for $\varphi_\alpha(\vec{x}, \vec{b}')$. Hence,

$$(\vec{a}_0 - \vec{a}_i, \dots, \vec{a}_{i-1} - \vec{a}_i, \vec{f} - \vec{a}_i, \vec{a}_{i+1} - \vec{a}_i, \dots, \vec{a}_\lambda - \vec{a}_i) \models \Sigma$$

i.e.

$$(\vec{e}_0 - \vec{e}_i, \dots, \vec{e}_{i-1} - \vec{e}_i, -\vec{e}_i, \vec{e}_{i+1} - \vec{e}_i, \dots, \vec{e}_\lambda - \vec{e}_i) \models \Sigma$$

This yields (d).

For (e), if α is groupless, then $\varphi_\alpha(\vec{x}, \vec{b})$ is not a torsor set and the result follows by Lemma 2.7. Otherwise, the set $X = \varphi_\alpha(\mathbb{U}, \vec{b}')$ defines a torsor set. Hence, $X - \vec{f} \sim \varphi_\alpha(\vec{x}, \vec{b})$ is a group set (note that $\vec{f}$ is in X). In the case of a group code, extend the Morley sequence $\{\vec{e}_i\}_{0 \le i \le \lambda}$ by an element $\vec{d}$. Therefore, the sequence $\vec{e}_0 + \vec{d}, \dots, \vec{e}_{i-1} + \vec{d}, \vec{e}_i - \vec{e}_j + \vec{d}, \vec{e}_{i+1} + \vec{d}, \dots, \vec{e}_\lambda + \vec{d}, \vec{d}$ is again Morley for $\varphi(\vec{x}, \vec{b})$. Hence,

$$\Sigma(\vec{e}_0, \dots, \vec{e}_i - \vec{e}_j, \dots, \vec{e}_\lambda).$$

Likewise, given H in $\mathrm{Inv}(\varphi(\vec{x}, \vec{b}))$, the sequence

$$\vec{e}_0 + \vec{d}, \dots, \vec{e}_{i-1} + \vec{d}, H\vec{e}_i + \vec{d}, \vec{e}_{i+1} + \vec{d}, \dots, \vec{e}_\lambda + \vec{d}, \vec{d}$$

is again Morley for $\varphi(\vec{x}, \vec{b})$. Therefore, $\Sigma(\vec{e}_0, \dots, H\vec{e}_i, \dots, \vec{e}_\lambda)$. The last point follows again from Lemma 2.7. □(Claim)

Take now some finite part Ψ'_α of Σ which implies (b), (c), (e1), (e2) and (e5) by compactness (note that (a) follows trivially).

If α is groupless, consider the following operations:

$$V_i(\vec{x}_0, \dots, \vec{x}_\lambda) = (\vec{x}_0 - \vec{x}_i, \dots, \vec{x}_{i-1} - \vec{x}_i, -\vec{x}_i, \vec{x}_{i+1} - \vec{x}_i, \dots, \vec{x}_\lambda - \vec{x}_i)$$

and let $\mathcal{V}$ be the subgroup generated by these operations (Observe that $\mathcal{V}$ has cardinality $(\lambda+1)!(\lambda+2)$).

Now, the formula

$$\Psi_\alpha(\vec{x}_0, \dots, \vec{x}_\lambda) = \bigwedge_{V \in \mathcal{V}} \Psi'_\alpha(V(\vec{x}_0, \dots, \vec{x}_\lambda))$$

satisfies (d), and lies also in Σ.

If α is an coset code, property (d) follows from (e3) and (e4). Hence, it is enough to find Ψ_α which satisfy the latter. Let $\mathcal{W}(\vec{x}_0, \dots, \vec{x}_\lambda)$ be the subgroup of $\mathrm{GL}_{\mathrm{n}_\alpha(\lambda+1)}(\mathbb{F}_p)$ generated by the operations mentioned in (e3) and (e4). Again, $\mathcal{W}$ is finite, and depends on $\mathrm{Inv}(\varphi(\vec{x}, \vec{b}))$. Note that $\lambda \ge m_\alpha$, hence $\vec{b}$ remains constant after applying these operations by (b). Set therefore:

$$\Psi_\alpha(\vec{x}_0, \dots, \vec{x}_\lambda) = \bigwedge_{W \in \mathcal{W}(\vec{x}_0, \dots, \vec{x}_\lambda)} \Psi'_\alpha(V(\vec{x}_0, \dots, \vec{x}_\lambda)),$$

which has the required properties. □

Definition 4.6. Let α, λ and Ψ_α be as above. A realization of Ψ_α is called a *difference sequence* for α.

Moreover, given a realization $\vec{e}_0, \ldots, \vec{e}_\lambda$ of Ψ_α, we denote by a *derived difference sequence* one obtained by composition of the following operations:

$$\vec{e}_0 - \vec{e}_i, \ldots, \vec{e}_{i-1} - \vec{e}_i, -\vec{e}_i, \vec{e}_{i+1} - \vec{e}_i, \ldots, \vec{e}_\lambda - \vec{e}_i.$$

If $\nu \leq \lambda$ and we use the above operation only for $i \leq \nu$ then we speak about a *ν-derived difference sequence* .

Note 4.7. A general transposition (ij) (and therefore all permutations of $\{0, \ldots, \lambda\}$) is obtained as follows:

$$\vec{e}_0, \ldots, \vec{e}_i, \ldots, \vec{e}_j, \ldots, \vec{e}_\lambda$$

$$(V_j) \qquad \vec{e}_0 - \vec{e}_j, \ldots, \vec{e}_i - \vec{e}_j, \ldots, -\vec{e}_j, \ldots, \vec{e}_\lambda - \vec{e}_j$$

$$(V_i) \quad \vec{e}_0 - \vec{e}_j - (\vec{e}_i - \vec{e}_j), \ldots, \vec{e}_j - \vec{e}_i, \ldots, -\vec{e}_j - (\vec{e}_i - \vec{e}_j), \ldots, \vec{e}_\lambda - (\vec{e}_j - (\vec{e}_i - \vec{e}_j) =$$
$$\vec{e}_0 - \vec{e}_i, \ldots, \vec{e}_j - \vec{e}_i, \ldots, -\vec{e}_i, \ldots, \vec{e}_\lambda - \vec{e}_i$$

$$(V_j) \qquad \vec{e}_0, \ldots, \vec{e}_j, \ldots, \vec{e}_i, \ldots, \vec{e}_\lambda$$

5. δ-pregeometry and red extensions

Let L^{morley} be the language which consist of $0, +, -$, and a relation symbol for every quantifier free L–formula. $\mathbb{U}$ becomes an L^{morley}–structure in a natural way. We extend this language to $L^* = L^{\text{morley}} \cup \{R\}$, where R is a unary predicate, which will yield the *red coloring.* The L^*-structures A we consider are additive subgroups of $\mathbb{U}$ with a distinguished additive subgroup $R(A)$ (viewed as an $\mathbb{F}_p$-vector space). The elements of $R(A)$ are the *red* points of A, the others *white.* We write $A \subset B$ if in the extended language we also have that $R(A) = R(B) \cap A$. We consider the following function, as introduced by B. Poizat in [11]:

$$\delta(A) = 2 \operatorname{trdeg} A - \dim R(A)$$

for finite A.

Note that δ satisfy the following (cf. [3]):

1. $\delta(0) = 0$,
2. $\delta\langle A, B\rangle + \delta(A \cap B) \leq \delta(A) + \delta(B)$.

If $\dim(R(A)/R(B))$ is finite[4], we define the *relative* δ-value of A over B by:

$$\delta(A/B) = 2 \operatorname{trdeg}(A/B) - \dim(R(A)/R(B)).$$

If also B is finite, we have

$$\delta(A/B) = \delta\langle A, B\rangle - \delta(B).$$

We will later make also use of the notation $\delta(\vec{a}/B)$ for $\delta(\langle\vec{a}\rangle/B)$.

[4] We do not assume that B is included in A.

Following [7], we say that Y is *self-sufficient* or *strong* in X (denoted as $Y \leq X$) if for all finite $A \subset X$, we have that $\delta(A/Y) \geq 0$. It is easy[5] to see that self-sufficiency is transitive. Moreover, the intersection of self-sufficient subsets of X is again self-sufficient and each subset S of X is contained in a smallest self-sufficient subset, its *self-sufficient closure* $\mathrm{cl}_X(S)$.

A proper extension $Y \leq Z$ is *minimal* if no $Y \subsetneq Y' \subsetneq Z$ is self-sufficient in Z. The set $Z \setminus Y$ must be finite, which allows us to express minimality by

$$\delta(Z/Y') < 0 \text{ for all } Y \subsetneq Y' \subsetneq Z.$$

We state a basic fact (whose proof is similar as in Lemma 4.1 in [3]):

Lemma 5.1. *Let $B \leq A$ be a minimal extension. We have one of the following cases:*

(1) *$A = \langle B, a\rangle$ for a white element a and $R(A) = R(B)$. Moreover, $\delta(A/B) = 0$ or 2, depending whether a is algebraic or transcendental over B.*
(2) *$A = \langle B, \vec{a}\rangle$ for a basis $\vec{a} = (a_1, \dots, a_n)$ of $R(A)$ over B. We have two subcases:*
   (a) *$n \geq 2$, $\delta(A/B) = 0$ and $\vec{a}$ is a generic element of a prealgebraic $X \subset \mathbb{U}^n$ defined over $\mathrm{acl}(B)$. In this case we call A a* minimal prealgebraic extension *of B.*
   (b) *$n = 1$, $\delta(A/B) = 1$ and a_1 is transcendental over B.*

Note that in (2) all new elements of A are transcendental over B. We call such an extension *transcendental.*

Proof. First assume that all red elements in A are in B. Let a be in $A \setminus B$. Then $\langle B, a\rangle$ is self-sufficient in A and by minimality it is A.
If A is algebraic over B, then $\delta(A/B) = 0$. Otherwise $\delta(A/B) = 2$. So we are in case (1).

Otherwise choose a basis $\vec{a} = (a_1, \dots, a_n)$ of $R(A)$ over B and set $A' = \langle B, \vec{a}\rangle$. Then $A' \leq A$ and by minimality $A' = A$: We are in case (2). There are two subcases:

(2a): There is a proper extension C of B in A such that $\delta(C/B) = 0$. By minimality we have $C = A$. Then $n = 2k$ where k is the transcendence degree of A over B. By minimality we have $2 \cdot \mathrm{trdeg}(\vec{a}/U, B) < n - \dim(U)$ for all non–trivial subspaces U of $\langle \vec{a}\rangle$. Choose an $\mathrm{acl}(B)$–definable set X of Morley rank k and Morley degree 1, which contains $\vec{a}$. Then $\vec{a}$ is generic in X and X is prealgebraic.

(2b): Otherwise every extension A'' of B in A with $\delta(A''/B) = 1$ is self-sufficient in A. Hence we get $A = \langle B, a_1\rangle$ and $\delta(A/B) = 1$. □

Theorem 4.4 now yields the following result:

Lemma 5.2. *Let $B \leq A$ minimal prealgebraic extension of B. Then there is a unique code α and parameters $\vec{b}$ in $\mathrm{acl}(B)$ such that A is generated over B by a generic red realization of $\varphi_\alpha(\vec{x}, \vec{b})$.* □

The final lemma of this section can be proved as Lemma 3A in [6] (see Lemma 4.5 in [3]). We will all throughout this paper use it in order to show that a realization of a code is generic over some set of parameters.

[5] Actually the following facts are formal consequences of (1) and (2).

Lemma 5.3. *Let α be a code, $\vec{b} \in \operatorname{acl}(B)$ and $\vec{a}$ be a red realization of $\varphi_\alpha(\vec{x}, \vec{b})$ which does not completely lie in B. Then, the following holds:*

(1) $\delta(\vec{a}/B) \leq 0$
(2) *If $\delta(\vec{a}/B) = 0$, $\vec{a}$ is a generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. In particular $\vec{a}$ is $\mathbb{F}_p$–linearly independent over* $\operatorname{acl}(B)$. □

6. A counting result

Definition 6.1. Let $\mathcal{K}$ be the class of all L^*-structures M (i.e colored subspaces of $\mathbb{U}$) such that $\emptyset \leq M$.

So $M \in \mathcal{K}$ iff $\delta(A) \geq 0$ for all finite subsets A of M. It is easy to see that $\mathcal{K}$ can be described by a set of universal sentences.[6]

If M belongs to $\mathcal{K}$ the self-sufficient closure of a finite subset of M is finite.

Definition 6.2. In $\mathcal{K}$ we define that M' is an amalgam of M and A over B, if

- M and A are self-sufficient in M',
- M and A are algebraically independent over B,
- $M' = \langle M, A \rangle$.

If in addition

- $M \cap A = B$,

we call M' a *free* amalgam.

Using standard arguments (cf.[10]) one shows:

Lemma 6.3. *If M, B, and A are in $\mathcal{K}$ and $B \leq M$ and $B \leq A$, there is an amalgam M' of M and A over B. If M or A are transcendental over B, we can find a free amalgam M'.* □

Definition 6.4. For structures in $\mathcal{K}$ a *difference sequence* for $\alpha \in \mathcal{C}$ of length λ is a *red* realization $\vec{e}_0, \ldots, \vec{e}_\lambda$ of $\Psi_\alpha(\vec{x_0}, \ldots, \vec{x_\lambda})$.

The next Lemma is the key tool in order to classify the structures which will be amalgamated with Fraïssé's method. This characterization will be useful for exhibiting the theory of the Fraïssé limit to be obtained in Section 8.

Lemma 6.5. *Given a code α and a natural number r, there is some $\lambda(r, \alpha) = \lambda \geq 0$ such that for every strong extension $M \leq N$ and every difference sequence $\vec{e}_0, \ldots, \vec{e}_\mu$ in N, with canonical parameter $\vec{b}$ and $\lambda \leq \mu$, either*

- *the canonical parameter of some λ-derived sequence of $\vec{e}_0, \ldots, \vec{e}_\mu$ lies in* $\operatorname{dcl}(M)$,

or

- *the sequence $\vec{e}_0, \ldots, \vec{e}_\lambda$ contains a Morley sequence of $\varphi_\alpha(\vec{x}, \vec{b})$ over M of length r.*

By "Morley sequence" we mean Morley sequence in the sense of $\mathrm{ACF_p}$.

[6] Her we use again the finiteness of $\mathbb{F}_p$.

Proof. If there are more than m_α many of the $\vec{e}_i$ in the same class of $R(N)^{n_\alpha}/R(M)^{n_\alpha}$, we subtract one of these elements from the others and obtain a derived sequence with m_α many elements in M, whose canonical base lies in $\mathrm{dcl}(M)$. So we assume that each class of $N^{n_\alpha}/M^{n_\alpha}$ contains at most m_α many $\vec{e}_i$. Work over $M' = M \cup \{\vec{e}_0, \ldots, \vec{e}_{m_\alpha - 1}\}$ and observe that $\vec{b} \in \mathrm{dcl}(M')$. All we need now from the sequence $(\vec{e}_i)$ is that the $\vec{e}_i$ realize $\varphi_\alpha(\vec{x}, \vec{b})$.

Set

$$s = \dim(\vec{e}_0, \ldots, \vec{e}_\lambda / \langle M' \rangle).$$

Then $\dim(\vec{e}_0, \ldots, \vec{e}_\lambda / M) \leq s + m_\alpha n_\alpha$. Thus by our assumption

$$\lambda + 1 \leq m_\alpha \, p^{(s + m_\alpha n_\alpha) n_\alpha}.$$

Consider the following sets of indices:

$$X_1 = \{ i \leq \lambda \mid \mathrm{trdeg}(\vec{e}_i / M', \vec{e}_0, \ldots, \vec{e}_{i-1}) = k_\alpha \},$$
$$X_2 = \{ i \notin X_1 \mid \dim(\vec{e}_i / \langle M', \vec{e}_0, \ldots, \vec{e}_{i-1} \rangle) > 0 \}.$$

It is clear that

$$s \leq |X_1| \, n_\alpha + |X_2| \, n_\alpha.$$

With the notation $\delta(i) = \delta(\vec{e}_i / \langle M', \vec{e}_0, \ldots, \vec{e}_{i-1} \rangle)$, Lemma 5.3 implies that $\delta(i) < 0$ if $x \in X_2$, and $\delta(i) = 0$ otherwise. Since $M \leq N$ we have

$$0 \leq \delta(\vec{e}_0, \ldots, \vec{e}_\lambda / M) = \delta(\vec{e}_0, \ldots, \vec{e}_{m_\alpha - 1} / M) + \sum_{i=1}^{\lambda} \delta(i) \leq \; m_\alpha n_\alpha - |X_2|.$$

If we put the three inequalities together, we obtain

$$\lambda + 1 \leq m_\alpha \, p^{((|X_1| + m_\alpha n_\alpha) n_\alpha + m_\alpha n_\alpha) n_\alpha}.$$

If λ is large enough, $|X_1| \geq r$ and $(\vec{e}_i)_{i \in X_1}$ is our Morley sequence. □

7. Leaving $\mathcal{K}^\mu$

We choose two finite-to-one functions μ^* and μ defined on $\mathcal{C}$ with values on $\mathbb{N}$ such that the following inequalities hold:

- $\mu(\alpha) \geq m_\alpha$
- $\mu^*(\alpha) \geq \max(\lambda(m_\alpha + 1, \alpha) + 1, n_\alpha + 1)$
- $\mu(\alpha) \geq \lambda(\mu^*(\alpha), \alpha) + 1$

Definition 7.1. The class $\mathcal{K}^\mu$ is the class of all L^*-structures M (i.e colored subspaces of $\mathbb{U}$) such that:

- $\emptyset \leq M$.
- No α in $\mathcal{C}$ has a difference sequence in M of length $\mu(\alpha) + 1$.

It is easy to see that $\mathcal{K}^\mu$, as $\mathcal{K}$, is axiomatizable by universal L^*–sentences. In fact, we have

Remark 7.2. For all α there is a universal sentence θ_α, which is true in $M \in \mathcal{K}$ iff M has no difference sequence for α of length $\mu(\alpha) + 1$.

All models of the L^*-theory T^μ we intend to construct will be in $\mathcal{K}^\mu$. We want that as many copies as possible of a prealgebraic extension of a strong subset of a model are realized. In this section we show that this is an elementary property of the model. The results are also important for the amalgamation in $\mathcal{K}^\mu$.

Lemma 7.3. *Assume M and M' are structures in $\mathcal{K}$. M' is a prealgebraic minimal extension of M, M is in $\mathcal{K}^\mu$ and M' is not in $\mathcal{K}^\mu$. Let $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)}$ be a difference sequence for a good code α in M', such that its canonical parameter $\vec{c}$ is in* $\mathrm{acl}(M)$.[7]

Then we find such a difference sequence $\vec{d}_0, \ldots, \vec{d}_{\mu(\alpha)}$ for α in M' with the same canonical parameter such that $\vec{d}_0, \ldots, \vec{d}_{\mu(\alpha)-1}$ are in M, $\vec{d}_{\mu(\alpha)}$ is an M-generic realization of $\varphi_\alpha(\vec{x}, \vec{c})$ and generates M' over M.

If we cannot find the new sequence by a permutation of the old one, then α is a group code and the new sequence is obtained using operations as $\vec{e}_j$ is replaced by some $H\vec{e}_j - \vec{e}_i$ where H is in $\mathrm{Inv}(\varphi_\alpha(\vec{x}, \vec{c}))$.

Furthermore α is the unique good code that describes M' over M.

Proof. Since M is in $\mathcal{K}^\mu$, there is some $\vec{e}_i$ not completely contained in M. Since M is strong in M' by Lemma 5.3 $\vec{e}_i$ is M-generic. Since M' is minimal over M it generates M' over M. If there is some other $\vec{e}_j$ not completely contained in M, then again $\vec{e}_j$ is M-generic and generates M' over M. Hence $\vec{e}_i = H\vec{e}_j - \vec{m}_j$ where H is $\mathrm{GL}_{\mathrm{n}_\alpha}(\mathbb{F}_p)$ and $\vec{m}_j$ is in M. Then $H\vec{e}_j - \vec{e}_i$ is in M. Since $\vec{e}_j$ is M-generic, we have

$$\vec{e}_j \underset{\vec{c}}{\downarrow} H\vec{e}_j - \vec{e}_i.$$

By the properties of a difference sequence it follows that α is a group code and H is in $\mathrm{Inv}(\varphi_\alpha(\vec{x}, \vec{c}))$. If we replace $\vec{e}_j$ by $H\vec{e}_j - \vec{e}_i$, then we obtain again a difference sequence with the same canonical parameter and this sequence has one more element in M. We can iterate the argument to obtain the assertion.

Finally every prealgebraic set that gives us M' over M determines a unique code by Theorem 4.4. All such prealgebraic sets can be transformed into each other using $\mathrm{GL}_{\mathrm{n}_\alpha}(\mathbb{F}_p)$ and translations. Hence there is a unique good code for this situation. □

Corollary 7.4. *Let M be in $\mathcal{K}^\mu$ and $M \le M'$ a minimal extension. If M' has linear dimension 1 over M, then M' is in $\mathcal{K}^\mu$.*

Otherwise, in the prealgebraic case, M' is in $\mathcal{K}^\mu$ if and only if none of the following two conditions holds:

a) *There is a code $\alpha \in \mathcal{C}$ and a difference sequence $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)}$ for α in M' such that:*
   (i) *$\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)-1}$ are contained in M.*
   (ii) *$M' = \langle M, \vec{e}_{\mu(\alpha)} \rangle$.*
   (iii) *In this case α is the unique good code that describes M' over M.*
b) *There exists a code $\alpha \in \mathcal{C}$ and a difference sequence for α in M' of length $\mu(\alpha) + 1$ with canonical parameter $\vec{b}$ with $\mu^*(\alpha)$ many elements which form a Morley sequence of $\varphi(\vec{x}, \vec{b})$ over M.*

Proof. Consider first the case where $\dim(M'/M) = 1$. If $R(M') = R(M)$, then M' does not contain new difference sequences, so M' is in $\mathcal{K}^\mu$. Now assume that $\dim(R(M')/M) = 1$, and that M' is not in $\mathcal{K}^\mu$ witnessed by some difference sequence $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)}$. If the canonical parameter $\vec{b}$ lies in $\mathrm{dcl}(M)$, since no $\vec{e}_i$ can be linearly independent over M, all e_i would be in M, which is not possible. Therefore,

[7] The Lemma implies $c \in \mathrm{dcl}(M)$.

since $\mu(\alpha) \geq \lambda(1,\alpha)$[8] and $M \leq M'$, some $\vec{e_j}$ is an M–generic realization of $\varphi(\vec{x},\vec{b})$, impossible.

Finally, let M'/M be prealgebraic. Clearly, if a) or b) hold, then M' is not in $\mathcal{K}^\mu$. Assume, for the converse, that M' is not in $\mathcal{K}^\mu$, witnessed by some good code α and a difference sequence $\{\vec{e}_i\}_{0\leq i\leq\mu(\alpha)}$ with canonical parameter $\vec{b}$. First, suppose that the canonical parameter of some derived difference sequence lies in $\mathrm{acl}(M)$. Then Lemma 7.3 yields case a) as desired.

Otherwise, since $\mu(\alpha) \geq \lambda(\mu^*(\alpha),\alpha)$, $\{\vec{e}_i\}_{0\leq i\leq\mu(\alpha)}$ contains a Morley sequence of $\varphi(\vec{x},\vec{b})$ over M of length $\mu^*(\alpha)$. Therefore, b) holds. □

Corollary 7.5. *For each good code α there is an $\forall\exists$–sentence χ_α such that a structure M in $\mathcal{K}^\mu$ satisfies χ_α iff M has no prealgebraic minimal extension in $\mathcal{K}^\mu$, which is given by α.*

Proof. Let $M' = \langle M,\vec{a}\rangle$, where $\vec{a}$ is a red M–generic realization of $\varphi_\alpha(\vec{x},\vec{b})$, for some $\vec{b} \in M$. If M' is not in $\mathcal{K}^\mu$, we are in the cases a) or b) of Corollary 7.4. In case a) M' contains a difference sequence of length $\mu(\alpha)+1$ for α, in case b) difference sequence of length $\mu(\beta)+1$ for a good code β, which contains a subsequence of length $\mu^*(\beta)$, linearly independent over M, so $\mu^*(\beta)n_\beta \leq n_\alpha$. Since μ^* is finite-to-1, only a finite set C_α of codes β can occur.

Set $C'_\alpha = C_\alpha \cup \{\alpha\}$. Then M has no prealgebraic minimal extension in $\mathcal{K}^\mu$, given by α iff

$$\forall\vec{b} \in M, \forall \text{ red } M\text{–generic realizations } \vec{a} \text{ of } \varphi_\alpha(\vec{x},\vec{b}) \bigvee_{\beta\in C'_\alpha} \langle M,\vec{a}\rangle \models \neg\,\theta_\beta.$$

Since the L–type of $\vec{a}$ over M is uniformly definable over $\vec{b}$, this can easily expressed by an $\forall\exists$–sentence. □

8. The Fraïssé model

We show now that the class $\mathcal{K}^\mu$ has the Amalgamation Property, and hence, we can obtain *rich* fields as introduced by Poizat in [10].

The following lemma replaces an easy argument in the case of black points by a more sophisticated one for our purposes.

Lemma 8.1. *Let A,B,M be structures in $\mathcal{K}^\mu$ where B is a strong substructure of M and of A. Let M' be a free amalgam of M and A over B. Assume that $\vec{e}_0,\ldots,\vec{e}_{\mu(\alpha)}$ is a difference sequence for a good code α in M'.*

Then there is a derived difference sequence of the above sequence with the canonical parameter in $\mathrm{acl}(M)$ *or in* $\mathrm{acl}(A)$.

Proof. We assume that the assertion of the lemma is not true. Let $\vec{b}$ be the canonical parameter of the sequence. If we apply Lemma 6.5, then, since $\mu^*(\alpha) \geq \lambda(m_\alpha+1,\alpha)+1$ we get a subsequence with more than $\lambda(m_\alpha+1,\alpha)$ elements that form a Morley sequence of $\varphi_\alpha(\vec{x},\vec{b})$ over M. From this again we get a subsequence of length $m_\alpha+1$ that is a Morley sequence over M and also over A. Write these

[8] We silently assume that λ is monotonous in the first argument.

elements as $\vec{e}_i$ for $0 \leq i \leq m_\alpha$. If $E = \{\vec{e}_0, \ldots, \vec{e}_{m_\alpha - 1}\}$, we have that $\vec{b} \in \mathrm{dcl}(E)$ and

$$\vec{e}_{m_\alpha} \mathop{\downarrow}_{\vec{b}} M, E, \qquad \vec{e}_{m_\alpha} \mathop{\downarrow}_{\vec{b}} A, E.$$

Write every element of E as the sum of a tuple in M and a tuple in A. Define E_M to be the set of all elements of M which occur, E_A similarly and set $E' = E_M \cup E_A$. Then also $\vec{b} \in \mathrm{dcl}(E')$ and, since E' and E are interdefinable over M and over A, we have

$$\vec{e}_{m_\alpha} \mathop{\downarrow}_{\vec{b}} M, E', \qquad \vec{e}_{m_\alpha} \mathop{\downarrow}_{\vec{b}} A, E'.$$

Whence

$$\vec{e}_{m_\alpha} \mathop{\downarrow}_{BE'} M, \qquad \vec{e}_{m_\alpha} \mathop{\downarrow}_{BE'} A.$$

Furthermore

$$M \mathop{\downarrow}_{BE'} A.$$

Write $\vec{e}_{m_\alpha} = \vec{m} + \vec{a}$ for $\vec{m} \in M$ and $\vec{a} \in A$. Then $\{\vec{e}_{m_\alpha}, \vec{m}, \vec{a}\}$ is a pairwise independent triple over B, E'. Whence $\varphi_\alpha(\vec{x}, \vec{b})$ is a group formula for a definable group G and $\vec{b}$ is the canonical parameter of G. Now, $\vec{a}$ is a generic element of an $\mathrm{acl}(B, E')$–definable coset of G and $\vec{b}$ is definable from the canonical base of $p = \mathrm{tp}(\vec{a}/\,\mathrm{acl}(B, E'))$. Note that

$$\vec{a} \mathop{\downarrow}_{BE_A} E'$$

so the canonical base of p is in $\mathrm{acl}(A)$, and we have $b \in \mathrm{acl}(A)$. This contradicts our assumption. □

A *self-sufficient embedding* of B in A is an isomorphism of B onto a self-sufficient subset of A.

Theorem 8.2. *The class $\mathcal{K}^\mu$ has the amalgamation property with respect to self-sufficient embeddings.*

Proof. Let $B \leq M$ and $B \leq A$ be structures in $\mathcal{K}^\mu$. We need to show that there is an extension M' of M in $\mathcal{K}^\mu$, with $M \leq M'$ and some $B \leq A' \leq M'$ such that A and A' are isomorphic over B. We may assume that both extensions $B \leq A$ and $B \leq M$ are minimal, by splitting them into minimal ones.

Case 1. $\dim(M/B) = 1$ or $\dim(A/B) = 1$. By Lemma 6.3 M and A can be strongly embedded in an amalgam M'. Since $\dim(M'/A) \leq 1$ or $\dim(M'/M) \leq 1$, Corollary 7.4 implies that $M' \in \mathcal{K}^\mu$.

Case 2. Both extensions M/B and A/B are prealgebraic. Let M' be a free amalgam of M and A over B, which exists by Lemma 6.3. We are done if M' belongs to $\mathcal{K}^\mu$. So assume that M' does not belong to $\mathcal{K}^\mu$. We need then show that M and A are isomorphic over B.

There is a good code α with a difference sequence $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)}$ in M'. By Lemma 8.1 and symmetry we may assume that its canonical parameter $\vec{b}$ lies in $\mathrm{acl}(M)$. By Lemma 7.3 we may assume that $\vec{e}_0, \ldots, \vec{e}_{\mu(\alpha)-1}$ are in M and $\vec{e}_{\mu(\alpha)}$ is an M–generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$ which generates M' over M.

Subcase 2.a We first assume that the canonical parameter for some $(\mu(\alpha)-1)$-derived difference sequence, is in $\mathrm{dcl}(B)$. Since this difference sequence has the same properties, we denoted it again by $\{\vec{e}_i\}_{0\le i\le\mu(\alpha)}$. By Lemma 5.3 there are two cases:

Subcase 2.a.1) $\vec{e}_{\mu(\alpha)}$ is in A. By minimality of A over B, we have that $A = \langle B, \vec{e}_{\mu(\alpha)}\rangle$. Since A is in $\mathcal{K}^\mu$, there exists an $\vec{e}_i$ which lies in M but not in B. It follows from $B \le M$ and Lemma 5.3 that $\vec{e}_i$ is B-generic, and isomorphic to A over B. This shows that M and A are isomorphic over B.

Subcase 2.a.2) $\vec{e}_{\mu(\alpha)}$ is an A-generic realization of $\varphi(\vec{x},\vec{b})$. Write $\vec{e}_{\mu(\alpha)} = \vec{m}+\vec{a}$, and note that $0 = \delta(\vec{e}_{\mu(\alpha)}/M) = \delta(\vec{a}/M) = \delta(\vec{a}/B)$, so $\vec{a}$ generates A over B (again by minimality of $B \le A$). Moreover, $\{\vec{e}_{\mu(\alpha)}, \vec{m}, \vec{a}\}$ is a B-independent triple. By Note 2.2, $-\vec{m}$ and $\vec{a}$ have the same type over B and hence we obtain the desired isomorphism.

Subcase 2.b No $(\mu(\alpha)-1)$-derived difference sequence has canonical basis $\vec{c}$ in $\mathrm{acl}(B)$. Write $\vec{e}_{\mu(\alpha)} = \vec{m}+\vec{a}$ (as above, $\vec{a}$ generates A over B). Note that only n_α many elements of a Morley sequence may fork with a fixed tuple of length n_α, so Lemma 6.5 (applied to $B \le M'$) gives us an $\vec{e}_i \in M$ which is an $B,\vec{m}$–generic realization of $\varphi_\alpha(\vec{x},\vec{c})$ since $\mu^*(\alpha) \ge n_\alpha + 1$. In this case, $\vec{e}_i - \vec{m}$ is B-isomorphic to $\vec{a}$. □

We call M in $\mathcal{K}^\mu$ *rich* if for any $B \le M$ finite and any finite extension $B \le A$ of members of $\mathcal{K}^\mu$, there is a self-sufficient substructure $A' \le M$ with $B \le A'$ and B-isomorphic to A.

Corollary 8.3. *There is a unique* (*up to isomorphism*) *countable rich structure M in $\mathcal{K}^\mu$. All rich structures are $L^*_{\infty,\omega}$-equivalent.*

9. Axiomatization

In this section we will show that rich fields in $\mathcal{K}^\mu$ are exactly the ω-saturated models of a specific theory T^μ. We work in the extended language L^*.

Let T^μ denote following axiom schemes:

Universal Axioms:

(1) Any model is a member of $\mathcal{K}^\mu$.

$\forall\exists$ **Axioms:**

(2) Any model is an algebraically closed field of characteristic p.

(3) Given a good code α and $\vec{b}$ in the model, the extension of M generated by a M-generic *red* realization of $\varphi_\alpha(\vec{x},\vec{b})$ does not belong to $\mathcal{K}^\mu$.

Corollary 7.5 shows that axiom (3) is elementary. The following result gives us an equivalent description of a rich model, and moreover, yields an explicit axiomatization of its theory.

Theorem 9.1. *An L^*-structure is rich if and only if it is an ω-saturated model of T^μ.*

Proof. The argument of the proof is similar to the one in [3] based on ideas from [10]. We first show that ω-saturated models of T are rich, and also that a rich model satisfies all axioms. Hence, by the above and ∞-equivalence of rich models, we obtain the other implication.

Let $M \models T^\mu$ be ω-saturated. Let $B \leq M$ and $B \leq A$ be finite L^*-structures in $\mathcal{K}^\mu$. We may assume that $B \leq A$ is minimal. We can distinguish four different cases, as given by Lemma 5.1:

A/B is algebraic: We are done by Axiom (2).

A/B is prealgebraic: Consider the free amalgam M' of M and A over B. In this case A is given over B by a red generic realization of a good code. Axiom (3) yields that M' does not belong to $\mathcal{K}^\mu$. By Theorem 8.2 A has a strong embedding over B into M.

A is generated over B by a *red* element that is transcendental over B: We will *approximate* this extension by prealgebraic minimal extensions $A_n = \langle B, x_1, x_2\rangle$, where the x_i are red, transcendental over B and $x_1^n = x_2$. The sequence A_n/B converges (in the space of *colored* L-types) to the extension A_∞/B, where $A_\infty = \langle B, a_1, a_2\rangle$, with a_1, a_2 red and algebraically independent over B. This extension decomposes into $B \leq \langle B, a_1\rangle \leq A_\infty$, which implies by Corollary 7.4 that A_∞ belongs to $\mathcal{K}^\mu$. So, since μ is finite to one, A_n belongs to $\mathcal{K}^\mu$ for large n. As shown above, we can find self-sufficient B-copies of A_n in M for large n. By saturation of M, A_∞ is also self-sufficiently embeddable over B. Since A is isomorphic to $\langle B, a_1\rangle$, we conclude that there is a self-sufficient B-copy of A in M.

A is generated over B by a *white* element that is transcendental over B and $R(A) = R(B)$: Consider for each n the extension $B \leq \langle B, z_1\rangle \leq \langle B, z_1, z_2\rangle = C_n$, where z_1 is red, $z_1^n = z_2$ is white and $R(C_n) = \langle R(B), z_1\rangle$. By the above each C_n can be strongly embedded in M. Therefore also the limit $C_\infty = \langle B, c_1, c_2\rangle$ can be strongly embedded in M. Since $\langle C, c_2\rangle \leq C_\infty$ is B–isomorphic to A, we are done.

Suppose now that M is a rich field. We first show that M is algebraically closed. Let $a \in \mathrm{acl}(M)$. Choose a finite set B in M such that a is in $\mathrm{acl}(B)$. Taking the self-sufficient closure of B in M, we can assume that $B \leq M$. Paint $\langle B, a\rangle \setminus B$ in white. It is clear that $\langle B, a\rangle$ is in $\mathcal{K}^\mu$ (since B is) and $B \leq \langle B, a\rangle$. By richness, we find a copy of a in M over B. This yields (2).

For Axiom (3), let α and $\vec{b}$ be as in the statement such that $\langle M, \vec{a}\rangle$ is in $\mathcal{K}^\mu$, where $\vec{a}$ is some generic red realization of $\varphi_\alpha(\vec{x}, \vec{b})$ over M. We show a contradiction. Choose some set $B \leq M$ containing $\vec{b}$. Again, $B \leq \langle B, \vec{a}\rangle$, and by richness, we get a B-copy of $\vec{a}$ in M, say $\vec{a}'$. Take now some finite $C \leq M$ containing $B \cup \vec{a}'$. We have that $C \leq \langle C, \vec{a}\rangle$. We can iterate and obtain a Morley sequence in M for $\varphi_\alpha(\vec{x}, \vec{b})$ of length $\mu(\alpha) + 2$. This yields a difference sequence for α of length $\mu(\alpha) + 1$ in M by property (a) of Ψ_α. □

Corollary 9.2. *T^μ is complete. Two tuples $\vec{a}$ and $\vec{a}'$ in two models M and M' have the same type iff there is an isomorphism $f : \mathrm{cl}(\vec{a}) \to \mathrm{cl}(\vec{a}')$ which maps $\vec{a}$ to $\vec{a}'$.*

Proof. Corollary 8.3 and Theorem 9.1 provides us a rich model M^* of T^μ. Let M be any model of T. By theorem 9.1 there is a rich $M' \equiv M$. So $M' \equiv_{\infty,\omega} M^*$, which proves completeness.

To prove the second statement choose ω–saturated elementary extensions $M \prec N$ and $M' \prec N'$. By Corollary 9.3 (we use only the part which does not rely on 9.2.) is $M \leq N$ and $M' \leq N'$, so "cl" does not change.

An isomorphism $f : \mathrm{cl}(\vec{a}) \to \mathrm{cl}(\vec{a}')$ is now part of a back-and-forth-system of partial isomorphisms. Whence f is a elementary map.

For the converse suppose that $\vec{a}$ and $\vec{a}'$ have the same type. There is an elementary map $f : \mathrm{cl}(\vec{a}) \to M'$ which maps $\vec{a}$ onto $\vec{a}'$. We write A' for $f(\mathrm{cl}(\vec{a}))$. Since A' has the same type as $\mathrm{cl}(\vec{a})$, A' is self-sufficient in M, and we can conclude that $A' = \mathrm{cl}(\vec{a}')$.

□

Corollary 9.3. *Let $M \subset N$ be an extension of two models of T. Then $M \prec N$ iff $M \leq N$.*

Proof. If $M \not\leq N$, there is an $a \in N$ with $\delta(a/M) < 0$. We find a finite $B \leq M$ with $\delta(a/B) < 0$. This is witnessed by the truth of an $L_1 \cup L_2$–formula $\varphi(a, \vec{b})$. $\varphi(x, \vec{b})$ is not satisfiable in M, whence $M \not\prec N$.

The converse $M \leq N \Rightarrow M \prec N$ follows directly from 9.2, since $M \leq N$ ensures that $\mathrm{cl}(a)$ is the same in M and N. □

10. Rank computations

In this section we show that T^μ has Morley rank 2. First we describe algebraic closure acl^* in models M of T^μ.

Definition 10.1. Let $M \models T^\mu$, B a subset of M. Then $\mathrm{cl}_d(B)$ is the union of all finite $A \subset M$ with $\delta(A/\mathrm{cl}(B)) = 0$.

It is easy to see that

$$\mathrm{cl}_d(B) = \{a \in M \mid \mathrm{d}(a/B) = 0\},$$

where we use the notation

$$\mathrm{d}(A/B) = \delta(\mathrm{cl}\langle A, B\rangle / \mathrm{cl}(B)).$$

Lemma 10.2. *Both closures acl^* and cl_d agree on models M of T^μ.*

Proof. We may assume that M is ω-saturated by Lemma 9.3. If B is finite, then so is $\mathrm{cl}(B)$, hence contained in $\mathrm{acl}^*(B)$. So we may assume that B is finite and self-sufficient in M.

First we show cl_d is part of acl^*. We consider $\delta(A/\mathrm{cl}(B)) = 0$. We decompose the extension A/B into a sequence of minimal extensions and use induction on the length of the sequence. Hence we have to consider A/B minimal $\delta(A/B) = 0$ with $B \leq M$. By Lemma 5.1 two cases may arise:

Case 1) A is algebraic over B in the field sense. But then $A \subseteq \mathrm{acl}^*(B)$.

Case 2) A is a prealgebraic extension of B.

By Theorem 4.4 there are a good code α and parameters $\vec{b}$ in $\mathrm{acl}(B)$ such that $A = \langle B, \vec{a}\rangle$ where $\vec{a}$ is a generic red solution of $\varphi_\alpha(\vec{x}, \vec{b})$. We show that $\varphi_\alpha(\vec{x}, \vec{b})$ has only finitely many red solutions, which implies $\vec{a} \in \mathrm{acl}^*(B)$ and therefore $A \subset \mathrm{acl}^*(B)$. Assume that there are infinitely many solutions. Then there is a solution $\vec{e_0}$ which is not contained in B. By Lemma 5.3 $\vec{e_0}$ is a B-generic realization of $\varphi_\alpha(\vec{x}, \vec{b})$. We have $\delta(\vec{e_0}/B) = 0$, and therefore $B' = \langle B, \vec{e_0}\rangle \leq M$. Since B' is again finite, we can find a B'–generic solution $\vec{e_1}$, etc. So we find an infinite Morley sequence $(\vec{e_i})$ for $\varphi_\alpha(\vec{x}, \vec{b})$ in M. As above, this contradicts Axiom (1).

Now consider an element a outside of $\mathrm{cl}_d(B)$. If $A = \mathrm{cl}(B, a)$, we have $\delta(A/B) > 0$. Decompose A/B into minimal extensions:

$$B = A_0 \leq A_1 \leq \cdots \leq A_n = A$$

There is some $i < n$ with $\delta(A_{i+1}/A_i) > 0$, which implies that $\dim(A_{i+1}/A_i) = 1$ by 5.1. We saw in 7.4 that free amalgams with transcendental strong extensions of dimension 1 cannot leave $\mathcal{K}^\mu$. So, since M is rich, there are infinitely many $A' \leq M$ which are over A_i isomorphic to A_{i+1}. By 9.2 they have all the same type over A_i. Hence A_{i+1} is not $*$–algebraic over A_i. Since A_{i+1} is algebraic over Ba, it follows that $a \notin \mathrm{acl}^*(B)$. □

Theorem 10.3. *T^μ has Morley rank 2. In particular, the generic type has rank 2 and the generic red type has rank 1. The algebraic closure in models of T^μ is cl_d.*

Proof. We consider an ω-saturated model M of T^μ inside a monster model. We compute $\mathrm{MR}^*(a/M)$ for elements a in the monster model.

We have

$$0 \leq \mathrm{d}(a/M) \leq \delta(a/M) \leq 2.$$

So there are three cases:

$\mathrm{d}(a/M) = 0$: Then a is algebraic over M, i.e. $a \in M$ and we have $\mathrm{MR}^*(a/M) = 0$.

$\mathrm{d}(a/M) = 1$: If a is red, it follows that $\langle M, a\rangle$ is self–sufficient. So $\mathrm{tp}^*(a/M)$ is uniquely determined. Since all other red types over M are algebraic, we conclude that $\mathrm{MR}^*(a/M) = 1$ and the red subgroup is strongly minimal. If a is not red, then $\mathrm{cl}(M, a)$ contains a red element c outside M. If follows $\langle M, c\rangle \leq \mathrm{cl}(M, a)$ and $\mathrm{d}(a/M, c) = 0$. This implies that a and c are L^*–interalgebraic over M and $\mathrm{MR}^*(a/M) = \mathrm{MR}^*(c/M) = 1$.

$\mathrm{d}(a/M) = 2$: Then $\langle M, a\rangle$ is self–sufficient. So $\mathrm{tp}^*(a/M)$ is uniquely determined, since the isomorphism type of of $\langle M, a\rangle$ is determined by $R(\langle M, a\rangle) = R(M)$. Since all other types over M have Morley rank ≤ 1, we have $\mathrm{MR}^*(a/M) \leq 2$. But $R(M)$ is an infinite subgroup of M of infinite index. Hence T^μ has Morley rank 2 and $\mathrm{MR}^*(a/M) = 2$. □

Remark 10.4. We note the following without proof.

(1) As any complete theory of fields of finite Morley rank T^μ is ω_1–categorical.
(2) Lindström's theorem implies that T^μ is model complete. This can be directly proved as in [4].

(3) For every natural number $r \geq 2$ there is a theory of a field of Morley rank r with an additive red subgroup of Morley rank $r-1$. This can be proved as in [3] using

$$\delta(A) = r \cdot \mathrm{trdeg}(A) - \dim(A).$$

(4) In models (K, R) of T^μ, we have

$$\mathbb{F}_p = \{x \in K \mid x \cdot R \subset R\}.$$

If we consider $\mathbb{F}_{p^n}$–vector spaces R and use $\mathbb{F}_{p^n}$–linear dimension, we can produce a field (K, R) of Morley rank 2, where

$$\mathbb{F}_{p^n} = \{x \in K \mid x \cdot R \subset R\}.$$

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