FIELDS WITH A DENSE-CODENSE LINEARLY INDEPENDENT MULTIPLICATIVE SUBGROUP

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Abstract. We study expansions of an algebraically closed field K or a real closed field R with a linearly independent subgroup G of the multiplicative group of the field or the unit circle group $\mathbb{S}(R)$, satisfying a density/codensity condition (in the sense of geometric theories). Since the set G is neither algebraically closed nor algebraically independent, the expansion can be viewed as "intermediate" between the two other types of dense/codense expansions of geometric theories: lovely pairs and H-structures. We show that in both the ACF and RCF cases, the resulting theory is near model complete and the expansion preserves many nice model theoretic conditions related to the complexity of definable sets such as stability and NIP. We also analyze the groups definable in the expansion.

1. Introduction

This paper brings together the expansions of geometric structures with dense-codense independent subsets introduced in [10] (in the o-minimal setting) and [6] (in the geometric case) and the study of algebraically closed and real closed fields with a multiplicative subgroup satisfying the Mann property [12], [14], [1]. Our main motivation has been to further explore the general notion of a dense-codense expansion to find, in some sense, a "middle ground" between the lovely pair construction [4] and the expansion with a dense-codense independent subset.

Recall that a (complete) theory is called *geometric* if in all of its models acl satisfies the exchange property (thus inducing a pregeometry) and the theory eliminates the $\exists^\infty$ quantifier. Models of such theories are called geometric structures. We call a subset $P(M)$ of a geometric structure M *dense-codense* if any infinite subset of M definable over a set A has a non-empty intersection with $P(M)$ and $\mathrm{acl}(A \cup P(M)) = \mathrm{scl}(A)$ (the *small closure* of A).

If one further imposes the requirement that $P(M)$ is an algebraically closed, we get a *lovely pair* of models [4], a common generalization of the notion of a beautiful pair in the strongly minimal case [19] and dense pair in the o-minimal case [11]. A somewhat opposite requirement we can impose would be for the set $P(M)$ to be algebraically independent. In this case we get the notion of an H-*structure* (using symbol H for the predicate), originally introduced in the o-minimal context by Dolich, Miller and Steinhorn [10], and then generalized to the geometric

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setting in [6]. Both lovely pair and H-structure expansions allow a good description of definable sets (they are near model complete) and preserve many important stability/simplicity-theoretic and combinatorial conditions (e.g. superstability, supersimplicity, NIP), and in the SU-rank 1 case, one gets a reasonable description of forking in the expanded language. In the case of H-structure expansion of an SU-rank 1 theory, one also gets a clear description of canonical bases in terms of those in the old language (and as a consequence, geometric elimination of imaginaries down to old ones) [6]. The situation is much less clear in the lovely pairs case, and in fact, one always gets new imaginaries in the expansion provided there is a definable group (the converse is also true in the stable case, see [18]). A major tool that enabled us to describe canonical bases in H-structure was the existence of a "projection" of a tuple $\vec{a}$ over the predicate H, namely a unique minimal tuple $\vec{h}$ in $H(M)$ such that is $\vec{a}$ is independent from $H(M) \cup B$ over $\vec{h} \cup B$. Such a tool is not available in the case of lovely pairs.

To summarize, both constructions allow one to control the new definable sets, but the control is much tighter in the case of H-structures due to independence of $H(M)$ (and, as a consequence, triviality of its geometry). The two expansions are related in a strong way, taking the algebraic closure of $H(M)$ in an H-structure, one gets a lovely pair.

When working in the context of a "geometric field" K (e.g. a model of ACF_0 or RCF), a reasonable trade-off between algebraic independence of the subset and it being algebraically closed is to consider a multiplicative subgroup $G(K)$ generated by $H(K)$. The result is a dense-codense subset of K "intermediate" between $H(K)$ and $P(K) = \mathrm{acl}(H(K))$. Thus, we expect the expanded structure $(K, G(K))$, a so-called G-*structure*, to behave in a way that resembles both lovely pairs and H-structures. On the other hand, an immediate consequence of this construction is the linear independence of $G(K)$, and as a result, the fact that it satisfies the Mann property. This brings us in the context of [12] (in the cases of ACF_0 and RCF) and [14] (in the case of ACF_0). The multiplicative subgroup generated by $H(K)$ has infinite rank, as opposed to the cases considered in [12], [1] where the groups have finite rank.

The main goal of this paper is to study this intermediate expansion, show that the theory obtained is near model complete and characterize the definable sets in the expansion. As with the other expansions, we show these new pairs preserve nice combinatorial conditions related to the complexity of definable sets such a stability and NIP. This paper is organized as follows:

In section 2 we introduce G-structures and prove some basic properties.

In section 3 we deal with the case where the base field is algebraically closed. We introduce the notion of G-bases which plays a role similar to the one that H-bases did for H-structures. We prove the resulting expansion is strictly stable and we use the notion of G-bases to characterize forking in the expansion. Forking in expansions by Mann groups was also studied by Göral in his Ph.D. thesis (see [14]).

In section 4 we study expansions of a real closed field together with the group generated by a dense independent subset of $R^{>0}$. In section 5 we study the pair coming from a model of RCF together with the group generated by a dense independent subset of the unit circle $\mathbb{S}(R)$ in R. In section 6 we show these two resulting theories are NIP and that the structure induced in G is weakly 1-based.

In section 7 we start an analysis of definable groups in pairs (R, G), where R is a real closed field and G has the Mann property and is either a subgroup of $R^{>0}$ or a subgroup of $\mathbb{S}(R)$. We consider the case where G is of finite rank and also the case where it is generated by a collection of independent dense-codense elements. Our results deal mostly with subgroups definable in the pair of 1-dimensional groups definable in the pure field language. We show, under this 1-dimensional assumption, that such groups are either small or definable in the pure field language. We also consider the group G seen as a definable subset of the pair (R, G). We prove that the definable subgroups of G in the pair can be defined directly in G and we describe G^{00}. In each of the expansions under consideration, we identify the compact group arising as a quotient G/G^{00} and construct explicitly a strong f-generic type for G.

2. G-structures: definition and first properties

Let $(K, +, \cdot, 0, 1, H)$ be a sufficiently saturated H-structure of ACF or RCF (see [6]). We can also add symbols for multiplicative inverse and order (in the case of RCF). Let $G(K)$ be the subgroup of $(K^{\times}, \cdot)$ generated by $H(K)$.

Let $T^G = Th(K, +, \cdot, 0, 1, G)$, where G is the new unary predicate symbol interpreted by $G(K)$.

Lemma 2.1. *$G(K)$ is linearly independent over $\mathbb{Q}$.*

Proof. Suppose

$$\Sigma_{i=1}^{n} \alpha_i h_1^{k_{i,1}} \dots h_m^{k_{i,m}} = 0,$$

where $\alpha_i \in \mathbb{Q}$, $h_j \in H(K)$, $k_{i,j} \in \mathbb{Z}$, h_j are distinct, and

$$(k_{i,1}, \dots, k_{i,m}) \neq (k_{l,1}, \dots, k_{l,m})$$

for $i \neq l$. Multiplying by a product of positive powers of h_i if needed, we may assume that all $k_{i,j} \geq 0$ for all i, j. Since h_i are algebraically independent, we have $\alpha_i = 0$ for all i.

□

Lemma 2.2. *$G(K)$ is dense-codense in $(K, +, \cdot, 0, 1)$ (in the sense of geometric structures).*

Proof. Follows from $H(K) \subset G(K) \subset \operatorname{acl}(H(K))$. □

Note that $(G(K), \cdot)$ is a free abelian group. We will now look at the algebraic closure restricted to G.

Lemma 2.3. *Suppose $a, g_1, \dots, g_k \in G(K)$ and $a \in \operatorname{acl}(g_1, \dots, g_k)$, witnessed by a polynomial equation*

$$p(x, y_1, \dots, y_k) = 0$$

with integer coefficients, of degree $n \geq 1$ in variable x. Let m be the maximal degree in which any of y_i appears in the polynomial p. Then for some $0 < r \leq n$, and $s_1, \dots, s_k \in \mathbb{Z}$ such that $|s_i| \leq 2m$, we have

$$a^r = g_1^{s_1} \dots g_k^{s_k}.$$

Proof. Suppose

$$p(a, g_1, \dots, g_k) = \Sigma_{i=0}^{n} (q_{i,1} w_{i,1} + \dots q_{i,l_i} w_{i,l_i}) a^i = \sum_{i=0}^{n} \sum_{j=0}^{l_i} q_{i,j} w_{i,j} a^i$$

where $q_{i,j} \in \mathbb{Z}$ and $w_{i,j}$ is a product of $g_1, \ldots, g_k$ each raised to a (nonnegative) integer power. Since $G(K)$ is linearly independent and $n \geq 1$, at least two of the terms $w_{i,j}a^i$ must cancel each other. We may assume that for each fixed i, all the $w_{i,j}$ are distinct. Thus, for some $0 \leq i_1 < i_2 \leq n$ and some $j_1 \leq l_{i_1}$ and $j_2 \leq l_{i_2}$, we have $w_{i_1,j_1}a^{i_1} - w_{i_2,j_2}a^{i_2} = 0$. Then $a^{i_2-i_1} = w_{i_1,j_1}(w_{i_2,j_2})^{-1}$, and the statement follows.

□

Lemma 2.4. *Let $(K^*, +, \cdot, 0, 1, G)$ be a sufficiently saturated model of the theory $T^G = Th(K, +, \cdot, 0, 1, G)$. Then*

1. *$G(K^*)$ is dense-codense and linearly independent*
2. *if $a, g_1, \ldots, g_k \in G(K^*)$ and $a \in \operatorname{acl}(g_1, \ldots, g_k)$, then for some $r \geq 1$, and $s_1, \ldots, s_k \in \mathbb{Z}$ we have $a^r = g_1^{s_1} \ldots g_k^{s_k}$.*

Proof. (1) Clear.
(2) By Lemma 2.3, the statement holds in $(K, +, \cdot, 0, 1, G)$, and is first-order axiomatizable. □

We denote by tp_G the type in the G-structure $(K, +, \cdot, 0, 1, G)$ (similarly for acl_G). For a tuple $\vec{g}$ of elements of G, let $\mathrm{tp}_{gr}(\vec{g})$ be the type of $\vec{g}$ in the sense of the theory T^{gr} of free abelian groups with infinite basis. Similarly, acl_{gr} refers to algebraic closure in T^{gr} and dcl_{gr} refers to definable closure in T^{gr}.

Lemma 2.5. *Let (K, G) be a model of T^G, and $n > 1$. Then any element $g \neq 1$ in $G(K)$ has no more than one nth root.*

Proof. If $a^n = b^n$, and $a, b \in G(K)$, then $(ab^{-1})^n = 1$. Since free abelian groups are torsion free, and this is a first order property, we conclude that $ab^{-1} = 1$, hence, $a = b$. □

Remark 2.6. *Note that by Lemma 2.4, in any model of T^G,* acl *restricted to G is obtained by first closing under multiplication and inverses, and then applying the nth roots (when they exist). By Lemma 2.5, it also follows that* $\operatorname{acl}(\vec{g}) \cap G \subset \operatorname{dcl}_{gr}(\vec{g})$ *for any $\vec{g} \in G$.*

On the other hand, the algebraic (equivalently, definable) closure operator in any model of the theory of free abelian groups (T^{gr}) is given by closing under multiplication, inverses and nth roots (when they exist). Thus, acl *restricted to G coincides with* $\mathrm{acl}_{gr} = \mathrm{dcl}_{gr}$.

Next, we show that in our construction, G has the Mann Property.

Lemma 2.7. *The subgroup $G = G(K)$ of $K^\times$ has the Mann Property. Same is true in any model of T^G.*

Proof. Let $a_1, \ldots, a_n \in \mathbb{Q}$ be nonzero, $n \geq 2$. Let $g_1, \ldots, g_n \in G$ be such that

$$a_1 g_1 + \ldots + a_n g_n = 1,$$

and $\Sigma_{i \in I} a_i g_i \neq 0$ for any nonempty subset I of $\{1, \ldots, n\}$. Since elements of G are linearly independent (over $\mathbb{Q}$), and $1 \in G$, we must have

$$g_1 = g_2 = \ldots = g_n = 1.$$

Thus, each equation as above, either has a unique solution (when $a_1 + \ldots + a_n = 1$), or no solution at all. Same is true in any model of T^G, since linear independence of G is a first order property. □

Definition 2.8. Let $X \subset K^n$. We say that X is *large* if for some $m \geq 1$ and a field definable function $f : K^{nm} \to K$, $f(X^m) = K$.

Remark 2.9. *Note that codensity condition (extension property) implies that G is small in all models of T^G.*

Let $G^{[n]}$ denote the subgroup of $G = G(K)$ consisting of nth powers of elements of G.

Lemma 2.10. *For each n, $G^{[n]}$ has an infinite index in G. Same is true in any model of T^G.*

Proof. Let $m, n \geq 2$, and let $a_1, \ldots, a_m$ be distinct elements of $H(K)$. Then $a_1, \ldots, a_m \in G$, and we claim that $a_i G^{[n]} \neq a_j G^{[n]}$ for $i \neq j$. Otherwise, $a_i a_j^{-1} \in G^{[n]}$, and thus,

$$a_i a_j^{-1} = h_1^{k_1 n} \cdot \ldots \cdot h_l^{k_l n},$$

for some distinct $h_1, \ldots, h_l \in H(K)$, and $k_1, \ldots, k_l \in \mathbb{Z}$. Since $H(K)$ is algebraically independent, both a_i and a_j must be among $h_1, \ldots, h_l$. We may assume that $h_1 = a_i$, $h_2 = a_j$. It follows that $k_1 n = 1$ and $k_2 n = -1$, a contradiction since $n \geq 2$ and $k_1, k_2 \in \mathbb{Z}$. Thus, $a_1, \ldots, a_m$ are in different $G^{[n]}$-cosets. Since m was arbitrary, the statement follows. Clearly, same is true in any model of T^G.

□

3. The case of ACF

We start by recalling the notions of Mann axioms and Γ-family from [12]. We assume that G is a multiplicative subgroup of $K^\times$ satisfying the Mann property.

Let Γ be a subgroup of $G = G(K)$. We say that (K, G) *satisfies Mann axioms of* Γ, if for any

$$a_1, \ldots, a_n \in \mathbb{Q}^\times$$

and $n \geq 2$, the equation

$$a_1 g_1 + \ldots + a_n g_n = 1$$

has a finite number of nondegenarate solutions in G (meaning $\Sigma_{i \in I} a_i g_i \neq 0$ for all nonempty $I \subset \{1, \ldots, n\}$), all of which are in Γ and we list them as:

$$\gamma_1 = (\gamma_{11}, \ldots, \gamma_{1n}), \ldots, \gamma_k = (\gamma_{k1}, \ldots, \gamma_{kn}).$$

The corresponding first order sentence in the language $L_{G,(\gamma')_{\gamma \in \Gamma}}$ of K expanded with unary predicate $G(-)$ and constants for elements of Γ is called a *Mann axiom of* Γ.

In the case when G is the group generated by an independent dense codense subset H, we can take $\Gamma = \{1\}$, since $(1, 1, \ldots, 1)$ is the only possible nondegenerate solution of the above equation.

Suppose now that K is an algebraically closed field. The theory $ACF(\Gamma)$ of a Γ-*family* [12] is axiomatized by the sentences expressing the following properties in the language $L_{G,(\gamma')_{\gamma \in \Gamma}}$:

1. K is an algebraically closed field (of a fixed characteristic);
2. G is subgroup of $K^\times$;
3. $\gamma \to \gamma' : \Gamma \to G$ is a group homomorphism;
4. $(K, G, (\gamma')_{\gamma \in \Gamma})$ satisfies the Mann axioms of Γ.

Fact 3.1. *([12], Theorem 6.8) Let $(K, G, (\gamma))$ and $(K', G', (\gamma))$ be two models of $ACF(\Gamma)$. Then*

$$(K, G, (\gamma)) \equiv (K', G', (\gamma)) \iff (G, (\gamma)) \equiv (G', (\gamma)),$$

as groups with distinguished elements.

Since, in the case when G is generated by a dense-codense independent set H, $\Gamma = 1$, T^G is axiomatized by saying :

(1) K is an algebraically closed field (of fixed characteristic);
(2) G is a multiplicative subgroup of K;
(3) G is linearly independent and satisfies the theory of free abelian groups.

Note that the axioms imply that G is dense in K (infinite dimensional) since it satisfies the axioms of free abelian groups and codense in K (infinite codimension) when the model is saturated since G is linearly independent.

Lemma 3.2. *Let (K, G) and (K', G') be sufficiently saturated models of T^G and let $g_1, \ldots, g_n \in G$, $g'_1, \ldots, g'_n \in G'$ be such that $(G, \vec{g}) \equiv (G', \vec{g}')$ (i.e. $\vec{g}$ and $\vec{g}'$ have the same type in the theory of free abelian groups). Then $\mathrm{tp}_G(\vec{g}) = \mathrm{tp}_G(\vec{g}')$.*

Proof. Let Γ and Γ' be subgroups of G and G' generated by $\vec{g}$ and $\vec{g}'$, respectively. The rest follows by Fact 3.1. □

Thus, there is no new structure induced on G in T^G (by formulas over $\emptyset$).

Note that by Corollary 6.2 of [12], T^G is stable. Since the theory of free abelian groups is not superstable, neither is T^G. One can see this directly, since the chain of groups $\{G^{[2^n]} : n \in \mathbb{N}\}$ is strictly descending. By contrast, both the theory of lovely (beautiful) pairs and the theory of H-structures of T are ω-stable of U-rank ω.

We expand the language of G-structures with new unary operational symbols $\{f_n(-) : n \geq 2\}$, interpreted as follows: if $a \in G(K)^{[n]}$ then $f_n(a) = b \in G(K)$ where b is such that $b^n = a$ (such b is unique by Lemma 2.5). Otherwise, $f_n(a) = 1$. Let T^{G+} be the resulting expansion of T^G. We will denote by tp_{G+} and $qftp_{G+}$ the complete type and the quantifier free type, respectively, in the expanded language.

Note that for any subset A of $G(K)$, closing A under products and inverses, and then under single applications of the functions $\{f_n(-) : n \geq 2\}$, results in a subgroup of $G(K)$ closed under operations $f_n(-)$, $n \geq 2$. Indeed, if $a \in G(K)^{[m]}$ and $b \in G(K)^{[n]}$, then $f_m(a) f_n(b) = f_{mn}(a^n b^m)$. Also, note that by Lemma 2.4, this closure operator coincides with (field theoretic) acl restricted to $G(K)$. By Remark 2.6, it also coincides with dcl_{gr} (equivalently, acl_{gr}). By and the fact that T^{gr} has q.e. relative to pp-formulas, T^{gr} has q.e. in the group language expanded by operations f_n.

Proposition 3.3. *Let $(K, G)^+$ and $(K', G)^+$ be two sufficiently saturated models of T^{G+} and let $\vec{a} \in K$, $\vec{a}' \in K'$ be G-independent tuples, such that*

$$qftp_{G+}(\vec{a}) = qftp_{G+}(\vec{a}').$$

Then $\mathrm{tp}_{G+}(\vec{a}) = \mathrm{tp}_{G+}(\vec{a}')$.

Proof. Let $c \in K$ be any element. We need to find $\vec{b} \in K$, $\vec{b}' \in K'$ and $c' \in K'$ such that $\vec{a}\vec{b}c$ and $\vec{a}'\vec{b}'c'$ are both G-independent, and $qftp_{G+}(\vec{a}\vec{b}c) = qftp_{G+}(\vec{a}'\vec{b}'c')$. We proceed by cases:

Case 1: $c \notin \operatorname{acl}(\vec{a}G(K))$. Take any $c' \in K'$ such that $c' \notin \operatorname{acl}(\vec{a}'G(K'))$. Then clearly both $\vec{a}c$ and $\vec{a}'c'$ are G-independent. We also claim that $qftp_{G+}(\vec{a}c) = qftp_{G+}(\vec{a}'c')$. To show this, it suffices to note that if $d \in \operatorname{acl}(\vec{a}c) \setminus \operatorname{acl}(\vec{a})$, then $d \notin G(K)$, and the same is true for any $d' \in \operatorname{acl}(\vec{a}'c') \setminus \operatorname{acl}(\vec{a}')$.

Case 2: $c \in G(K) \setminus \operatorname{acl}(\vec{a})$. Then c is non-algebraic over $G(\vec{a})$ in the sense of T^{gr}, and $G(\vec{a})$ and $G(\vec{a}')$ have the same type in the sense of T^{gr} (group type). Let $c' \in G(K')$ be such $G(\vec{a})c$ and $G(\vec{a}')c'$ have the same group type. Since $c' \in G(K)$ and $c' \notin \operatorname{acl}(G(\vec{a}'))$, from $\vec{a}' \downarrow_{G(\vec{a}')} c'$, we get $c' \notin \operatorname{acl}(\vec{a}')$. Then $\vec{a}c$ and $\vec{a}'c'$ have the same field-theoretic type. Also, whenever $d \in \operatorname{acl}(\vec{a}c) \setminus \operatorname{acl}(G(\vec{a})c)$, we have $d \notin G(K)$, and same is true for $\vec{a}'$, c' and d'. This shows that $qftp_{G+}(\vec{a}c) = qftp_{G+}(\vec{a}'c')$. Clearly, $\vec{a}c$ and $\vec{a}'c'$ are G-independent.

Case 3: $c \in \operatorname{acl}(\vec{a})$. Note that $G(\operatorname{acl}(\vec{a})) = \operatorname{acl}(G(\vec{a})) \cap G(K)$, and the same is true of $\vec{a}'$. Thus, by Lemma 2.4, $G(\operatorname{acl}(\vec{a}))$ is obtained by closing $G(\vec{a})$ under multiplication and operations f_n (and same is true of $\vec{a}'$). Then since $qftp_{G+}(\vec{a}) = qftp_{G+}(\vec{a}')$, we can extend a partial field isomorphism from $\vec{a}$ to $\vec{a}'$, to a partial field isomorphism from $\vec{a}G(\operatorname{acl}(\vec{a}))$ to $\vec{a}G(\operatorname{acl}(\vec{a}'))$, and then to one from $\operatorname{acl}(\vec{a})$ to $\operatorname{acl}(\vec{a}')$, as needed. Also, clearly, $\operatorname{acl}(\vec{a})$ and $\operatorname{acl}(\vec{a}')$ are both G-independent.

□

Corollary 3.4. *The theory T^G is near model complete. That is, if $(K, G) \models T^G$, every $\mathcal{L}_G$-formula $\psi(\vec{y})$ is equivalent to a boolean combinations of formulas of the form $\exists x_1 \in G \dots \exists x_n \in G\phi(\vec{x}, \vec{y})$, where ϕ is an $\mathcal{L}$-formula.*

Proof. Given a sufficiently saturated model (K, G) of T^G and two tuples $\vec{a}_1$ and $\vec{a}_2$ in K satisfying the same formulas of the form $\exists x_1 \in G \dots \exists x_n \in G\phi(\vec{x}, \vec{y})$, where ϕ is an $\mathcal{L}$-formula (call such formulas G-*existential*), we can extend both tuples to $\vec{a}_1'$ and $\vec{a}_2'$ which are G-independent and still satisfy the same G-existential formulas. Then $qftp_{G+}(\vec{a}_1') = qftp_{G+}(\vec{a}_2')$. The rest follows by Proposition 3.3.

□

In the setting of H-structures one of the main tools is the existence of an H-basis. We will now see that a similar construction works for our setting.

Proposition 3.5. *Let $(K^*, +, \cdot, 0, 1, G)$ be a sufficiently saturated model of the theory $T^G = Th(K, +, \cdot, 0, 1, G)$ and let $\vec{a} = (a_1, \dots, a_n)$, $\vec{b} = (b_1, \dots, b_m) \in G$. Then as elements in the field $(K, +, \cdot, 0, 1)$, we have $\vec{a} \downarrow_{\operatorname{dcl}_{gr}(\vec{a}) \cap \operatorname{dcl}_{gr}(\vec{b})} \vec{b}$.*

Proof. Assume there is an $i \leq m$ such that $b_i \in \operatorname{acl}(a_1, \dots, a_n, b_1, \dots, b_{i-1})$. Then by Lemma 2.4 there exists integers $s_1, \dots, s_k, r_1, \dots, r_i$ with $r_i \neq 0$ such that $b_i^{r_i} = a_1^{s_1} \cdots a_n^{r_n} \cdot b_1^{r_1} \cdots b_{i-1}^{r_{i-1}}$. Let $d = a_1^{s_1} \cdots a_n^{r_n} = b_1^{-r_1} \cdots b_{i-1}^{-r_{i-1}} b_i^{r_i}$, then $d \in \operatorname{dcl}_{gr}(\vec{a}) \cap \operatorname{dcl}_{gr}(\vec{b})$ and $b_i \in \operatorname{acl}(d, b_1, \dots, b_{i-1})$ as we wanted. □

In particular, if $C \subset G$ and $\vec{a}, \vec{b} \in G$, we get $\vec{a} \downarrow_{\operatorname{dcl}_{gr}(C\vec{a}) \cap \operatorname{dcl}_{gr}(C\vec{b})} \vec{b}C$

The elements in G need not have all positive roots, but the behave similar to modules over $\mathbb{Q}$, in the sense that we can assign to them a good notion of dimension:

Lemma 3.6. *Let $(K^*, +, \cdot, 0, 1, G)$ be a model of the theory $T^G = Th(K, +, \cdot, 0, 1, G)$ and let $a_1, \dots, a_n \in G$, $C \subset G$ be such that $C = \operatorname{dcl}_{gr}(C)$. Then whenever $b_1, \dots, b_{n+1} \in \operatorname{dcl}_{gr}(a_1, \dots, a_n, C)$ (so for each i, $b_i \in G$), we have that for some $1 \leq i \leq n+1$, $b_i \in \operatorname{dcl}(b_1, \dots, b_{i-1}, b_{i+1}, \dots, b_{n+1}C)$.*

Proof. By Lemma 2.4 for each j there exists $c_j \in C$ integers $s_j^1, \dots, s_j^n, r_j$ with $r_j \neq 0$ such that $c_j b_j^{r_j} = a_1^{s_j^1} \cdots a_n^{s_j^n}$, We call $\vec{v}_j = (s_j^1/r_j, \dots, s_j^n/r_j)$ a *set of coordinates* of b_j with respect to the set $\{a_1, \dots, a_n\}$ over C. Then the set of coordinates $\vec{v}_1, \dots, \vec{v}_{n+1}$ is a linearly dependent set of vectors in the vector space $\mathbb{Q}^n$ and thus we can write one of them as a linear combination in terms of the other ones. Assume that

$$\vec{v}_{n+1} = (p_1/q_1)\vec{v}_1 + \dots, +(p_n/q_n)\vec{v}_n$$

which can be rewritten as
$(q_1 \cdots q_n)(s_{n+1}^1, \dots, s_{n+1}^n) =$
$(p_1 r_{n+1} q_2 \cdots q_n)(s_1^1/r_1, \dots, s_1^n/r_1) + \dots + p_n r_{n+1} q_1 \cdots q_{n-1}(s_n^1/r_n, \dots, s_n^n/r_n)$ and finally $(a_1^{s_{n+1}^1} \cdots a_n^{s_{n+1}^n} c_{n+1}^{-1})^{q_1 \cdots q_n} = b_{n+1}^{q_1 \cdots q_n \dot{r}_{n+1}} \in \mathrm{dcl}(b_1, \dots, b_n, C)$. □

Observation 3.7. *Let $(K^*, +, \cdot, 0, 1, G)$ be a model of the theory $T^G = Th(K, +, \cdot, 0, 1, G)$, let $g_1, \dots, g_n \in G$ and $C \subset G$ be such that $C = \mathrm{acl}_{gr}(C)$. Then whenever $B \leq \mathrm{dcl}_{gr}(g_1, \dots, g_n, C)$, $C \subset B$, is a definably closed subset, then there is a finite set $\{b_1, \dots, b_k\} \subset B$ of size $\leq n$ that generates B over C. Furthermore, if the minimal size of a set of generators for B over C is n, then $B = \mathrm{dcl}_{gr}(b_1, \dots, b_n, C)$.*

Proof. The proof is very similar to the one of Lemma 3.6. Assume first that the elements in $\{g_1, \dots, g_n\}$ are independent in the group sense over C. For each $b \in B$ there exists integers $s_b^1, \dots, s_b^n, r_b$ with $r_b \neq 0$ and $c_b \in C$ such that $b^{r_b} c_b = g_1^{s_b^1} \cdots g_n^{s_b^n}$ and write $\vec{v}_b = (s_b^1/r_b, \dots, s_b^n/r_b)$ for a set of coordinates of b with respect to the set $(g_1, \dots, g_n)$ over C. The family $\{\vec{v}_b : b \in B\}$ generates a finite dimensional subspace of $\mathbb{Q}^n$ and thus it has a finite basis. As before, from the basis it is easy to extract the desired set $\{b_1, \dots, b_k\} \subset B$ of size $\leq n$ that generates B.

For the furthermore part, if the minimal set of generators is of size n, then the space generated by the family $\{\vec{v}_b : b \in B\}$ over C coincides with $\mathbb{Q}^n$ and thus $B = \mathrm{dcl}_{gr}(g_1, \dots, g_n, C)$. □

Proposition 3.8. *Let $(K^*, +, \cdot, 0, 1, G)$ be a sufficiently saturated model of the theory $T^G = Th(K, +, \cdot, 0, 1, G)$, let $\vec{a} \in K$ and $C \subset K$ be G-independent. Let $\vec{g}, \vec{h} \in G(K)$ be tuples of minimal length such that $\vec{a} \downarrow_{C\vec{g}} G(K)$ and $\vec{a} \downarrow_{C\vec{h}} G(K)$. Then $\mathrm{dcl}_{gr}(\vec{g}\ G(C)) = \mathrm{dcl}_{gr}(\vec{h}\ G(C))$.*

Proof. Let n be the length of the tuples $\vec{g}$ and $\vec{h}$.

Write $\vec{a} = \vec{a}_0 \vec{a}_1$ where $\vec{a}_0$ is a tuple acl-independent over $G(K)C$ and $\vec{a}_1 \in \mathrm{acl}(\vec{a}_0 GC)$. If $\vec{a}_1 = \emptyset$ there is nothing to prove, so we may assume that $\vec{a}_1 \neq \emptyset$.

By Lemma 3.5

$$\vec{g} \quad \underset{\mathrm{dcl}_{gr}(\vec{g}G(C)) \cap \mathrm{dcl}_{gr}(\vec{h}G(C))}{\downarrow} \quad \vec{h}$$

Let $\vec{k}$ be a basis (with respect to dcl_{gr}) of $\mathrm{dcl}_{gr}(\vec{g}G(C)) \cap \mathrm{dcl}_{gr}(\vec{h}G(C))$ over $G(C)$. By Observation 3.7, $|\vec{k}| \leq |\vec{g}|$.

Case 1. Assume $|\vec{k}| = |\vec{g}|$. By Observation 3.7 we get that $\mathrm{dcl}_{gr}(\vec{g}\ G(C)) = \mathrm{dcl}_{gr}(\vec{k}G(C)) = \mathrm{dcl}_{gr}(\vec{h}G(C))$ as we wanted.

Case 2. Assume that $|\vec{k}| < |\vec{g}|$. By the definition of $\vec{k}$ we get $\vec{g} \downarrow_{\mathrm{dcl}_{gr}(\vec{k}G(C))} \vec{h}$. Since C is G-independent we get $\vec{g} \downarrow_{C\, \mathrm{dcl}_{gr}(\vec{k}G(C))} \vec{h}$. Since $\vec{a}_0$ is acl-independent over $G(K)C$, $\vec{g} \mathop{\smile\!\!\!\!\!\!\mid}_{\vec{a}_0 C\, \mathrm{dcl}_{gr}(\vec{k}G(C))} \vec{h}$. Since $\vec{a}_1 \in \mathrm{acl}(\vec{a}_0\vec{g}C)$ and $\vec{a}_1 \in \mathrm{acl}(\vec{a}_0\vec{h}C)$ we get $\vec{a}_1 \in \mathrm{acl}(\vec{a}_0\vec{k}C)$, contradicting the minimality of $|\vec{g}|$.

□

Definition 3.9. Let $(K^*, +, \cdot, 0, 1, G)$ be a sufficiently saturated model of the theory $T^G = Th(K, +, \cdot, 0, 1, G)$, let $\vec{a} \in K$ and let $C \subset G$ be G-independent. Let $\vec{g} \in G$ be a tuple of minimal length such that $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}_{\vec{g}C} G$. Then we call the tuple $\vec{g}$ a *G-basis* of $\vec{a}$ with respect to C. By Proposition 3.8 all such possible choices for subtuples are interdefinable over $G(C)$. We will use the notation $\mathbf{GB}(\vec{a}/C)$ for $\mathrm{dcl}_{gr}(\vec{g}C)$, and will often view it as an infinite tuple (so that we can talk about its type). For an arbitrary set A, we let $\mathbf{GB}(A/C)$ be the union of $\mathbf{GB}(\vec{a}/C)$ for all finite tuples $\vec{a}$ of elements of A. When we take $C = \emptyset$ (so $\mathrm{acl}_{gr}(G(C)) = \{1\}$) we simply write $\mathbf{GB}(\vec{a})$.

Remark 3.10. *Let $C \subset K$. Then* $\mathrm{acl}_G(C) = \mathrm{acl}(C \cup \mathbf{GB}(C))$.

We are now ready to describe (non-)forking in T^{G+}. We will denote by $\mathop{\smile\!\!\!\!\!\!\mid}^G$ independence in the sense of T^G (or T^{G+}). We use $\mathop{\smile\!\!\!\!\!\!\mid}$ for independence in the sense of ACF and $\mathop{\smile\!\!\!\!\!\!\mid}^{gr}$ for independence in the sense of T^{gr}.

Lemma 3.11. *(working in a sufficiently saturated model $(K, G)^+$ of T^{G+})*
Suppose $C \subset B \subset K$ and $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}^G_C B$. Then

$$\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}_{C\ G(K)} B\ G(K)$$

and

$$\mathbf{GB}(\vec{a}C) \mathop{\smile\!\!\!\!\!\!\mid}^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(B).$$

Proof. Suppose first that $\vec{a} \not\mathop{\smile\!\!\!\!\!\!\mid}_{C\ G(K)} B\ G(K)$. We may assume that $\vec{a} = a_0\vec{a}'$, $C = \vec{c}$, $B = \vec{b}\vec{c}$, where each of $\vec{a}'$ and $\vec{c}\vec{b}$ is a tuple independent over $G(K)$, and $a_0 \in \mathrm{acl}(\vec{a}'\vec{b}\vec{c}G(K))$ and $a_0 \notin \mathrm{acl}(\vec{a}'\vec{c}G(K))$. Take an independent sequence $I = (\vec{b}_i : i \in \omega)$ in K, where $\vec{b}_0 = \vec{b}$, $\vec{b}_i$ have the same length as $\vec{b}$, and $\bigcup \vec{b}_i$ is independent over $G(K)\vec{a}'\vec{c}$. Then, by Proposition 3.2, I is indiscernible (in the sense of T^{G+}) over $\vec{c}$. Let $p(\vec{x}, \vec{y}\vec{z}) = \mathrm{tp}_{G+}(\vec{a}, \vec{b}\vec{c})$. Then $\bigcup p(\vec{x}, \vec{b}_i\vec{c})$ is inconsistent, a contradiction with $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}^G_{\vec{c}} \vec{b}$.

Now, suppose $\mathbf{GB}(\vec{a}C) \not\mathop{\smile\!\!\!\!\!\!\mid}^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(B)$. Let $B' = \mathbf{GB}(B)$, $A' = \mathbf{GB}(\vec{a}C)$, $C' = \mathbf{GB}(C)$. Let $(B'_i : i \in \kappa)$ be a sufficiently long indiscernible sequence in $\mathrm{tp}_{gr}(B'/C')$ such that there is no A'' such that $\mathrm{tp}_{gr}(A'', B'_i, C') = \mathrm{tp}_{gr}(A', B', C')$ for all i. Note that $C \mathop{\smile\!\!\!\!\!\!\mid}_{C'} G(K)$, and $(B'_i : i \in \kappa)$ is indiscernible over C' in the sense of T. Then there exists $C^* \subset K$ such that $\mathrm{tp}(C^*, B'_i, C') = \mathrm{tp}(C, B', C')$ for all i. Taking conjugates over C', we may assume that $C^* = C$. By Erdos-Rado, we get a sequence $(B'_i : i \in \omega)$ in $G(K)$, indiscernible over C in the sense of T and indiscernible over C' in the sense of T^{gr}, with $\mathrm{tp}(B'_i, C) = \mathrm{tp}(B', C)$, $\mathrm{tp}_{gr}(B'_i, C') = \mathrm{tp}_{gr}(B', C')$ such that there is no A'' such that $\mathrm{tp}_{gr}(A'', B'_i, C') = \mathrm{tp}_{gr}(A', B', C')$ for all i. Note that $(B'_i : i \in \omega)$ is indiscernible over C in the sense of T^G, and there is no A'' such that $\mathrm{tp}_G(A'', B'_i, C') = \mathrm{tp}_G(A', B', C')$ for all i. This shows that $A' \not\mathop{\smile\!\!\!\!\!\!\mid}^G_C B'$. Since $A' \subset \mathrm{dcl}_G(\vec{a}C)$ and $B' \subset \mathrm{dcl}(B)$, this contradicts $\vec{a} \mathop{\smile\!\!\!\!\!\!\mid}^G_C B$.

□

Lemma 3.12. *Let $B, C \subset K$. Then* $\mathbf{GB}(BC) = \mathbf{GB}(B) \cup \mathbf{GB}(C/\operatorname{acl}_G(B))$

Proof. First see the G-basis as tuples and write $\vec{g}_C = \mathbf{GB}(C/\operatorname{acl}_G(B))$, $\vec{g}_B = \mathbf{GB}(B)$. Then we have by definition $B \perp_{\vec{g}_B} G$ and $C \perp_{\operatorname{acl}_G(B)\vec{g}_C} G$. By Remark 3.10 we get that $\operatorname{acl}_G(B) = \operatorname{acl}(B \cup \vec{g}_B)$, so $C \perp_{B\vec{g}_B\vec{g}_C} G$. By transitivity $BC \perp_{\vec{g}_B\vec{g}_C} G$ and thus $\mathbf{GB}(BC) \subset \mathbf{GB}(B) \cup \mathbf{GB}(C/\operatorname{acl}_G(B))$.

Now let $\vec{g}_{BC} = \mathbf{GB}(BC)$ seen as a tuple. Then we have by definition $BC \perp_{\vec{g}_{BC}} G$ and we get $B \perp_{\vec{g}_{BC}} G$, so $\mathbf{GB}(B) \subset dcl(\vec{g}_{BC})$. We also get $C \perp_{\vec{g}_{BC}B} G$ and thus $C \perp_{\vec{g}_{BC}\vec{g}_B B} G$ and thus by Lemma 3.10 we obtain $C \perp_{\vec{g}_{BC}\operatorname{acl}_G(B)} G$ and so $\mathbf{GB}(C/\operatorname{acl}_G(B)) \subset \vec{g}_{BC}$.

□

Lemma 3.13. *(working in a sufficiently saturated model $(K, G)^+$ of T^{G+})*

Let $C \subset D \subset K$ be such that C, D are G-independent. If $\vec{a} \perp_{C\ G(K)} D\ G(K)$ and $\mathbf{GB}(\vec{a}C) \perp^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(D)$ then $\vec{a} \perp^{G}_{C} B$.

Proof. We may write $\vec{a} = (\vec{a}_1, \vec{a}_2) \in K$ so that $\vec{a}_1$ is an independent tuple over $CG(K)$, and $\vec{a}_2 \in \operatorname{acl}(G(K)C\vec{a}_1)$. Since $\vec{a} \perp_{C\ G(K)} D\ G(K)$, $\vec{a}_1$ is an independent tuple over $DG(K)$.

Claim $\operatorname{tp}_G(\vec{a}/D)$ does not divide over C.

Let $p(\vec{x}, D) = tp(\vec{a}_1, D)$. Let $\{D_i : i \in \omega\}$ be an $\mathcal{L}_G$-indiscernible sequence over C, with $D_0 = D$. Since $\vec{a}_1$ is independent over D, $tp(\vec{a}_1, D)$ does not divide over C and $\cup_{i\in\omega} p(\vec{x}, D_i)$ is consistent. We can find $\vec{a}_1' \models \cup_{i\in\omega} p(\vec{x}, D_i)$ such that $\{\vec{a}_1' D_i : i \in \omega\}$ is indiscernible and $\vec{a}_1'$ is independent over $\cup_{i\in\omega} D_i$. By the generalized extension property, we may assume that $\vec{a}_1'$ is independent over $\cup_{i\in\omega} D_i G(K)$. Note that $\vec{a}_1 D$ is G-independent and $\vec{a}_1' D_i$ is also G-independent for any $i \in \omega$. So by Proposition 3.3 $\operatorname{tp}_G(\vec{a}_1 D) = \operatorname{tp}_G(\vec{a}_1' D_i)$ for any $i \in \omega$.

Now let $\vec{g} = \mathbf{GB}(\vec{a}/C)$ (viewed as a tuple) and let $q(\vec{y}, G(D)) = tp_{gr}(\vec{g}, G(D))$. Note that $\vec{g}$ is an independent tuple over $\vec{a}_1 D$ (as well as an independent tuple over $\vec{a}_1 C$). Since $\{G(D_i) : i \in \omega\}$ is an $\mathcal{L}_G$-indiscernible sequence, then $\{G(D_i) : i \in \omega\}$ is also an $\mathcal{L}_{gr}$-indiscernible sequence. By $\mathbf{GB}(\vec{a}C) \perp^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(D)$ and Lemma 3.12 we get $\mathbf{GB}(\vec{a}/C) \perp^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(D)$ so there is $\vec{g}' \models \cup_{i\in\omega} q(\vec{y}, G(D_i))$. In particular, for each i, $\vec{g}'$ is an $\mathcal{L}$-independent tuple over $G(D_i)$. Since $D_i \perp_{G(D_i)} G(K)$, the tuple $\vec{g}'$ is $\mathcal{L}$-independent tuple over D_i.

Since $\vec{a}_1'$ is an independent tuple over GD_i for each i, we get that $\vec{a}_1' \perp G(K) D_i$, so $\vec{a}_1' \perp_{\vec{g}'} D_i$ and thus $\vec{a}_1' \vec{g}'$ is an independent tuple over D_i. In particular, for each i we get

$$\operatorname{tp}(\vec{a}_1' \vec{g}' D_i) = \operatorname{tp}(\vec{a}_1 \vec{g} D)$$

Since D_i is G-independent, we have $D_i \perp_{G(D_i)} G(K)$, so

$$D_i \perp_{\vec{g}' G(D_i)} G(K)$$

and

$$D_i \vec{g}' \perp_{\vec{g}' G(D_i)} G(K).$$

On the other hand $\vec{a}'_1 \mathop{\smile\!\!\!\!\!\!\,|}\, G(K)D_i$ implies $\vec{a}'_1 D_i \vec{g}' \mathop{\smile\!\!\!\!\!\!\,|}_{\vec{g}' G(D_i)} G(K)$. Thus $\vec{a}'_1 D_i \vec{g}'$ is G-independent for each $i \in \omega$ and

$$\mathrm{tp}_{gr}(\vec{g}' G(D_i)) = \mathrm{tp}_{gr}(\vec{g} G(D))$$

This last equation together with $\mathrm{tp}(\vec{g}, \vec{a}_1, D) = \mathrm{tp}(\vec{g}', \vec{a}'_1, D_i)$ and Proposition 3.3 imply that $\mathrm{tp}_G(\vec{g}, \vec{a}_1, D) = \mathrm{tp}_G(\vec{g}', \vec{a}'_1, D_i)$ for each $i \in \omega$. This shows that $\mathrm{tp}_G(\vec{a}_1, \vec{g}/D)$ does not divide over C and since $\vec{a} \in \mathrm{acl}(\vec{a}_1, \vec{g}C)$ we get that $\mathrm{tp}(\vec{a}/D)$ does not divide over C. □

Puting all the results together we get:

Theorem 3.14. *Let $C \subset D \subset K$ be such that C, D are G-independent and $\vec{a} \in K^n$. Then $\vec{a} \mathop{\smile\!\!\!\!\!\!\,|}^G_C D$ if and only if $\vec{a} \mathop{\smile\!\!\!\!\!\!\,|}_{C\ G(K)} D\ G(K)$ and $\mathbf{GB}(\vec{a}C) \mathop{\smile\!\!\!\!\!\!\,|}^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(D)$.*

Corollary 3.15. *The definable subset $G(K)$ is 1-based. One can see this using the fact that forking independence restricted to G is modular (see Proposition 3.5 and Theorem 3.14). We can also see this directly, by Lemma 3.2 the definable subsets of G are definable directly from the group structure.*

Finally, we compare our characterization of independence with the one obtained by Göral in [14]. The setting used in [14] includes a larger class of examples than the ones studied in this paper. For example, some of the Mann groups studied in [14] are not linearly independent.

Corollary 3.16. *(Theorem 2.12 [14]) Let A, B, C be such that $A = \mathrm{acl}_G(A)$, $B = \mathrm{acl}_G(B)$, $C = \mathrm{acl}_G(C)$ and $A \cap B = C$. Then $A \mathop{\smile\!\!\!\!\!\!\,|}^G_C B$ if and only if $AG(K) \mathop{\smile\!\!\!\!\!\!\,|}_{C\ G(K)} BG(K)$.*

Proof. Let A, B, C be as in the statement. By Theorem 3.14, if $A \mathop{\smile\!\!\!\!\!\!\,|}^G_C B$, then $AG(K) \mathop{\smile\!\!\!\!\!\!\,|}_{C\ G(K)} BG(K)$. Assume then that $AG(K) \mathop{\smile\!\!\!\!\!\!\,|}_{C\ G(K)} BG(K)$. It remains to show:

Claim $\mathbf{GB}(A) \mathop{\smile\!\!\!\!\!\!\,|}^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(B)$.

We know from Proposition 3.5 that $\mathbf{GB}(A) \mathop{\smile\!\!\!\!\!\!\,|}^{gr}_{dcl_{gr}(\mathbf{GB}(A)) \cap dcl_{gr}(\mathbf{GB}(B))} \mathbf{GB}(B)$. Since $A = \mathrm{acl}_G(A)$, $\mathbf{GB}(A)$ is interdefinable with $G(A)$. Similarly $\mathbf{GB}(B)$ is interdefinable with $G(B)$, so $dcl_{gr}(\mathbf{GB}(A)) \cap dcl_{gr}(\mathbf{GB}(B))$ is interdefinable with $G(A) \cap G(B)$. We get then that $\mathbf{GB}(A) \mathop{\smile\!\!\!\!\!\!\,|}^{gr}_{G(A) \cap G(B)} \mathbf{GB}(B)$.

On the other hand, $A \cap B = C$, so $G(C) \subset G(A) \cap G(B)$. And if $d \in G(A) \cap G(B)$, then $d \in A \cap B = C$, so $d \in G(C)$. We get then $G(A) \cap G(B) = G(C)$. Note that as before, $\mathbf{GB}(C)$ is interdefinable with $G(C)$. Thus $\mathbf{GB}(A) \mathop{\smile\!\!\!\!\!\!\,|}^{gr}_{G(C)} \mathbf{GB}(B)$ and $\mathbf{GB}(A) \mathop{\smile\!\!\!\!\!\!\,|}^{gr}_{\mathbf{GB}(C)} \mathbf{GB}(B)$.

□

As opposed to the work of Göral, our approach puts an emphasis on G-bases and independence at the level of G, this part is absent in the description above since G is 1-based. The notion of G-bases is a natural generalization of the H-bases that appear in [6] and our description of forking can also be seen as a natural generalization of the characterizations that work for H-structures and lovely pairs.

Finally, we would like to point out that Göral also includes a description of definable groups in this expansion.

4. The case of RCF

Here K is assumed to be an RCF, and we may also assume that $H \subset K^{>0}$ (so H is dense in $K^{>0}$ only). Then also $G \subset K^{>0}$. As before, G satisfies the Mann axioms for $\Gamma = \{1\}$. As in [12], if Γ is any subgroup of $K^{>0}$ satisfying the Mann property, we can define $RCF(\Gamma)$ to be the theory axiomatized by the sentences expressing the following properties in the language $L_{G,(\gamma')_{\gamma\in\Gamma}}$:

(1) K is a real closed ordered field and G is a dense subgroup of $K^{>0}$;
(2) $\gamma \to \gamma' : \Gamma \to G$ is a group homomorphism;
(3) $(K, (\gamma')_{\gamma\in\Gamma})$ satisfies the ordering axioms of Γ, i.e. the inequalities of the form

$$k_1\gamma_1 + \ldots + k_n\gamma_n > 0$$

or

$$k_1\gamma_1 + \ldots + k_n\gamma_n \leq 0$$

that hold in K (where $k_i \in \mathbb{Z}$ and $\gamma_i \in \Gamma$);
(4) $(K, G, (\gamma')_{\gamma\in\Gamma})$ satisfies the Mann axioms of Γ.

Note that since in our case $\Gamma = \{1\}$, (2) and (3) are trivial, and (4) follows from G being linearly independent over $\mathbb{Q}$.

Fact 4.1. *([12], Theorem 7.1) Let $(K, G, (\gamma))$ and $(K', G', (\gamma))$ be two models of $RCF(\Gamma)$. Then $(K, G, (\gamma)) \equiv (K', G', (\gamma))$ if and only if $[p]G = [p]G'$ (indexes of subgroups of p-powers) for every prime number p, and for each $\gamma \in \Gamma$ and each $n > 0$,*

$$\gamma \in G^{[n]} \iff \gamma \in G'^{[n]}.$$

Thus, in our case, T^G (with the restriction of G to $K^{>0}$) is axiomatized by saying:

(1) K is a real closed field;
(2) G is a multiplicative subgroup of $K^{>0}$;
(3) G is dense in K;
(4) for any $n > 1$, $G^{[n]}$ has infinite index in G;
(5) G is linearly independent.

Note that codensity (extension property) is not an explicit part of this axiomatization. However, by Lemma 2.4, any saturated model of T^G will satisfy the codensity property.

We will now extract some more information from the proof of [12] to show that G-independent types are easy to characterize. We will need, the following fact: since G is dense in $R^{>0}$, so is $G^n = \{g^n : g \in G\}$ so G as an ordered group is *regularly dense ordered abelian group.* A discussion of these groups is carried in Section 7 of [12]. In particular, we will need:

Fact 4.2. *Let $(G, \cdot, \leq)$ be a regularly dense ordered abelian group and let $(g_1, \ldots, g_k)$, $(g'_1, \ldots, g'_k) \in G$. Write $\operatorname{tp}_{gr}(g_1, \ldots, g_k)$ for the type in the language of ordered groups. Then $\operatorname{tp}_{gr}(g_1, \ldots, g_k) = \operatorname{tp}_{gr}(g'_1, \ldots, g'_k)$ if and only if*

(1) *$g_1^{n_1} \cdots g_k^{n_k} < 1$ if and only if $g_1'^{n_1} \cdots g_k'^{n_k} < 1$ for all $n_1, \ldots, n_k \in \mathbb{Z}$.*
(2) *$g_1^{n_1} \cdots g_k^{n_k} \in G^m$ if and only if $g_1'^{n_1} \cdots g_k'^{n_k} \in G^m$ for all $n_1, \ldots, n_k \in \mathbb{Z}$, $m \geq 2$.*

Theorem 4.3. *(see [12], Theorem 7.1) Let (K, G) and (K', G') be two models of T^G. Then whenever $\vec{a} = \vec{b}\vec{g} \in K$, $\vec{a}' = \vec{b}'\vec{g}' \in G'$ are two G-independent tuples such that $\vec{g} = G(\vec{a})$, $\vec{g}' = G(\vec{a}')$, and*

1. *$(G, \vec{g}) \equiv (G', \vec{g}')$ (the types of $\vec{g}$ and $\vec{g}'$ agree in the sense of ordered groups)*
2. *$\mathrm{tp}_{RCF}(\vec{a}) = \mathrm{tp}_{RCF}(\vec{a}')$ (their types agree in the sense of real closed fields)*

Then $\mathrm{tp}_G(\vec{a}) = \mathrm{tp}_G(\vec{a}')$.

Proof. We may assume that both structures are $\aleph_1$-saturated. We will build a back and forth system between the two structures, i.e. we will prove that if $c \in K$ then there is $c' \in K'$ and $\vec{d} \in G$, $\vec{d'} \in G'$, such that $\vec{a}\vec{d}c$ and $\vec{a}'\vec{d}c'$ are again G-independent, $c \in G$ if and only if $c' \in G'$, and (1,2) above hold for these new tuples.

Case 1: $c \notin \mathrm{dcl}_{RCF}(\vec{a}G)$. By the extension (codensity) property, we can find $c' \in K'$ such that $c' \notin \mathrm{dcl}_{RCF}(\vec{a}'G'$ and $\mathrm{tp}_{RCF}(c/\vec{a}) = \mathrm{tp}_{RCF}(c'/\vec{a}')$ (i.e. c and c' realize the corresponding cuts over $\mathrm{dcl}_{RCF}(\vec{a})$ and $\mathrm{dcl}_{RCF}(\vec{a}')$, respectively). Then clearly both $\vec{a}c$ and $\vec{a}'c'$ are G-independent, and (1,2) hold for the new tuples. In this case, $\vec{d}$ and $\vec{d'}$ are both empty tuples.

Case 2: $c \in \mathrm{dcl}_{RCF}(\vec{a})$. Take $c' \in K'$ such that $\mathrm{tp}_{RCF}(c', \vec{a}') = \mathrm{tp}_{RCF}(c, \vec{a})$. If $c \in G$, then $c \in \mathrm{dcl}_{RCF}(\vec{g})$, hence also $c \in \mathrm{dcl}_{gr}(\vec{g})$ (in the sense of the group G). Let $\psi(x, \vec{y})$ be a formula in the language of ordered groups witnessing it (essentially, it says that x is the (positive) nth root of a product of elements of $\vec{y}$ and/or their inverses). The same formula, viewed as a formula in the language of rings, is also witnessing $x \in \mathrm{dcl}_{RCF}(\vec{y})$. Then clearly, $c' \in G'$, since c' is the unique element of K' satisfying $\psi(x, \vec{g}')$ in K', and $\psi(x, \vec{g}')$ does have a solution in G'. Similarly, if $c' \in G'$, then $c \in G$. Clearly, both $\vec{a}c$ and $\vec{a}'c'$ are G-independent, and (1,2) hold for the new tuples. In this case again, $\vec{d}$ and $\vec{d'}$ are both empty tuples.

Case 3: $c \in G \setminus \mathrm{dcl}_{RCF}(\vec{a})$.

We need to find $c' \in G'$ such that $\mathrm{tp}_{RCF}(c'/\vec{a}') = \mathrm{tp}_{RCF}(c/\vec{a})$ and $\mathrm{tp}_{gr}(c'/\vec{g}') = \mathrm{tp}_{gr}(c/\vec{g})$.

First, by $\aleph_1$-saturation of G', we can find $c'' \in G'$ be such that $\mathrm{tp}_{gr}(c''/\vec{g}') = \mathrm{tp}_{gr}(c/\vec{g})$. Next, as in [12] we need to refine the way we choose g_1' in order to make it compatible with the field structure.

Let $h : \mathrm{dcl}_{RCF}(\vec{a}) \to \mathrm{dcl}_{RCF}(\vec{a}')$ be a partial ordered field isomorphism specified by $h(\vec{a}) = \vec{a}'$.

Let

$$p(x, \mathrm{dcl}_{RCF}(\vec{a}')) = h(\mathrm{tp}_{RCF}(c/\,\mathrm{dcl}_{RCF}(\vec{a}))).$$

Then, by o-minimality, $p(x, \mathrm{dcl}_{RCF}(\vec{a}'))$ is determined by a cut together with the fact that x is transcendental over $\vec{a}'$. By saturation we can find an interval (q, r) inside this cut and by using the density of G' we may assume the interval has endpoints in G'.

Recall that both G and G' are regularly dense. Furthermore, since the structures are saturated, the set of divisible elements in G is also dense in K and the set of divisible elements in G' is also dense in K'. By choosing $t \in G'$ divisible such that $t \in (q/c'', r/c'')$ then $tc'' \in (q, r)$, and therefore, $\mathrm{tp}_{RCF}(c, \vec{a}) = \mathrm{tp}_{RCF}(tc'', \vec{a}'))$

Claim $\mathrm{tp}_{gr}(c'', \vec{g}') = \mathrm{tp}_{gr}(tc'', \vec{g}')$ in the sense of regularly dense ordered abelian groups.

Here we use Fact 4.2. We know that for any element $z \in \mathrm{dcl}_{gr}(\vec{g}')$, zc'' has an n-th root in G' if and only if $zc''t$ has an n-th root in G' and that the cut determined by $\text{dcl}_{gr}(\vec{g}')$ in the sense of ordered groups is determined by the cut defined by $\text{dcl}_{RCF}(\vec{a}')$ in the sense of ordered fields. The claim follows easily from these observations.

Since, $\text{tp}_{gr}(c, \vec{g}) = \text{tp}_{gr}(c'', \vec{g}')$, we get $\text{tp}_{gr}(c, \vec{g}) = \text{tp}_{gr}(tc'', \vec{g}'))$.

Thus, we can take $c' = c''t$. The new tuples are still G-independent (since we are adding elements of $G(K)$ or $G(K')$) and $(1, 2)$ hold as well. Here again $\vec{d}$ and $\vec{d}'$ are empty.

Case 4: $c \in \text{dcl}_{RCF}(\vec{a}G)$. Let $\vec{d} \in G$ be such that $c \in \text{dcl}_{RCF}(\vec{a}\vec{d}$. We may assume that $\vec{d}$ is a minimal such tuple. By iterating case 3, we can find $\vec{d}' \in G'$ such that $\text{tp}_{RCF}(\vec{a}\vec{d}) = \text{tp}_{RCF}(\vec{a}'\vec{d}')$ and $\text{tp}_{gr}(\vec{g}\vec{d}) = \text{tp}_{gr}(\vec{g}'\vec{d}')$. Now, apply case 2 to find the required c'.

□

As in the previous section, we derive near model completeness of T^G from the above proposition.

Corollary 4.4. *The theory T^G is near model complete. That is, if $(R, G) \models T^G$, every $\mathcal{L}_G$-formula is equivalent to a boolean combinations of G-existential formulas.*

Proof. Given a sufficiently saturated model (R, G) of T^G and two tuples $\vec{a}_1$ and $\vec{a}_2$ in R satisfying the same G-existential formulas, we can extend both tuples to $\vec{a}_1'$ and $\vec{a}_2'$ which are G-independent and still satisfy the same G-existential formulas. By Fact 4.2, equality of group types of $G(\vec{a}_1')$ and $G(\vec{a}_2')$ follows from equality of their G-existential types. The rest follows by Proposition 4.3.

□

Lemma 4.5. *Let (K, G) and (K', G') be sufficiently saturated models of T^G and let $g_1, \ldots, g_n \in G$, $g_1', \ldots, g_n' \in G'$ be such that:*

1. *$g_1^{n_1} \cdot \ldots \cdot g_k^{n_k} \in G^m$ if and only if $g_1'^{n_1} \cdot \ldots \cdot g_k'^{n_k} \in G^m$ for all $n_1, \ldots, n_k \in \mathbb{Z}$, $m \geq 2$.*
2. *For any semialgebraic set V definable over $\emptyset$, $\vec{g} \in V$ if and only if $\vec{g}' \in V$.*

Then $\text{tp}_G(\vec{g}) = \text{tp}_G(\vec{g}')$. Moreover, if $\vec{b} \in K$ is such that $\vec{g}\vec{b}$, $\vec{g}'\vec{b}$ are G-independent, $G(\vec{g}\vec{b}) = \vec{g}$, $G(\vec{g}\vec{b}) = \vec{g}'$ and for any semialgebraic set V definable over $\vec{b}$, $\vec{g} \in V$ if and only if $\vec{g}' \in V$ then $\text{tp}_G(\vec{g}/\vec{b}) = \text{tp}_G(\vec{g}'/\vec{b})$.

Proof. Note that $g_1^{n_1} \cdot \ldots \cdot g_k^{n_k} = 1$ can be expressed as $g_1, \ldots, g_n$ belonging to an algebraic set definable over $\emptyset$. The result follows from Fact 4.2 and Theorem 4.3.

□

This shows that the only new structure induced on G without parameters is given by polynomial (in fact, linear) inequalities with integer coefficients. With outside parameters, the induced new structure is given by traces of semialgebraic sets.

Finally, we prove a technical result relating L_G-definable sets in one variable to L_G-definable sets up to small sets. An analogous result appears as Proposition 3.12 in [6] and was used as an essential ingredient to understand imaginaries and to study the preservation of NTP_2 and NTP_1 [3] in dense-codense expansions of geometric structures.

Proposition 4.6. *Let (R, G) be a G-structure and let $Y \subset R$ be $\mathcal{L}_G$-definable. Then there is $X \subset M$ $\mathcal{L}$-definable such that $Y \triangle X$ is small.*

Proof. If Y is small or cosmall, the result is clear, so we may assume that both Y and $M \setminus Y$ are large. Assume that Y is definable over $\vec{a}$ and that $\vec{a}$ is G-independent. Let $b \in Y$ be such that $b \notin \operatorname{scl}(\vec{a})$ and let $c \in M \setminus Y$ be such that $c \notin \operatorname{scl}(\vec{a})$. Then $b\vec{a}$, $c\vec{a}$ are G-independent, while $\operatorname{tp}_G(b\vec{a}) \neq \operatorname{tp}_G(c\vec{a})$. Then by Lemma 4.3, $\operatorname{tp}_{RCF}(b\vec{a}) \neq \operatorname{tp}_{RCF}(c\vec{a})$, so there is X_{bc} a semialgebraic set such that $b \in X_{bc}$ and $c \notin X_{bc}$. By compactness, we may first assume that X_{bc} only depends on $\operatorname{tp}_{RCF}(b/\vec{a})$ and applying compactness again we may assume that X_{bc} does not depend on $\operatorname{tp}_{RCF}(b/\vec{a})$ and we will call it simply X. Thus for $b' \in Y$ and $c' \in M \setminus Y$ not in the small closure of $\vec{a}$, we have $b' \in X$ and $c' \in M \setminus X$. This shows that $Y \triangle X$ is small. □

Since definable subsets of R in o-minimal theories are finite unions of points and intervals, we get:

Corollary 4.7. *Let $Y \subset R$ be definable in L_G. Then there is a finite partition $-\infty = a_0 < a_1 < \cdots < a_m = \infty$ of R such that Y is either small or cosmall in (a_{i-1}, a_i) for $i = 1, \ldots, m$. Furthermore, if Y is definable from $\vec{d}$, where $\vec{d}$ is G-independent, so is the partition $-\infty = a_0 < a_1 < \cdots < a_m = \infty$.*

Since 1-dimensional cells are definably homeomorphic to an open interval, we also get:

Corollary 4.8. *Let $X \subset R^n$ be L-definable and 1-dimensional and let $Y \subset X$ be L_G-definable. Then there is a finite partition of X into cells $\{C_i : i \leq m\}$ such that each C_i is either a point or a 1-dimensional cell and $C_i \cap Y$ is small or cosmall in $C_i \cap X$ for $i = 1, \ldots, m$. Furthermore, if Y is definable from $\vec{d}$, where $\vec{d}$ is G-independent, so is the partition $\{C_i : i \leq m\}$.*

5. Adding a subgroup of $\mathbb{S}$

In this section we consider a version of our construction of a G-structure, where we add a subgroup G of the unit circle $\mathbb{S}$, generated by an algebraically independent dense set. Here $\mathbb{S}$ is viewed as a definable subgroup of the multiplicative group $(\mathbb{C}^\times, \cdot)$. The resulting theory has a lot in common with the expansion studied by Belegradek and Zil'ber in [1]. The difference in our setting is that the resulting group is of infinite rank, is torsion free and linearly independent.

The field of complex numbers is defined in the usual way in $(\mathbb{R}, +, \times, 0, 1)$. Thus, $\mathbb{C} = \mathbb{R}^2$, with operations

$$(x, y) + (x', y') = (x + x', y + y'), \quad (x, y) \cdot (x', y') = (xx' - yy', xy' + x'y).$$

For $A \subset \mathbb{C}$ let $A_{re} = \{x : (x, y) \in A \text{ for some } y\}$.

Similarly, for any real closed field K define $C(K) = K^2$ with the corresponding addition and multiplication. Define $\mathbb{S}(K) = \{(x, y) | x, y \in K, x^2 + y^2 = 1\}$ as a subgroup of the multiplicative group $C(K)^\times$. As noted in [1], for $z = (x, y)$ in $S(K)$, we have $z^2 - 2xz + 1 = 0$ and $(z - x)^2 + y^2 = 0$. This implies that when restricted to $S(K)$, algebraic closure in the sense of $C(K)$ coincides with the algebraic closure in the sense of K.

Choose an algebraically independent countable dense subset H_0 of the interval $(-1, 1)$ of $\mathbb{R}$. Then let

$$H = \{(h, \sqrt{1 - h^2}) : h \in H_0\}.$$

Note that $H \subset \mathbb{S}$ (in fact, just the upper semicircle) and $H_{re} = H_0$.

Lemma 5.1. *H is an algebraically independent subset of the field of complex number $\mathbb{C}$.*

Proof. Since for any $z = x + yi \in \mathbb{S}$ we have $z + \frac{1}{z} = 2x$ (in $\mathbb{C}$), z and $(x, 0)$ are interalgebraic in $\mathbb{C}$. Since H_{re} is algebraically independent (in $\mathbb{R}$), so is H (in $\mathbb{C}$). □

Let G be the multiplicative subgroup of $\mathbb{S}$ generated by H. Note that G is a free abelian group (hence, torsion free).

Lemma 5.2. *G is linearly independent in $\mathbb{C}$ over $\mathbb{Q}$.*

Proof. The proof is identical to that of Lemma 2.1. □

Next lemma shows that there is no new structure imposed by algebraic equations (in the sense of $\mathbb{C}$) on G.

Lemma 5.3. *For any polynomial $f(z_1, \dots, z_n)$ with integer coefficients there is a quantifier free formula $\theta_f(\vec{z})$ in the multiplicative group language such that for any $\vec{g} \in G$ we have $f(\vec{g}) = 0$ if and only if $(G, \cdot) \models \theta_f(\vec{g})$.*

Proof. Let $f(\vec{z}) = \Sigma_{i=1}^{N} k_i z_1^{r_{1,i}} \dots z_n^{r_{n,i}}$, where all $k_i \neq 0$ and all the monomials are distinct. Since G is linearly independent, for any $\vec{g} \in G$, if $f(\vec{g}) = 0$, then there is a partition of the set $\{1, 2, \dots, n\}$ into disjoint subsets $I_1, \dots, I_m$ such that for any $l \leq m$ we have

$$\Sigma_{i \in I_l} k_i = 0$$

and

$$g_1^{r_{1,i}} \dots g_n^{r_{n,i}} = g_1^{r_{1,j}} \dots g_n^{r_{n,j}}$$

for all $i, j \in I_l$. Taking disjunction over all such partitions we get the desired quantifier free formula.

□

Note that the formula θ_f does not need parameters from G.

Lemma 5.4.

(1) *For any n, $G^{[n]}$ is dense in $\mathbb{S}$ in the following sense: for any $-1 < a < b < 0$ or $0 < a < b < 1$ there exist $(g_1, g_2), (g_1', g_2') \in G^{[n]}$ such that $a < g_1, g_1' < b$, g_2 is between $\sqrt{1 - a^2}$ and $\sqrt{1 - b^2}$, and g_2' is between $-\sqrt{1 - a^2}$ and $-\sqrt{1 - b^2}$.*

(2) *For any $(g_1, g_2) \in G$, the coset $(g_1, g_2)G^{[n]}$ is dense in $\mathbb{S}$ and $\left((g_1, g_2)G^{[n]}\right)_{re}$ is dense in $[-1, 1]$.*

Proof. (1) First, note that since $G^{[n]}$ is closed under complex conjugation (in $\mathbb{S}$, $\frac{1}{z} = \bar{z}$), it suffices to show only the existence of (g_1, g_2). We can write $(a, \sqrt{1 - a^2}) = e^{i\theta_1}$ and $(b, \sqrt{1 - a^2}) = e^{i\theta_2}$ where $0 < \theta_i < \pi$. Choose $h = \cos\theta_0 \in H_0$ such that $\frac{\theta_1}{n} < \theta_0 < \frac{\theta_2}{n}$ (it exists by the density of H_0 in $(-1, 1)$). Then $(g_1, g_2) = e^{in\theta_0}$ is the required element of $G^{[n]}$.

(2) By (1), $\{\theta | e^{i\theta} \in G^{[n]}\}$ is dense in $[0, 2\pi]$. Clearly, for any $\theta' \in [0, 2\pi]$, so is $\{\theta + \theta' | e^{i\theta} \in G^{[n]}, \theta + \theta' < 2\pi\} \cup \{\theta + \theta' - 2\pi | e^{i\theta} \in G^{[n]}, \theta + \theta' \geq 2\pi\}$. It follows that for any $e^{i\theta} \in G$ the coset $e^{i\theta}G^{[n]}$ is dense in $\mathbb{S}$. The rest is clear.

□

Note also that the analogue of Lemma 2.3 holds in present setting as well. Moreover, we also have:

Lemma 5.5. *Let $(K, +, \cdot, 0, 1, G)$ be a sufficiently saturated model of the theory $T^G = Th(\mathbb{R}, +, \cdot, 0, 1, G)$. Then*

(1) *$G(K)$ is linearly independent*
(2) *if $a, g_1, \ldots, g_k \in G(K)$ and $a \in \mathrm{acl}(g_1, \ldots, g_k)$ (in the sense of K or $C(K)$), then for some $r \geq 1$, and $s_1, \ldots, s_k \in \mathbb{Z}$ we have $a^r = g_1^{s_1} \ldots g_k^{s_k}$.*

Proof. The proof of (1) and (2) is identical to that of Lemma 2.4. □

If we want to emphasize that G is the subgroup of $\mathbb{C}$ generated by H as defined above, we will use the notation $G = G(\mathbb{R})$.

We will now introduce a variant of the axioms (0-7) from Proposition 3.1 of [1], formulated for a structure K in the language of ordered rings, with a binary predicate G.

(1) K is a real closed field;
(2) $G(K)$ is a subgroup of $\mathbb{S}(K)$;
(3) $G(K)$ is elementarily equivalent to $G(\mathbb{R})$ (as an abelian group);
(4) For any $g \in G(K)$ and any $n > 1$, the coset $gG(K)^{[n]}$ is dense in $\mathbb{S}(K)$;
(5) For any $g \in G(K)$ and any $n > 1$, the set $\left(gG(K)^{[n]}\right)_{re}$ is dense in $[-1, 1]$;
(6) For any finite set of polynomials over $\mathbb{Z}$

$$\{f_i(x, y_1, y_1', \ldots, y_n, y_n', z_1, \ldots, z_m) | i \leq k\}$$

of positive degree, and for any $\vec{c} \in K^m$, the intersection $F_1 \cap \ldots \cap F_k$ is dense in K, where

$$F_i = \{a \in K | f_i(a, \vec{b}, \vec{c}) \neq 0 \text{ for all } \vec{b} \in G(K)^n, \text{ such that for some } d \in K \; f_i(d, \vec{b}, \vec{c}) \neq 0\}.$$

(7) For every polynomial $f(\vec{z})$ over $\mathbb{Z}$, for any tuple $\vec{z}$ in $G(K)$ $f(\vec{z}) = 0$ holds in $C(K)$ exactly when $\theta_f(\vec{z})$ holds in the group $G(K)$.

Lemma 5.6. *The axioms (1-7) hold in* $(\mathbb{R}, +, \cdot, 0, 1, G)$.

Proof. Axioms (1-3) hold trivially.

Axioms (4,5) follow by Lemma 5.5. Axiom (6) holds since any interval in $\mathbb{R}$ is uncountable. Axiom (7) holds by Lemma 5.3. □

Note that (7) implies that $G(K)$ is linearly independent over $\mathbb{Q}$. Note also that (4) and (5) can easily be shown to be equivalent for any subgroup of $\mathbb{S}$. We include both of them for convenience.

Note that G_{re} is a subset of the interval $(-1, 1]$. Since G is closed under complex conjugates (complex reciprocals), for each $a \in G_{re}$ both $a^+ = (a, \sqrt{1 - a^2})$ and $a^- = (a, -\sqrt{1 - a^2})$ belong to G. For a tuple $\vec{e} = (e_1, \ldots, e_n)$ of elements of G_{re} let $\vec{e}^+ = (e_1^+, \ldots, e_n^+)$ be the corresponding tuple of elements of G. We will also use the notation a_{re} for the first coordinate of any $a \in \mathbb{S}(K)$. Given a structure (K, G) as above, for any set $A \subset K$, we say that A is *G_{re}-independent* if A is independent from G_{re} over $A \cap G_{re}$. Clearly, $\mathrm{dcl}(G_{re}) = \mathrm{dcl}(G)$ in K (in the ring language).

The next result is a modification of Theorem 4.3 for the new setting.

Theorem 5.7. *Let (K, G) and (K', G) be two models of axioms (1-6). Then whenever $\vec{a} = \vec{b}\vec{g} \in K$, $\vec{a}' = \vec{b}'\vec{g}' \in K'$ are two G_{re}-independent tuples such that $\vec{g} = G_{re}(\vec{a})$, $\vec{g}' = G_{re}(\vec{a}')$, and (writing G for $G(K)$ and G' for $G'(K)$)*

(1) *$(G, \vec{g}^+) \equiv (G', \vec{g}'^+)$ (the types of $\vec{g}^+$ and $\vec{g}'^+$ agree in the sense of multiplicative groups)*

(2) $\text{tp}_{RCF}(\vec{a}) = \text{tp}_{RCF}(\vec{a}')$ *(their types agree in the sense of real closed fields)*
Then $\text{tp}_G(\vec{a}) = \text{tp}_G(\vec{a}')$.

Proof. We may assume that both structures are $\aleph_1$-saturated. We will build a back and forth system between the two structures, i.e. we will prove that if $c \in K$ then there is $c' \in K'$ and $\vec{d} \in G_{re}$, $\vec{d'} \in G'_{re}$, such that $\vec{a}\vec{d}c$ and $\vec{a}'\vec{d}c'$ are again G_{re}-independent, $c \in G_{re}$ if and only if $c' \in G'_{re}$, and properties (1) and (2) above hold for these new tuples.

Case 1: $c \notin \text{dcl}_{RCF}(\vec{a}G)$. By axiom (6) and saturation, we can find $c' \in K'$ such that $c' \notin \text{dcl}_{RCF}(\vec{a}'G')$ and $\text{tp}_{RCF}(c/\vec{a}) = \text{tp}_{RCF}(c'/\vec{a}')$ (i.e. c and c' realize the corresponding cuts over $\text{dcl}_{RCF}(\vec{a})$ and $\text{dcl}_{RCF}(\vec{a}')$, respectively). Then clearly both $\vec{a}c$ and $\vec{a}'c'$ are G-independent, and properties (1) and (2) hold for the new tuples. In this case, $\vec{d}$ and $\vec{d'}$ are both empty tuples.

Case 2: $c \in \text{dcl}_{RCF}(\vec{a})$. Take $c' \in K'$ such that $\text{tp}_{RCF}(c', \vec{a}') = \text{tp}_{RCF}(c, \vec{a})$. If $c \in G_{re}$, then c^+ belongs to $G \cap \text{dcl}_{RCF}(\vec{g})$. By Lemma 5.5 part (2), there is a formula $\psi(x, \vec{y})$ in the language of groups witnessing $c^+ \in \text{dcl}_{gr}(\vec{g}^+)$ which says that x is the (positive) nth root of a product of elements of $\vec{y}$ and/or their inverses. The same formula, viewed as a formula in the language of rings, is also witnessing $(x, \sqrt{1-x^2}) \in \text{dcl}_{RCF}(\vec{y})$. Then clearly, $c'^+ \in G'$, since c'^+ is the unique element of K' satisfying $\psi(x, \vec{g}'^+)$ in K', and $\psi(x, \vec{g}'^+)$ does have a solution in G'. Then $c' \in G'_{re}$. Similarly, if $c' \in G'_{re}$, then $c \in G_{re}$. Clearly, both $\vec{a}c$ and $\vec{a}'c'$ are G_{re}-independent, and (1,2) hold for the new tuples. In this case again, $\vec{d}$ and $\vec{d'}$ are both empty tuples.

Case 3: $c \in G_{re} \setminus \text{dcl}_{RCF}(\vec{a})$.

We need to find $c' \in G'_{re}$ such that $\text{tp}_{RCF}(c'/\vec{a}') = \text{tp}_{RCF}(c/\vec{a})$ and $\text{tp}_{gr}(c'^+/\vec{g}'^+) = \text{tp}_{gr}(c^+/\vec{g}^+)$.

First, by $\aleph_1$-saturation of G', we can find $d \in G'$ be such that $\text{tp}_{gr}(d/\vec{g}'^+) = \text{tp}_{gr}(c^+/\vec{g}^+)$. Next, as in [12] we need to refine the way we choose d in order to make it compatible with the field structure.

Let $h : \text{dcl}_{RCF}(\vec{a}) \to \text{dcl}_{RCF}(\vec{a}')$ be a partial ordered field isomorphism specified by $h(\vec{a}) = \vec{a}'$.

Let

$$p(x, \text{dcl}_{RCF}(\vec{a}')) = h(\text{tp}_{RCF}(c/\,\text{dcl}_{RCF}(\vec{a}))).$$

Then, by o-minimality, $p(x, \text{dcl}_{RCF}(\vec{a}'))$ is determined by a cut (inside $[-1, 1]$) together with the fact that x is transcendental over $\vec{a}'$. By saturation we can find an interval (q, r) inside this cut and by using the density of G'_{re} in $[-1, 1]$ (special case of axiom (5)) we may assume $q, r \in G'_{re}$.

Note that by saturation and axiom (5), for any $h \in G'$, the set $\left(h(G')^{div}\right)_{re}$ is dense in $[-1, 1]$. Thus, we can find $t \in G'$ divisible, such that $(td)_{re} \in (q, r)$ and td belongs to the upper semicircle.

Thus, $\text{tp}_{RCF}(c, \vec{a}) = \text{tp}_{RCF}((td)_{re}, \vec{a}'))$

Claim $\text{tp}_{gr}(d, \vec{g}'^+) = \text{tp}_{gr}(td, \vec{g}'^+)$ in the sense of free abelian groups.

We know that for any element $z \in \text{dcl}_{gr}(\vec{g}'^+)$, zd has an n-th root in G' if and only if zdt has an n-th root in G' . The claim follows easily from this observation.

Since, $\text{tp}_{gr}(c^+, \vec{g}^+) = \text{tp}_{gr}(d, \vec{g}'^+)$, we get $\text{tp}_{gr}(c^+, \vec{g}^+) = \text{tp}_{gr}(td, \vec{g}'^+))$.

Thus, we can take $c' = (td)_{re}$. The new tuples are still G-independent (since we are adding elements of $G(K)$ or $G(K')$) and properties (1) and (2) hold as well. Here again $\vec{d}$ and $\vec{d'}$ are empty.

Case 4: $c \in \mathrm{dcl}_{RCF}(\vec{a}G_{re})$. Let $\vec{d} \in G_{re}$ be such that $c \in \mathrm{dcl}_{RCF}(\vec{a}\vec{d})$. We may assume that $\vec{d}$ is a minimal such tuple. By iterating case 3, we can find $\vec{d}' \in G'$ such that $\mathrm{tp}_{RCF}(\vec{a}\vec{d}) = \mathrm{tp}_{RCF}(\vec{a}'\vec{d}')$ and $\mathrm{tp}_{gr}(\vec{g}^+\vec{d}^+) = \mathrm{tp}_{gr}(\vec{g}'^+\vec{d}'^+)$. Now, apply case 2 to find the required c'. □

Corollary 5.8. *The theory $T^{GS} = Th(\mathbb{R}, +, \cdot, 0, 1, G)$ is axiomatized by (1-6).*

Remark 5.9. *Using the approach of Gunaydin from* [15]*, Section 8.2 (with the corresponding notion of density for oriented abelian groups and the fact that in our setting $\Gamma = \{1\}$) one can give an alternative axiomatization of T^{GC}:*
(1) K is a real closed ordered field;
(2) G is a dense subgroup of $\mathbb{S}(K)$;
(3) G is linearly independent in $C(K)$;
(4) G is torsion free;
(5) for any $n > 1$, $G^{[n]}$ has infinite index in G.

Corollary 5.10. *The theory T^{GS} is near model complete in the following sense: if $(K, G) \models T^{GS}$, every $\mathcal{L}_G$-formula is equivalent to a boolean combinations of G_{re}-existential formulas over the language $\mathcal{L}$ expanded with the unary operation symbol for the positive square root function $f(x) = \sqrt{\max(x, 0)}$.*

Proof. Given a sufficiently saturated model (K, G) of T^{GS} and two tuples $\vec{a}_1$ and $\vec{a}_2$ in R satisfying the same G_{re}-existential formulas, we can extend both tuples to $\vec{a}_1'$ and $\vec{a}_2'$ which are G_{re}-independent and still satisfy the same G_{re}-existential formulas. Since $G(R)$ is a model of the theory of free abelian groups, equality of group types of $G_{re}(\vec{a}_1')^+$ and $G_{re}(\vec{a}_2')^+$ follows from equality of their G-existential types. Since for any $g \in G_{re}$ we have $g^+ = (g, \sqrt{1 - g^2})$, any quantifier free formula in the group language involving $\vec{g}^+$ (where $\vec{g}$ is a tuple in G_{re}) can be replaced with a G_{re}-existential formula over the language $\mathcal{L}$ expanded with $f(x) = \sqrt{\max(x, 0)}$. The rest follows by Proposition 5.7.

□

Finally we obtain a description of definable sets up to small sets that will be useful to describe definable groups.

Proposition 5.11. *Let (K, G) be a models of axioms (1-6). $Y \subset K$ be $\mathcal{L}_G$-definable. Then there is $X \subset K$ $\mathcal{L}$-definable such that $Y \triangle X$ is small.*

Proof. If Y is small or cosmall, the result is clear, so we may assume that both Y and $K \setminus Y$ are large. Assume that Y is definable over $\vec{a}$ and that $\vec{a}$ is G-independent. Let $b \in Y$ be such that $b \notin scl(\vec{a})$ and let $c \in K \setminus Y$ be such that $c \notin scl(\vec{a})$. Then $b\vec{a}$, $c\vec{a}$ are G-independent, while $\mathrm{tp}_G(b\vec{a}) \neq \mathrm{tp}_G(c\vec{a})$. Then by Theorem 5.7, $\mathrm{tp}_{RCF}(b\vec{a}) \neq \mathrm{tp}_{RCF}(c\vec{a})$, so there is X_{bc} a semialgebraic set such that $b \in X_{bc}$ and $c \notin X_{bc}$. Now proceed by compactness just as in Proposition 4.6 to obtain the desired X. □

Since definable subsets of R in o-minimal theories are finite unions of points and intervals, we get:

Corollary 5.12. *Let $Y \subset K$ be definable in L_G. Then there is a finite partition $-\infty = a_0 < a_1 < \cdots < a_m = \infty$ of R such that Y is either small or cosmall in (a_{i-1}, a_i) for $i = 1, \ldots, m$. Furthermore, if Y is definable from $\vec{d}$, where $\vec{d}$ is G-independent, so is the partition $-\infty = a_0 < a_1 < \cdots < a_m = \infty$.*

Since 1-dimensional cells are definably homeomorphic to an open interval, we also get:

Corollary 5.13. *Let $X \subset K^n$ be L-definable and 1-dimensional and let $Y \subset X$ be L_G-definable. Then there is a finite partition of X into cells $\{C_i : i \leq m\}$ such that each C_i is either a point or a 1-dimensional cell and $C_i \cap Y$ is small or cosmall in $C_i \cap X$ for $i = 1, \ldots, m$. Furthermore, if Y is definable over $\vec{d}$, where $\vec{d}$ is G-independent, so is the partition $\{C_i : i \leq m\}$.*

6. NIP and weak 1-basedness of G

In this section we will study the expansions of $\mathbb{R}$ that were introduced in sections 4 and 5; in particular we will show that in both cases, $Th(\mathbb{R}, G)$ has NIP. We will also show, under both frameworks, that G with the induced structure is geometric and weakly 1-based (linear in the sense of geometric structures).

6.1. **NIP.** There are several results about the preservation of NIP in expansions of geometric theories, among them in lovely pairs, H-structures [6, 2, 7] and tame expansions of real closed fields [16, 7]. More importantly to us, Günaydin and Hieronymi showed in [16] that NIP holds in expansions of real closed fields with a dense group with the Mann property whose n-th powers have finite index in the group [16]. All these results fall into the more general framework of Chernikov and Simon on adding predicates to an NIP theory [8].

The proof presented in [16] uses the fact that every definable subset of G^n is a boolean combination of the trace of a semialgebraic set in G^n with cosets of subgroups of G^n. We will use this same idea in our setting. On the other hand, the subgroups considered in [16] had finite index in G and once G is added to the language, the cosets of those groups are definable over $\emptyset$. This last part does not hold in our setting, for example, for the expansion considered in section 4, the subgroups $G^{[m]} = \{g \in G : \exists h \in G\ g = h^m\}$ have infinite index in G for $m > 1$, they have unboundedly many cosets in G and those cosets in general are not definable over $\emptyset$. Instead we will combine the description of definable sets from [16] and then use the criterion from [8].

We will start by considering the expansion $Th(K, G)$, where G is a dense subgroup of $K^{>0}$ as introduced in section 4, later we will describe the changes we need to apply to our arguments to include the case of a subgroup of $\mathbb{S}$ studied in section 5. We start with some definitions.

Definition 6.1. For $m \geq 1$, let $G^{[m]} = \{g \in G : \exists h \in G \quad g = h^m\}$.

Let $\vec{k} = (k_1, \ldots, k_n) \in \mathbb{Z}^n$ and for each $m \geq 1$ define $G_{m,\vec{k}} := \{(g_1, ..., g_n) \in G^n : g_1^{k_1} \cdot \ldots \cdot g_n^{k_n} \in G^{[m]}\}$ and $G_{\vec{k}} := \{(g_1, ..., g_n) \in G^n : g_1^{k_1} \cdot \ldots \cdot g_n^{k_n} = 1\}$

For each $n \geq 1$, let $\mathcal{D}(n)$ be the collection of finite intersections of the groups $G_{m,\vec{k}}$ and $G_{\vec{k}}$ where $\vec{k} \in \mathbb{Z}^n$.

Theorem 6.2. *$Th(K, G)$ has NIP.*

Proof. We apply the criterion from [8, Thm 2.4].

We begin by showing that every $\mathcal{L}_G$-formula $\phi(\vec{x}, \vec{y})$ has NIP over $G(\vec{x})$. Assume otherwise, so there is an $\mathcal{L}_G$-formula $\phi(\vec{x}, \vec{y})$, $I = (\vec{b}_i : i \in \omega)$ an indiscernible sequence of elements in $G(K)$ and a G-independent tuple $\vec{a} \in K$ such that $\phi(\vec{b}_i, \vec{a})$ holds iff i is even. Then by Lemma 4.5 we have that there is a finite collection of

$\mathcal{L}$-formulas $\psi_i(\vec{x},\vec{z})$ and $\mathcal{L}_{gr}$-formulas $\theta_i(\vec{x},\vec{w})$ and a pair of tuples $\vec{d},\vec{c}$ such that $\phi(\vec{x},\vec{a}) \wedge G(\vec{x})$ holds if and only if $[\vee_i \psi_i(\vec{x},\vec{d}) \wedge \theta_i(\vec{x},\vec{c})] \wedge G(\vec{x})$ holds. Then one of the conjuctions $\psi_i(\vec{x},\vec{d}) \wedge \theta_i(\vec{x},\vec{c})$ has the IP. To simplify the notiation we will remove the subindex i from the formulas.

Thus either the $\mathcal{L}$-formula $\psi(\vec{x},\vec{y})$ has the IP or $\theta(\vec{x},\vec{w})$ has IP. Since $\psi(\vec{x},\vec{y})$ defines a semialgebraic set, it is NIP. On the other hand, consider the subgroup $H \in D(n)$ which is definable over $\emptyset$, where $n = |\vec{x}|$. If all elements in $I = (\vec{b}_i : i \in \omega)$ are in the same coset of H, then any formula defining a boolean combination of cosets of H that holds for $\vec{b}_0$ holds for the whole sequence. On the other hand, assume the elements in $I = (\vec{b}_i : i \in \omega)$ are in different cosets of H. Then any formula defining a single coset of H holds for at most one element in the sequence and a formula defining the complement of a single coset of H holds for cofinitely many elements in the sequence. Thus the formula $\theta(\vec{x},\vec{c})$ does not have the IP.

Finally, by Theorem 4.3 every formula in (K,G) is equivalent to a boolean combination of existential formulas over G. This fact together with Theorem 2.4 [8] shows that $Th(M,G)$ also has NIP. □

Proposition 6.3. *The theory $Th(K,G)$ is not strongly dependent.*

Proof. We will use cosets of the subgroups $G^{[p]}$, where p is a prime, to witness the failure of strong dependence. Let $\{h_{i,j} : i,j \in \omega\}$ be different elements in the set of generators of G. Consider the formula $\varphi_i(x,h_{i,j}) = x \in h_{i,j}G^{[p_i]}$ where p_i is the $i-th$ prime and construct the array $\{\varphi_i(x,h_{i,j}) : i,j \in \omega\}$.

Note that whenever j,k are different, $h_{i,j}/h_{i,k} \notin G^{[p_i]}$, so the cosets $h_{i,j}G^{[p]}$, $h_{i,k}G^{[p]}$ are disjoint and thus the formulas $\{\varphi_i(x,b_{i,j}) : j < \omega\}$ are 2-inconsistent. This proves that the formulas in each row of the array are 2-inconsistent.

On the other hand, fix different primes $p_1,\ldots,p_s$ and consider the intersection $C = h_{1,1}G^{[p_1]} \cap h_{2,1}G^{[p_2]} \cap \cdots \cap h_{s,1}G^{[p_s]}$. We want to show that the intersection is non-empty. Assume that $g = h_{1,1}^{x_1}h_{2,1}^{x_2}\ldots h_{s,1}^{x_s}$. Then if $g \in C$, we must have that $x_1 \equiv 1 \mod p_1$, $x_1 \equiv 0 \mod p_2,\ldots,x_1 \equiv 0 \mod p_s$, similarly, $x_2 \equiv 0 \mod p_1$, $x_2 \equiv 1 \mod p_2$,…,$x_2 \equiv 0 \mod p_s$, etc. Now apply the Chinese remainder theorem to get the desired exponents $x_1,\ldots,x_s$. This shows that C is non-empty (in fact it has infinitely many solutions). This proves that the paths down on the array are consistent.

□

Theorem 6.4. *Assume $G \leq \mathbb{S}$ is a dense subgroup of $\mathbb{S}$ as studied in section 5. Then $Th(K,G)$ has NIP.*

Proof. By Theorem 5.7 the description of definable sets is the same as in the previous case, so the same proof will work. □

It is shown in [5] that when G is *divisible*, then G with the induced structure from R is weakly 1-based. We will now show that the same result holds when G is the group generated by a collection of dense-codense algebraically independent elements.

Let (R_0,G_0) be a model of T^G and let $(R,G) \succeq (R_0,G_0)$ be $|R_0|^+$-saturated.

Let T^* be the theory of the ordered multiplicative group G expanded with the traces of semialgebraic sets over R_0. We will denote the expanded structure by G^*. Let acl^* denote the algebraic closure operator in G^*. Note that T^* is a reduct of the expansion of T^G with constants from R_0.

Proposition 6.5. *The theory T^* of the group G with the induced structure from R_0 is geometric. Moreover,* $\mathrm{acl}^*(-) = \mathrm{dcl}_{gr}(- \cup G_0)$.

Proof. First, we will show that T^* eliminates $\exists^\infty$. Consider a formula $\phi(x, \vec{y})$ in the language of T^*. It suffices to show that the property "$\phi(x, \vec{a})$ has finitely many solutions" (in G) is definable in T^*. We may assume that $\phi(x, \vec{y})$ is a disjunction of the formulas of the form $G(x\vec{y}) \wedge \psi(x, \vec{y}, \vec{r}_0) \wedge \theta(x, \vec{y}, \vec{g}_0)$, where $\psi(x, \vec{y}, \vec{r}_0)$ is a formula in the language of real closed fields and $\theta(x, \vec{y}, \vec{g}_0)$ is a formula in the language of multiplicative groups defining a boolean combination of cosets of groups in $\mathcal{D}(n+1)$ (where n is the length of $\vec{y}$) and $\vec{g}_0$ are parameters from G_0. Clearly, it suffices to assume that $\phi(x, \vec{y})$ is a single such conjunction, and that $\theta(x, \vec{y}, \vec{g}_0)$ represents a single coset of a group in $\mathcal{D}(n+1)$ or its complement.Then for any $\vec{a} \in G$, $\theta(G, \vec{a}, \vec{g}_0)$ is either a coset of a group in $D(1)$ or its complement, hence either a singleton, or an empty set or an infinite dense set. Thus to say that $\phi(x, \vec{a})$ has finitely many solutions, we need to express that $\psi(R, \vec{y}, \vec{r}_0)$ has no interior (in R) or $\theta(G, \vec{y}, \vec{g}_0)$ has no more than one element. The former can be expressed in R by

$$\chi(\vec{y}, r_0) = \forall x_1 \forall x_2 (x_1 < x_2 \to \exists z (x_1 < z \wedge z < x_2 \wedge \neg\psi(z, \vec{y}, \vec{r}_0))),$$

which, when restricted to G, becomes a formula in T^*, and saying that $\theta(G, \vec{y}, \vec{g}_0)$ has no more than one element is also a formula in T^*.

Now, suppose $a, \vec{b} \in G$, and $a \in \mathrm{acl}^*(\vec{b})$. Then for some formulas $\psi(x, \vec{y}, \vec{r}_0)$ and $\theta(x, \vec{y}, \vec{g}_0)$ as above, a satisfies $\psi(x, \vec{b}, \vec{r}_0) \wedge \theta(x, \vec{b}, \vec{g}_0)$, and this formula has finitely many solutions (in G).

Case 1: The formula $\psi(x, \vec{b}, \vec{r}_0)$ defines a finite set (in R). Then $a \in \mathrm{dcl}_{RCF}(\vec{b}\vec{r}_0)$. Since R_0 is G-independent, we have $a \in \mathrm{dcl}_{RCF}(\vec{b}G_0)$, and thus, $a \in \mathrm{dcl}_{gr}(\vec{b}G_0)$ (since dcl_{RCF} restricted to G is dcl_{gr}).

Case 2. The formula $\psi(x, \vec{b}, \vec{r}_0)$ defines an infinite set so it must have interior. Then $\theta(x, \vec{b}, \vec{g}_0)$ must define a finite set and so a belongs to $\mathrm{dcl}_{gr}(\vec{b}\vec{g}_0)$.

This shows that $\mathrm{acl}^* = \mathrm{dcl}_{gr}(-\cup G_0)$. Since dcl_{gr} satisfies the exchange property, so does acl^*.

□

Proposition 6.6. *The theory T^* is weakly one-based. Moreover, the pregeometry induced by* acl^* *is modular, and its associated geometry is a projective geometry over $\mathbb{Q}$.*

Proof. To show weak one-basedness it suffices to show modularity (projectivity) of acl^*. Thus, suppose $a, b, c_1, \ldots, c_n \in G$ and $a \in \mathrm{acl}^*(b\vec{c})$. By the description of acl^*, for some integer $r, s, t_1, \ldots, t_n$ we have

$$a^r = b^t c_1^{t_1} \ldots c_n^{t_n} d,$$

where $d \in G_0$. Let $c_0 = c_1^{t_1} \ldots c_n^{t_n}$. Then clearly $c_0 \in \mathrm{acl}^*(\vec{c})$ and $a \in \mathrm{acl}^*(b, c_0)$.

Taking the divisible hull of G and viewing it as an additive group, we can also see that acl^* is induced by the linear span with the elements of G_0 added as constants. Since each element of the divisible hull is a rational multiple of an element of G, the associated geometry of acl^* is a projective geometry over $\mathbb{Q}$.

□

Remark 6.7. *Since the* acl^**-dimension in G^* is witnessed by positive Boolean combinations of cosets of definable subgroups, the structure G^* is weakly abelian (see* [5]*). This gives another proof of weak one-basedness of T^*.*

Next, we will look at the structure induced on a dense subgroup G of $\mathbb{S}$, as studied in section 5. Let (R_0, G_0) be a model of T^{GS}, (so, $G_0 \subset \mathbb{S}(R_0) \subset R_0^2$) and let $(R, G) \succeq (R_0, G_0)$ be $|R_0|^+$-saturated.

Let T^* be the theory of the multiplicative group $G \subset \mathbb{S}(R)$ expanded with the traces of semialgebraic sets over R_0. As before, we will denote the expanded structure by G^*, and acl^* will denote the algebraic closure operator in G^*. By Theorem 5.7, the type of a tuple $((a_1, a_1'), \ldots, (a_n, a_n')) \in G(R)^n$ over R_0 in the sense of the theory T^{GS} is determined by

$$\mathrm{tp}_{RCF}(a_1, a_1' \ldots, a_n, a_n'/R_0)$$

and

$$\mathrm{tp}_{gr}((a_1, |a_1'|), \ldots, (a_n, |a_n'|)/G_0).$$

Since $\mathrm{tp}_{RCF}(a_1, a_1' \ldots, a_n, a_n'/R_0)$ specifies the signs of $a_1', \ldots, a_n'$, we can drop the absolute values.

As before, we get the following result.

Proposition 6.8. *The theory T^* of the group G with the induced structure from R_0 is geometric. Moreover,* $\mathrm{acl}^*(-) = \mathrm{dcl}_{gr}(- \cup G_0)$.

Proof. First, we will show that T^* eliminates $\exists^\infty$. Consider a formula

$$\phi((x, x'), (y_1, y_1'), \ldots, (y_n, y_n'))$$

in the language of T^*.

It suffices to show that the property "$\phi((x, x'), (a_1, a_1'), \ldots, (a_n, a_n'))$ has finitely many solutions" (in G) is definable in T^*. We may assume that ϕ is a disjunction of the formulas of the form

$$G(x, x') \wedge \bigwedge G(y_i, y_i') \wedge \psi(x, x', y_1, y_1', \ldots, y_n, y_n', \vec{r}_0) \wedge \theta((x, x'), (y_1, y_1'), \ldots, (y_n, y_n'), \vec{r}_0)\vec{g}_0),$$

where ψ is a formula in the language of real closed fields and θ is a formula in the language of multiplicative groups defining a boolean combination of cosets of groups in $\mathcal{D}(n+1)$ (where n is the length of $\vec{y}$) and $\vec{g}_0$ are parameters from G_0. Clearly, it suffices to assume that $\phi(x, \vec{y})$ is a single such conjunction, and that $\theta(x, \vec{y}, \vec{g}_0)$ represents a single coset of a group in $\mathcal{D}(n+1)$ or its complement.Then for any $\vec{a} \in G$, $\theta(G, \vec{a}, \vec{g}_0)$ is either a coset of a group in $D(1)$ or its complement, hence either a singleton, or an empty set or an infinite dense subset of G. Thus, to say that $\phi(x, \vec{a})$ has finitely many solutions, we need to express that $\psi(R^2, \vec{y}, \vec{r}_0)$ intersected with $\mathbb{S}(R)$ has no interior in $\mathbb{S}(R)$, or $\theta(G, \vec{y}, \vec{g}_0)$ has no more than one element. It is clear that the last statement is first order and is expressed by a formula in T^*. The former statement can be expressed in R by saying that for any $-1 < a < b < 0$ or $0 < a < b < 1$ there exist $(s, s'), (t, t') \in \psi(R^2, \vec{y}, \vec{r}_0)$ such that $a < s, t < b$, s' is between $\sqrt{1-a^2}$ and $\sqrt{1-b^2}$, and t' is between $-\sqrt{1-a^2}$ and $-\sqrt{1-b^2}$. When restricted to G, this statement becomes a formula in T^*.

Claim $\mathrm{acl}^*(-) = \mathrm{dcl}_{gr}(- \cup G_0)$

Suppose $(a, a'), (b_1, b_1'), \ldots, (b_n, b_n') \in G$, and

$$(a, a') \in \mathrm{acl}^*((b_1, b_1'), \ldots, (b_n, b_n')).$$

Then for some formulas

$$\psi(x, x', y_1, y_1', \ldots, y_n, y_n', \vec{r}_0)$$

and

$$\theta((x, x'), (y_1, y_1'), \ldots, (y_n, y_n'), \vec{r}_0)$$

as above, (a, a') satisfies

$$\psi(x, x', b_1, b'_1, \ldots, b_n, b'_n), \vec{r}_0) \wedge \theta((x, x'), (b_1, b'_1), \ldots, (b_n, b'_n), \vec{g}_0),$$

and this formula has finitely many solutions (in G).

Case 1: The formula $\psi(x, x', b_1, b'_1, \ldots, b_n, b'_n), \vec{r}_0) \wedge x^2 + x'^2 = 1$ defines a finite set (in R^2). Then $a \in \mathrm{dcl}_{RCF}(b_1, \ldots, b_n, \vec{r}_0)$. Since R_0 is G_{re}-independent, we have $a \in \mathrm{dcl}_{RCF}(b_1, \ldots, b_n, (G_0)_{re})$. Thus, there is a non-constant polynomial $p(x, \vec{y}, \vec{t})$ with rational coefficients such that $a \in \mathrm{dcl}_{RCF}(\vec{b}, \vec{g}_0)$ is witnessed by $p(a, \vec{b}, \vec{g}_0) = 0$, for some $\vec{g}_0 \in (G_0)_{re}$. We may assume that $\vec{b}\vec{g}_0$ is an algebraically independent tuple (in the sense of RCF).

Replacing each variable v in $p(x, \vec{y}, \vec{t})$ with $\frac{1}{2}(v + \frac{1}{v})$ and passing to a common denominator, we get a complex polynomial $q(z, \vec{u}, \vec{w})$, with z appearing with positive degree, such that

$$q((a, a'), (b_1, b'_1), \ldots, (b_n, b'_n), (g_1, g'_1), \ldots, (g_k, g'_k)) = 0.$$

Since $\vec{b}\vec{g}_0$ is algebraically independent in R, so is the tuple

$$(y_1, y'_1), \ldots, (y_n, y'_n), (t_1, t'_1), \ldots, (t_k, t'_k)$$

(in the sense of the field $\mathbb{C}(R)$), and thus,

$$q((x, x'), (b_1, b'_1), \ldots, (b_n, b'_n), (g_1, g'_1), \ldots, (g_k, g'_k)) = 0$$

has finitely many solutions in $G(R)$.

By Lemma 5.3, there exists a formula

$$\theta_q((x, x'), (y_1, y'_1), \ldots, (y_n, y'_n), (t_1, t'_1), \ldots, (t_k, t'_k))$$

in the language of multiplicative groups, such that $\theta_d(G^{1+n+k})$ is the zero set of q on G^{1+n+k}. Then θ_q witnesses $(a, a') \in \mathrm{dcl}_{gr}((b_1, b'_1), \ldots, (b_n, b'_n), G_0)$.

Case 2. The formula $\psi(x, x', b_1, b'_1, \ldots, b_n, b'_n), \vec{r}_0) \wedge x^2 + x'^2 = 1$ defines an infinite set, so it must have interior (in $\mathbb{S}(R)$). Then $\theta((x, x'), (b_1, b'_1), \ldots, (b_n, b'_n), \vec{g}_0)$ must define a finite set (otherwise it is dense) and so a belongs to $\mathrm{dcl}_{gr}(\vec{b}\vec{g}_0)$.

This shows that $\mathrm{acl}^* = \mathrm{dcl}_{gr}(- \cup G_0)$. Since dcl_{gr} satisfies the exchange property, so does acl^*.

□

7. Definable groups in (R, G)

We begin this section with the study of definable groups in pairs of the form (R, P), where R is a real closed field and P is a new predicate where P stands for a small subset of R or of a power of R. We use the usual notation, $\mathcal{L}$ stands for the language of real closed fields and $\mathcal{L}_P$ for $L \cup \{P\}$. Some properties of groups definable in expansions similar to the ones we are considering were studied by Eleftheriou, Günaydin and Hieronymi in [13]. In particular, they conjectured:

Let $(F, *)$ be a L_P-definable group. Then there is a short exact sequence $0 \to B \to U \to K \to 0$ and a map $\tau : U \to H$ where

- U is $\vee$-definable.
- B is $\vee$-definable in L with $\dim(B) = ldim(F)$.

- K is definable and small.

- $\tau : U \to F$ is a surjective group homomorphism and

- all maps involved are $\vee$-definable.

While the conjecture remains open in our setting, in this section, we present some positive results when F is the subgroup of an $\mathcal{L}$-definable one dimensional group. Of course, we will need some assumptions on the expansion:

Partition Assumption. Let $Y \subset R$ be definable in $\mathcal{L}_P$. Then there is a finite partition $-\infty = a_0 < a_1 < \cdots < a_m = \infty$ of R such that Y is either small or cosmall in (a_{i-1}, a_i) for $i = 1, \ldots, m$.

Generalized Partition Assumption. Let $X \subset R^n$ be $\mathcal{L}$-definable and 1-dimensional and let $Y \subset X$ be L_G-definable. Then there is a finite partition of X into cells $\{C_i : i \leq m\}$ such that each C_i is either a point or a 1-dimensional cell and $C_i \cap Y$ is small or cosmall in $C_i \cap X$ for $i = 1, \ldots, m$.

Note that these assumptions hold when G is as in section 4 (see Corollary 4.7 and Corollary 4.8) and also as in section 5 (see Corollary 5.12 and Corollary 5.13). There are many other expansion that also satisfy this assumptions, for example dense pairs, H-stuctures, and expansions with groups satisfying the Mann propertry of finite index inside $R^{>0}$ (the work of Günaydin and van den Dries [12]) and inside $\mathbb{S}$ (the work of Belegradek and Zil'ber [1]).

We will start with torsion free groups, where the argument is more transparent:

Lemma 7.1. *Assume $F \leq (R, +)$ is $\mathcal{L}_P$-definable and the Partition Assumption holds. Then either F is small or $F = (R, +)$.*

Proof. Assume $F \leq (R, +)$ is $\mathcal{L}_P$-definable. If F is small there is nothing to prove. Assume otherwise. Then by the Partition Assumption there is a partition $-\infty = a_0 < a_1 < \cdots < a_m = \infty$ of R and an interval (a_i, a_{i+1}) such that F is cosmall in (a_i, a_{i+1}). Now let $f \in F \cap (a_i, a_{i+1})$, then $(F \cap (a_i, a_{i+1})) - f$ is cosmall in $(a_i, a_{i+1}) - f$ and contains 0. Thus, after possibly going to a different partition, we may assume that (a_i, a_{i+1}) contains 0.

Claim $F \cap (a_i, a_{i+1}) = (a_i, a_{i+1})$

Assume otherwise and let $c \in (a_i, a_{i+1}) \setminus F$. Then the coset $c + F$ is cosmall in $(c + a_i, c + a_{i+1})$. Since (a_i, a_{i+1}) is an open set containing 0, $(a_i, a_{i+1}) \cap (c + a_i, c + a_{i+1})$ is an open interval and both F and $c + F$ are cosmall in the interval, so they must intersect, a contradiction.

Thus $F \cap (a_i, a_{i+1}) = (a_i, a_{i+1})$, in particular the elements $a_i/2, a_{i+1}/2$ belong to F and thus both elements a_i, a_{i+1} belong to F.

Now consider $(a_i, a_{i+1}) + (a_i, a_{i+1}) = (2a_i, 2a_{i+1})$ which is a subset of F. Since $F \cap (a_{i+1}, a_{i+2}) \supset (a_{i+1}, 2a_{i+1}) \cap (a_{i+1}, a_{i+2})$, the set F must be cosmall in the interval (a_{i+1}, a_{i+2}). Similarly, it is easy to show that F must be cosmall in the interval and (a_{i-1}, a_i). It follows as in the claim that $F \cap (a_{i+1}, a_{i+2}) = (a_{i+1}, a_{i+2})$ and $F \cap (a_{i-1}, a_i) = (a_{i-1}, a_i)$. Proceeding inductively we get that $F = R$. $\square$

A similar argument applies to multiplicative groups:

Lemma 7.2. *Assume $F \leq (R^{>0}, \cdot)$ is $\mathcal{L}_P$-definable and assumption 1 holds. Then either F is small or $F = (R^{>0}, \cdot)$.*

Proof. We just change, in the proof above, $+$ for $\cdot$, 0 for 1, multiplication by 2 for squaring and dividing by 2 for taking a square root. □

Finally, we prove the argument even works for general 1-dimensional groups:

Lemma 7.3. *Assume that $(T, \cdot)$ is a 1-dimensional $\mathcal{L}$-definable group and let $F \leq (T, \cdot)$ be $\mathcal{L}_P$-definable. Also assume that the Generalized Partition Assumption holds. Then either F is small or F is $\mathcal{L}$-definable.*

Proof. Let $F \leq (T, \cdot)$ be $\mathcal{L}_G$-definable of dimension 1. If F is small there is nothing to prove. Assume otherwise. Since we need to see $(T, \cdot)$ as topological group, we give $(T, \cdot)$ the t-topology [17]. Then we can write $T = U_1 \cup \cdots \cup U_r$ where the sets U_i are definable and there is a definable bijection between U_i and some open subset V_i in R.

Then by the generalized partition assumption, there is a partition of U_j into cells $\{C_i^j : i \leq m_j\}$ such that each C_i^j is either a point or a 1-dimensional cell and $C_i^j \cap F$ is small or cosmall in C_i for $i = 1, \ldots, m$. Then for some pair of indexes i, j, the cell C_i^j is 1-dimensional and $C_i^j \cap F$ is cosmall.

Let $f \in C_i^j \cap F$ and apply to T the left translation by f^{-1}, which is a continuous bijection in T. Then the set $D_F = f^{-1} \cdot (C_i^j \cap F)$ is 1-dimensional, contains the identity and is cosmall in the t-open set $D = f^{-1} \cdot C_i^j$.

Claim $F \cap D = D$. Assume otherwise and let $d \in D \setminus F$. Then the coset dF is cosmall in $d \cdot D$. Since D is a 1-dimensional t-open set containing e, $d \cdot D$ is also a t-open set and they intersect in a t-open set where both F and $d \cdot F$ are cosmall, a contradiction.

Thus F contains an open set around the identity in the t-topology. Since translation is a homeomorphism, we get that F is t-open.

Assume now that that C_l^k is another 1-dimensional t-open set such that $F \cap C_l^k \neq \emptyset$. Then the intersection is t-open non-empty and thus not small. It follows as in the claim that the intersection must agree with C_l^k. We get that F is the union of the 1-dimensional t-open sets C_l^k that it intersects and maybe some extra points and so it must be $\mathcal{L}$-definable. □

Now we will study the definable subgroups of G inside pairs of the form (R, G), where R is a real closed field and G is either a dense subgroup of $R^{>0}$ with the Mann property (of infinite rank as in as in section 4 or of finite rank as studied by Günaydin and van den Dries) or G is a dense subgroup of $\mathbb{S}(R)$ with the Mann property (of infinite rank as in as in section 5 or of finite rank as studied by Belegradek and Zil'ber [1]).

Recall that the definable subsets of G^n in the pair (R, G) are finite unions of intersections of semialgebraic sets with realizations of group-formulas $\theta(\vec{x}, \vec{c})$, where $\theta(\vec{x}, \vec{c})$ defines a finite boolean combination of cosets of subgroups of the form $F \in \mathcal{D}(n)$, where $n = |\vec{x}|$. In particular, all groups $F \in \mathcal{D}(n)$ are definable. We will try to simplify the picture:

Lemma 7.4. *Whenever $(n, k) = 1$, $G_{n,k} = G^{[n]}$.*

Proof. We may work in the model (R, G) where G is generated by the independent set H. Assume first that $g \in G^{[n]}$, so there is $g_1 \in G$ such that $g = g_1^n$. Then $g^k = (g_1^k)^n$ so $g \in G_{n,k}$. On the other hand, assume that $g = h_1^{s_1} \ldots h_l^{s_l}$ where h_i stand for generators of G which are algebraically independent and we may assume that all h_i are different and thus algebraically independent. If $g \in G_{n,k}$, then $g^k \in G^{[n]}$, so $g^k = h_1^{ks_1} \dots h_l^{ks_l}$ has an n-th root. Since $(n,k) = 1$ each of $s_1, \dots, s_l$ is divisible by n and thus $g \in G^{[n]}$. □

Lemma 7.5. *Whenever $(n,k) = 1$, $G^{[n]} \cap G^{[k]} = G^{[nk]}$.*

Proof. We may work in the model (R, G) where G is generated by the independent set H. Clearly $G^{[nk]} \subset G^{[n]} \cap G^{[k]}$. Assume that $g = h_1^{s_1} \dots h_l^{s_l}$ where h_i stand for generators of G which are algebraically independent and we may assume that all h_i are different and thus algebraically independent. If $g \in G^{[n]} \cap G^{[k]}$ then each of $(s_1, \dots, s_l)$ is divisible by n and by k, so they are also divisible by nk. □

We now deal with the expansion (R, G), where $G \leq R^{>0}$ is dense and has the Mann property. This includes the case studied in 4 and in [12]

Lemma 7.6. *Let $S \leq G$ be $\mathcal{L}_G$-definable. Then $S \in \mathcal{D}(1)$.*

Proof. Let $S \leq G$ be $\mathcal{L}_G$-definable, then by Lemma 4.5 we can write $S = ((a_1, a_2) \cap B_1) \cup ((a_2, a_3) \cap B_2) \cup \cdots \cup ((a_n, a_{n+1}) \cap B_l) \cup F$ where each B_i is a boolean combination of cosets of groups in $\mathcal{D}(1)$, all $a_i > 0$, the intervals (a_i, a_{i+1}) are disjoint and F is a finite set of points.

Assume that R is the standard real field and $S \leq R^{>0}$ is a subgroup. Then either S is just the identity, infinite cyclic or dense. Assume that $S \neq \{e\}$ so it must be infinite and thus for some i, $(a_i, a_{i+1}) \cap B_i$ is infinite. Since the non-finite boolean combination of cosets of groups in $\mathcal{D}(1)$ are dense, $B_i \cap (a_i, a_{i+1})$ is dense in (a_i, a_{i+1}) and thus S must be dense in $R^{>0}$. Note that being dense is a first order property and thus this property is true in all models of the theory.

Let $c \in (a_1, a_2) \cap B_1$, then after multiplying by c^{-1} we may assume that $1 \in (a_1, a_2) \cap B_1$. Since S is dense in $R^{>0}$, then for every l, S^l is dense in $R^{>0}$ and in a saturated model S^{div}, the group consisting of the divisible elements of S, is also dense in $R^{>0}$. Since being dense is a first order property, this is also true in all models of the theory.

Claim B_1 and B_2 differ by a finite set.

Let $t \in (a_2, a_3) \cap B_2$ be divisible. First observe that $t^{-1} G^{[n]} = G^{[n]}$ and that for every coset $dG^{[n]}$, we also get $t^{-1} dG^{[n]} = dG^{[n]}$, so $t^{-1} B_2$ differs from B_2 by at most a finite set. Then $t^{-1} \cdot (a_2, a_3) \cap B_2 = (a_2/t, a_3/t) \cap t^{-1} B_2$ also contains the identity and $(a_2/t, a_3/t)$ is an open set around the identity. Let $c_1 = \max(a_1, a_2/t)$, $c_2 = \min(a_2, a_3/t)$. Then on the open set (c_1, c_2), we have $S \cap (c_1, c_2) = (c_1, c_2) \cap B_1 = (c_1, c_2) \cap t^{-1} B_2$, so B_1 and $t^{-1} B_2$ agree on an open interval and thus they differ by at most a finite set. The claim follows from this result.

Thus after modifying the finite set F, and using the density of S, we may assume all the sets B_i are the same and write $S = ([(b_1, b_2) \cap B] \setminus F_1) \cup F_2$, where F_1, F_2 are finite sets. Again using the density of S in $R^{>0}$, we get that $b_1 = 0$, $b_2 = \infty$ and thus $S = B \cup F$ which is clearly an element in $\mathcal{D}(1)$

□

Finally we see how to modify the arguments in the previous proof to deal with the expansion (R, G), where $G \leq \mathbb{S}(R)$ is dense and has the Mann property. This includes the case studied in section 5 and in [1].

Lemma 7.7. *Let $U \leq G$ be $\mathcal{L}_G$-definable. Then $U \in \mathcal{D}(1)$ or U is finite.*

Proof. Let $U \leq G$ be $\mathcal{L}_G$-definable, then by Corollary 5.13, we can write $U = (C_1 \cap B_1) \cup (C_2 \cap B_2) \cup \cdots \cup (C_n \cap B_n) \cup F$ where each B_i is a boolean combination of cosets of groups in $\mathcal{D}(1)$, the collection C_i is a family of 1-dimensional cells that are disjoint and F is a finite set of points. Assume first that R is the standard real field and $U \leq \mathbb{S}(R)$ is a subgroup. It is well known that the subgroups of $\mathbb{S}(R)$ are either finite or dense, since this is a first order property the same results holds in all models of the theory. If U if finite we get the desired result. If U is infinite, then it is dense and we proceed in a similar way as in the previous proof by showing that all B_i differ by at most a finite set and that the C_i form a partition of $\mathbb{S}(R)$ up to a finite set. □

7.1. **Dynamics.** In this section we study some dynamical properties of G inside the expansion (R, G), where G is a dense codense subgroup of $R^{>0}$ (respectively a subgroup of $\mathbb{S}(R)$) and has the Mann property. We follow the presentation from [9]. First note that in all settings under consideration G is abelian, so it is definably amenable. Our goals are to find explicit invariant measures and strong f-generic types, characterize G^{00} and the quotient G/G^{00}.

7.1.1. *Dynamics: the infinite index case.* Assume first that $G \leq R^{>0}$ and $(R, G) \models T^G$ are as in section 4. In particular G is dense, has the Mann property and the groups $G^{[n]}$ have infinite index in G.

Lemma 7.8. *Let (K, G) be an $\aleph_1$-saturated model model of T^G and consider the type $p(x) = x \in G \cup \{x > g : g \in G\} \cup \{x \notin gW : g \in G, W \in \mathcal{D}(1)\}$. Let μ_p be the measure centered in p. Then μ_p is G-invariant. Furthermore $G^{00} = G$ and $Stab(p) = G$.*

Proof. For any $a \in K$, the formula $\varphi(x) = (x > a) \wedge (x \in G)$ has μ_p-measure 1. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x > a) \wedge (g^{-1}x \in G) = (x > ag) \wedge (x \in G)$ also has μ_p-measure 1. Similarly for $a \in G$ and $W \in \mathcal{D}(1)$, the formula $\varphi(x) = (x \notin aW) \wedge (x \in G)$ has μ_p-measure 1. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x \notin aW) \wedge (g^{-1}x \in G) = (x \notin gaW) \wedge (x \in G)$ also has measure 1. By the description of definable subsets of G from Lemma 4.5 we get that μ_p is G-invariant.

Thus the group G is definably amenable. Also note that since each of the groups $G_{n,k}$ has infinite index in G, we get that $G^{00} = G$. Since for any $g \in G$, $g \cdot p = p$, the orbit of p under the action of G is a singleton, $Stab(p) = G^{00} = G$ and $p(x)$ is strongly f-generic. □

For any $n \geq 2$, the map that sends $g \in G$ to $g^n \in G^{[n]}$ is a definable group isomorphism, so the group $G^{[n]}$ is also definable amenable, ${G^{[n]}}^{00} = G^{[n]}$ and has strong f-generics. Thus the the lemma also holds for all $W \in \mathcal{D}(1)$ and by Lemma 7.7 for all definable subgroups of G.

Question 7.9. *Is the lemma above true for $W \in \mathcal{D}(1)$ just assuming that G has the Mann property and that all groups $G_{n,k}$ have infinite index in G?*

7.1.2. *Dynamics: the finite index case.* We now assume that we are in the setting studied in [12]. So G has the Mann property and that all groups $G^{[n]}$ have finite index in G. There are several examples of Mann-multiplicative subgroups with this property, for example $2^{\mathbb{Q}}3^{\mathbb{Z}}$, $2^{\mathbb{Q}}3^{\mathbb{Z}}5^{\mathbb{Z}}$, etc. In this setting, as opposed to the previous case, all $G_{n,k}$ have finite index in G and thus G^{00} is the subgroup of divisible elements.

Lemma 7.10. *Let (K, G) be an $\aleph_1$-saturated model model of T^G and consider the type $p(x) = x \in G \cup \{x > g : g \in G\} \cup \{x \in G^{[n]} : n \geq 2\}$. Let μ_p be the measure centered in p. Then μ_p is G^{00}-invariant.*

Proof. For any $a \in K$, the formula $\varphi(x) = (x > a) \wedge (x \in G)$ has μ_p-measure 1. If $g \in G$, then the translate $g\varphi(x) = (g^{-1}x > a) \wedge (g^{-1}x \in G) = (x > ag) \wedge (x \in G)$ also has μ_p-measure 1. Similarly the formula $\varphi(x) = (x \in G^{[n]})$ has μ_p-measure 1. If $g \in G^{00}$, then the translate $g\varphi(x) = (g^{-1}x \in aG^{[n]}) = (x \in gG^{[n]}) = (x \in G^{[n]})$ also has μ_p-measure 1. By the description of definable subsets of G we get that μ_p is G^{00}-invariant. □

Thus the group G is definable ameanable. Note that it follows from the previous proof for any $g \in G^{00}$, $g \cdot p = p$ and that the orbit of p under the action of G has cardinality at most $2^{\aleph_0}$, namely the orbit is determined by choosing a coset of each $G^{[n]}$ (and there are finitely many such cosets) for every n.

The quotient group G/G^{00} is a profinite group. For every positive pair of integers n, m such that n divides m, we get a natural map $f_{nm} : G/G^{[n]} \to G/G^{[m]}$ defined by $f_{nm}(aG^{[n]}) = aG^{[m]}$. The group G/G^{00} is the inverse limit of this system. Note that the quotients $G/G^{[n]}$ are finite and thus they are not ordered groups. Similarly the profinite group G/G^{00} is not ordered and thus G/G^{00}, even when it is infinite, is not a model of the theory of G.

Example 7.11. *Consider the multiplicative group $G = 2^{\mathbb{Q}}3^{\mathbb{Z}}$ that is dense in $\mathbb{R}^{>0}$ and has the Mann property (see* [12]*). Then $G/G^{[n]} = \mathbb{Z}/n\mathbb{Z}$ and G/G^{00} is the infinite compact group of profinite integers.*

7.1.3. *Dynamics: subgroups of $\mathbb{S}$ with the Mann property.* Finally we study the dynamics in two other settings. First we consider the case studied by Belegradek and Zilber in [1] which deal with dense subgroups of $\mathbb{S}$ that have the Mann property that have finite rank. Under these assumptions, all groups $G^{[n]}$ have finite index in G and G^{00} contains the intersection $\cap_n G^{[n]}$ as well as the group of infinitesimals around 1.

Recall that the shortest arc in the circle between $e^{-i\pi/4}$ and $e^{i\pi/4}$ is homeomorphic to $(-1, 1)$ through a map that sends $(1, 0)$ to the point 0. Thus we can identify the elements infinitesimally close to $(1, 0)$ with the elements infinitesimally close to 0 and give the elements infinitesimally close to $(1, 0)$ in $\mathbb{S}$ an order compatible with the group operation. We will use the order below:

Lemma 7.12. *Let (K, G) be an $\aleph_1$-saturated model of T^G and consider the Haar measure μ of G that asigns to an arc its length normalized by 2π. Then μ is G-invariant and the measure of $G^{[n]}$ is one over the index of $G^{[n]}$ inside G. Consider the type $p(x) = x \in G \cup \{x > a : a \in G, a \text{ infinitesimal }\} \cup \{x < 1/n : n \geq 1\} \cup \{x \in G^{[n]} : n \geq 2\}$. Let μ_p be the measure centered in p. Then μ_p is G^{00}-invariant.*

Proof. The first part is clear, as the length of an arc (even if intersected with G) is preserved under rotations. Also, if the index of G^n on G is k, then the cosets can be permuted by multiplying by elements in G, so they all have the same measure and the result follows.

Now we will check that μ_p is G^{00}-invariant. If $g \in G^{00}$, then g is infinitesimal and x is infinitesimal and so is gx, thus $gx < 1/n : n \geq 1$. Also if $a \in G$ is infinitesimal, and g is infinitesimal, then g^{-1} and ag^{-1} are also infinitesimal and $x > ag^{-1}$ implies $gx > a$. Finally if $x \in G^n$ and $g \in G^n$, then $xg \in G^n$, so if $g \in G^{00}$ and x is divisible, then gx is divisible. □

Example 7.13. *Consider the multiplicative group $G = \mathbb{U}$ of the roots of unity interpreted as a subset of $\mathbb{R}^2$. It is a divisible group but it has torsion points, for every $n \in \mathbb{N}$, the collection of n torsion points are precisely the n-th roots of unity. In a saturated model, the interpretation of $\mathbb{U}$ includes points without torsion that are dense inside the torsion points and $\mathbb{U}^{00}$ are those points infinitesimally close to* 1. *In this setting, $\mathbb{U}/\mathbb{U}^{00}$ is the group $\mathbb{S}$.*

This is not surprising, as a pure group, $\mathbb{U}$ is the interpretation of $\mathbb{S}$ in the field $\overline{\mathbb{Q}}$ and in a saturated model $\tilde{\mathbb{R}}$, the quotient $\mathbb{U}(\tilde{\mathbb{R}})/\mathbb{U}^{00}(\tilde{\mathbb{R}})$ correponds to the quotient of $\mathbb{S}(\tilde{\mathbb{R}})/\mathbb{S}^{00}(\tilde{\mathbb{R}})$ which is known to be $\mathbb{S}(\mathbb{R})$. One can also check directly using the work of Szmielew [20] *that as pure groups, $\mathbb{U}$ is elementary equivalent to $\mathbb{S}$.*

Now we consider the case studied in section 5 which deals with dense subgroups of $\mathbb{S}$ that have the Mann property that have infinite rank. Now all groups $G^{[n]}$ have infinite index in G and G^{00} is the group of infinitesimals around 1.

Lemma 7.14. *Let (K, G) be an $\aleph_1$-saturated model model of T^G and and consider the Haar measure μ of G that asigns to an arc its length normalized by 2π. Then μ is G-invariant and the measure of $G^{[n]}$ is zero. Consider the type $p(x) = x \in G \cup \{x > a : a \in G, a \text{ infinitesimal } x < 1/n : n \geq 1\} \cup \{x \notin aG^{[n]} : n \geq 2, a \in G\}$. Let μ_p be the measure centered in p. Then μ_p is G^{00}-invariant.*

Proof. The proof is very similar to the one of the previous lemma and we leave it to the reader. □

Question 7.15. *Assume F is small, bounded and definable in $(R, K) \models T^G$, where (R, K) is $\aleph_1$-saturated and $G \leq \mathbb{S}$ has the Mann property (regarless if it has finite or infinite rank). Does it hold that is $F \equiv F/F^{00}$?, i.e. Does a partial version of Pillay's conjecture hold in this setting?*

8. Questions

There are several questions that arised from this work and its relationship with other papers. Here we list some:

(1) Do similar results hold for elliptic curves? That is, if we add a dense-codense family of independent elements inside an elliptic curve defined in a real closed field R and we call G the group they generate (with the elliptic curve operation) is the theory of the pair (R, G) NIP and near model complete?
(2) Assume $G \leq R^{>0}$ has the Mann property and $F \leq (R, +)$ is small. Is $F = \{0\}$?
(3) Are all small groups definable in expansions of R by a multiplicative group with the Mann property definably amenable?
(4) Is $Th(R, G)$ rosy when G is an in section 4 or in section 5?

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