Bagatelle in finite Morley rank

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Abstract. An analog of Steinberg's theorem on centralizers of semi-simple elements, in the context of groups of finite Morley rank.

Introduction. Parallel to an ambitious classification program, the study of torsion in groups of finite Morley rank aims at direct structural analogies with algebraic groups. We bring a small contribution to this line of thought.

Acknowledgements. The result is a souvenir from the newly opened Matematik Köyü at Şirince whose founders we wish to thank (and congratulate).

Lemma (cf. [ST75, Corollary 2.16(b)]) *If G is a connected group of finite Morley rank of odd type and $i \in I(G)$ then $C(i)/C^\circ(i)$ has exponent at most* 2.

Proof. We rely on two principles. The first is the connectedness of centralizers of decent tori in connected groups of finite Morley rank [AB08, Theorem 1], also independently obtained by Frécon. The second is a *torality* result [AB08, Fact 4.1] due to Burdges and Cherlin: given a connected group of odd type and a 2-element α, α belongs to every maximal 2-torus of $C^\circ(\alpha)$. Let us start here.

By torality, there is a maximal 2-torus S containing i. A Frattini argument implies $C(i) = C^\circ(i) \cdot N_{C(i)}(S)$, so by a standard torsion-lifting principle [AC08, Fact 2.5] it suffices to show that if α is a p-element of $N_{C(i)}(S)$, then $\alpha^2 \in C^\circ(i)$.

Assume first $p \neq 2$. Then a quick computation yields $S = C_S(\alpha) \oplus [S, \alpha]$, and both factors are connected. As $i \in C_S(\alpha)$, $T = d(C_S(\alpha))$ is a non-trivial decent torus. Therefore $\alpha \in C(T) = C^\circ(T) \leq C^\circ(i)$ and we are done.

Suppose now that α is a 2-element; we may clearly assume $\alpha^2 \neq 1$. A Frattini argument implies that i normalizes a maximal 2-torus Σ of $C^\circ(\alpha)$; by torality $\alpha \in \Sigma$. Now under the action of i, $\Sigma = \Sigma_+ + \Sigma_-$ with obvious notations, and the intersection is finite. Hence $\Sigma = \Sigma_+^\circ + \Sigma_-^\circ$. Let us express $\alpha = \alpha_+ + \alpha_-$ along this new decomposition. It follows $\alpha_+^2 = \alpha^2 \in \Sigma_+^\circ$. So $T = d(\Sigma_+^\circ)$ is a non-trivial decent torus, and $i \in C(T) = C^\circ(T) \leq C^\circ(\alpha_+^2) = C^\circ(\alpha^2)$.

It remains to note that whenever α and β are 2-elements of a connected group of odd type, then $\alpha \in C^\circ(\beta) \Leftrightarrow \beta \in C^\circ(\alpha)$. Either is indeed equivalent to having a 2-torus containing both α and β, by torality. $\square$

References.

[AC08] Tuna Altınel and Jeffrey Burdges. On analogies between algebraic groups and groups of finite Morley rank. To appear, 2008.

[St75] Robert Steinberg. Torsion in Reductive Groups. *Advances in Mathematics,* 15:63-92, 1975.