Definably complete and Baire structures

Version 4.1

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Abstract

We consider definably complete and Baire expansions of ordered fields: every definable subset of the domain of the structure has a supremum and the domain can not be written as the union of a definable increasing family of nowhere dense sets. Every expansion of the real field is definably complete and Baire. So is every o-minimal expansion of a field. The converse is clearly not true. However, unlike the o-minimal case, the structures considered form an elementary class. In this context we prove a version of Kuratowski-Ulam's Theorem and some restricted version of Sard's Lemma.

1 Introduction

We recall that a subset A of a topological space X is said to be meager if there exists a collection $\{Y_i : i \in \mathbb{N}\}$ of nowhere dense subsets of X such that $A \subseteq \bigcup_{i\in\mathbb{N}} Y_i$. The Baire Category Theorem states that every open subset of $\mathbb{R}$ (with the usual topology) is not meager, i.e. $\mathbb{R}$ is a Baire space.

The notion of Baire space is clearly not first order. Here we consider a similar (*definable*) notion, which instead is preserved under elementary equivalence, and which coincides with the classical notion over the real numbers (this is made precise in Section 2).

The (first order) structures we consider are definably complete expansions of ordered fields. Definable completeness (see Definition 1.3) is a weak version of Dedekind completeness, which is preserved under elementary equivalence. The study of definably complete structures, which is mainly due to C. Miller, follows that of o-minimal structures; the aim is to develop an analogue to the theory of o-minimality in a more general situation, where the assumptions have been considerably weakened, but, unlike o-minimality, they are first order. It is shown in [Miller01], [Servi07], [Fratarc06] that, as in the o-minimal case, (a definable version of) most results of elementary real analysis can be proved in every definably complete expansion of an ordered field.

However, to obtain less elementary results one would need some more sophisticated machinery, in the direction of Sard's Lemma and Fubini's Theorem. Both of the quoted classical results refer to a notion of smallness (having measure zero), which has no natural translation in our context. We consider instead a topological notion of smallness (being meager), propose a definable version of this notion and carry out a theory of definably complete and Baire structures. In this context we prove an analogue to Fubini's Theorem (the Kuratowski-Ulam Theorem 4.1) and a very restricted form of an analogue to Sard's Lemma (Theorem 6.17). Notice that it is not known whether every definably complete structure is definably Baire.

The main motivation for the study of definably complete and Baire structures is to generalize the o-minimality results present in [Wilkie99] and [Kar–Mac99], to the situation where the base field is not necessarily $\mathbb{R}$. In [Wilkie99], the author proves that, given an expansion $\mathcal{R}$ of the real field with a family of C^∞ functions, if there are bounds (uniform in the parameters) on the number of connected components of quantifier free definable sets, then $\mathcal{R}$ is o-minimal. In particular, thanks to a well known finiteness result in [Khov91], the structure generated by all real Pfaffian functions is o-minimal. In [Kar–Mac99], the authors generalize Wilkie's theorem (by weakening the smoothness assumption) in a way which allows them to derive the following result (originally due to Speissegger, see [Speiss99]): the Pfaffian closure of an o-minimal expansion of the real field is o-minimal.

In a subsequent paper we use the results obtained in this paper to prove that, given a definably complete and Baire expansion $\mathbb{K}$ of an ordered field with a family of C^∞ functions, if there are bounds (uniform in the parameters) on the number of connected components of quantifier free definable sets, then $\mathbb{K}$ is o-minimal. By using our restricted version 6.17 of Sard's Lemma, we then proceed to prove the analogue to Khovanskii's finiteness result in the context of definably complete and Baire structures. We derive the o-minimality of every definably complete and Baire expansion of an ordered field with any family of definable Pfaffian functions. Finally, we prove that the relative Pfaffian closure of an o-minimal structure $\mathbb{K}_0$ into a definably complete and Baire expansion $\mathbb{K}$ of $\mathbb{K}_0$ is o-minimal. This latter result, whose proof is shaped on the one present in [Kar–Mac99], can be compared with the main result in [Fratarc06], where instead Speissegger's method was followed.

This work contributes to the study of ordered structures which satisfy properties implied by (but not equivalent to) o-minimality; hence it has a natural collocation in the framework depicted in [DMS08].

1.1 Preliminaries and notation

Throughout this paper, $\mathbb{K}$ is a (first-order) structure expanding an ordered field. We use the word "definable" as a shorthand for "definable in $\mathbb{K}$ with parameters from $\mathbb{K}$".

For convenience, on $\mathbb{K}^m$ instead of the usual Euclidean distance we will use the equivalent distance

$$d : (x, y) \mapsto \max_{i=1,\dots,m} |x_i - y_i|.$$

For every $\delta > 0$ and $x \in \mathbb{K}^m$, we define

$$B^m(x; \delta) := \{y \in \mathbb{K}^m : d(x, y) < \delta\},$$

the open "ball" of center x and "radius" δ; we will drop the superscript m if it is clear from the context.

Notation 1.1. Let $X \subseteq Y \subseteq \mathbb{K}^n$, with Y definable. We write $\mathrm{cl}_Y(X)$ (or simply $\overline{X}$ if Y is clear from the context) for the topological closure of X in Y, $\mathrm{int}_Y(X)$ (or simply $\mathring{X}$) for the interior part of X in Y, and $\mathrm{bd}_Y(X) := \overline{X} \setminus \mathring{X}$ for the boundary of X (in Y).

Notation 1.2. We define $\Pi^{m+n}_n : \mathbb{K}^{m+n} \to \mathbb{K}^m$ as the projection onto the first m coordinates. If $A \subset \mathbb{K}^{m+n}$ and $x \in \mathbb{K}^m$, we denote by A_x the fibre of A over x, i.e. the set $\{y \in \mathbb{K}^n : \ (x, y) \in A\}$.

1.2 Definably complete structures

Definition 1.3. An expansion $\mathbb{K}$ of an ordered field is called **definably complete** if every definable subset of $\mathbb{K}$ has a supremum in $\mathbb{K} \cup \{\pm\infty\}$.

Generalities on definably complete structures can be found in [Servi07] and [Miller01].

Definition 1.4. $X \subseteq \mathbb{K}^m$ is **definably compact** (d-compact for short) if it is definable, closed in $\mathbb{K}^m$ and bounded.

We order $\mathbb{K}^m$ lexicographically. We will denote by N a definable subset of $\mathbb{K}^m$ which is cofinal in the lexicographic ordering.

Lemma 1.5 (Miller)**.** *X is definably compact iff for every $(Y(y))_{y\in N}$ definable decreasing family of closed non empty subsets of X, we have $\bigcap_y Y(y) \neq \emptyset$.*

Definition 1.6. Let $f : N \to \mathbb{K}^n$ be definable. Define $\operatorname{acc}_{y\to\infty} f(y)$ (and write for simplicity $\operatorname{acc} f$) to be the set of accumulation points of f; that is, $x \in \operatorname{acc} f$ iff

$$(\forall r \in \mathbb{K}^m)(\forall \varepsilon > 0)(\exists y > r)\ y \in N \ \&\ d(f(y), x) < \varepsilon.$$

Lemma 1.7. *If X is definably compact, then for all definable N and for all $f : N \to X$ definable we have $\operatorname{acc} f \neq \emptyset$.*

It is not clear if the converse of the above lemma is true.

Definition 1.8. Let $(A(y))_{y\in N}$ be a definable family of non-empty subsets of $\mathbb{K}^n$. Define $\operatorname{acc}_{y\to\infty} A(y)$ (and write for simplicity $\operatorname{acc} A$) to be the set of accumulation points of A; that is, $x \in \operatorname{acc} A$ iff $(\forall r \in \mathbb{K}^m)$ $(\forall \varepsilon > 0)$ $(\exists y > r)$ $y \in N$ and $d(A(y), x) < \varepsilon$.

Note that $\operatorname{acc} A = \bigcap_y \overline{\left(\bigcup_{z\geq y} A(z)\right)}$.

Remark 1.9. Let $(A(t))_{0<t\in\mathbb{K}}$ be a definable family of subsets of $\mathbb{K}^m$, and $G := \bigcup_{t>0} A(t) \times \{t\}$. Then, $\operatorname{acc}_{t\to 0} A(t) = \left(\operatorname{cl}_{\mathbb{K}^{m+1}}(G)\right)_0 := \{x \in \mathbb{K}^m : (x, 0) \in \overline{G}\}$.

Lemma 1.10. *X is definably compact iff for all A definable family of non-empty subsets of X we have $X \cap \operatorname{acc} A \neq \emptyset$.*

Proof. First assume that X is d-compact. Let $Y(y) := \overline{\left(\bigcup_{z\geq y} A(y)\right)}$. Then $(X \cap Y(y))$ is a definable decreasing family of closed subsets of X. By Lemma 1.5, $\bigcap_y Y(y) \neq \emptyset$, and we are done.

Conversely, assume that X is not d-compact. By Lemma 1.5, there exists a definable decreasing family $Y := (Y(y))_{y\in\mathbb{K}^m}$ of closed subsets of X such that $\bigcap_y Y(y) = \emptyset$. However, since Y is decreasing, $X \cap \operatorname{acc} Y = \bigcap_y Y(y)$, and we are done. □

Proof of Lemma 1.7. Define $A(y) := \{f(y)\}$. By Lemma 1.10, $\operatorname{acc} A$ is non-empty. Note that $\operatorname{acc} A = \operatorname{acc} f$. □

Lemma 1.11. *Let $C \subset \mathbb{K}^n$ be a nonempty d-compact set, and let $V := \{V(t) :\ t \in I\}$ be a definable open cover of C. Then, there exists $\delta_0 \in \mathbb{K}^+$ (a* **Lebesgue number** *for V and C) such that, for every subset $X \subseteq C$ of diameter smaller than δ_0, there exists $t \in I$ such that $X \subseteq V(t)$.*

Proof. Suppose for a contradiction that

$$\forall \delta > 0\ \exists y \in C\ \forall t \in I \quad B(y;\delta) \not\subseteq V(t).$$

For every $\delta > 0$, define

$$Y(\delta) := \{y \in C : \forall t \in I \;\; B(y;\delta) \not\subseteq V(t)\}.$$

Note that $\bigl(Y(\delta)\bigr)_{\delta>0}$ is a definable family of subsets of C, increasing as δ decreases. Let y_0 be an accumulation point for the family $\bigl(Y(\delta)\bigr)_{\delta>0}$, as $\delta \to 0$ (which exists by Lemma 1.10).

Let $t_0 \in I$ and $\delta_0 > 0$ such that $B(y_0; 2\delta_0) \subseteq V(t_0)$. Let $y \in Y(\delta_0)$ such that $|y - y_0| < \delta_0$. Therefore, $B(y;\delta_0) \subseteq B(y_0; 2\delta_0) \subseteq V(t_0)$, contradicting the fact that $y \in Y(\delta_0)$. □

We will often use without further comment the following result:

Lemma 1.12 (Miller). *Let $f : \mathbb{K}^n \to \mathbb{K}^m$ be a definable continuous function and let $C \subset \mathbb{K}^n$ be d-compact. Then $f(C)$ is d-compact.*

Definition 1.13. A n-dimensional definable embedded $\mathcal{C}^\infty$ $\mathbb{K}$-manifold $V \subseteq \mathbb{K}^d$ (which we will simply call n-dimensional $\mathbb{K}$-manifold) is a definable subset V of $\mathbb{K}^d$, such that for every $x \in V$ there exists a definable neighbourhood $U(x)$ of x (in $\mathbb{K}^d$), and a definable $\mathcal{C}^\infty$ diffeomorphism $f_x : U(x) \simeq \mathbb{K}^d$, such that $U(x) \cap V = f_x^{-1}\bigl(\mathbb{K}^n \times \{0\}\bigr)$.

Remark 1.14. Note that a $\mathbb{K}$-manifold V can always be written as the intersection of a definable closed set and a definable open set. In fact, let $\delta : V \to \mathbb{K}^+$ be a definable map, such that, for every $x \in V$, $B\bigl(x, \delta(x)\bigr) \subseteq U(x)$. Let $U := \bigcup_{x\in V} B\bigl(x, \delta(x)\bigr)$; then, $V = \overline{V} \cap U$.

Note moreover that the dimension n of a $\mathbb{K}$-manifold V is uniquely determined by V, because $\mathbb{K}^n$ and $\mathbb{K}^{n'}$ are locally diffeomorphic iff $n = n'$.

2 Meager sets

Let $X \subseteq Y \subseteq \mathbb{K}^n$, with Y definable.

Definition 2.1. X is **nowhere dense** (in Y) if $\mathrm{int}_Y(\mathrm{cl}_Y(X)) = \emptyset$. X is **definably meager** (in Y) if there exists a *definable increasing* family $(A(t))_{t\in K}$ of nowhere dense subsets of Y, such that $X \subseteq \bigcup_t A(t)$. We will call such a family $\bigl(\mathrm{cl}_Y(A(t))\bigr)_{t\in\mathbb{K}}$ a **witness** of the fact that X is definably meager. X is **definably residual** (in Y) if $Y \setminus X$ is definably meager. If Y is clear from the context, we will simply say that X is nowhere dense (resp. definably meager, definably residual).

Notice that, if $(A(t))_{t\in\mathbb{K}}$ is a witness of the fact that X is meager in $\mathbb{K}^n$, then also the family

$$\left(\overline{A(t)} \cap [-|t|, |t|]^n\right)_{t\in\mathbb{K}}$$

is a witness, hence we may always assume that each $A(t)$ is d-compact.

Notice also that we do *not* require that a meager set is definable.

The subsets of Y, with the operations Δ (symmetric difference) and $\cap$, form a commutative ring; the definably meager subsets of Y form an ideal of this ring.

Definition 2.2. Y is **definably Baire** if every non-empty open definable subset of Y is not meager (in Y).

Note that if $\mathbb{K}$ has countable cofinality, then X is definably meager (Baire, respectively) in $\mathbb{K}^n$ if X is meager (Baire, respectively) in the usual topological sense. In general, the converse is not true: for instance, if $\mathbb{K}$ is a countable o-minimal structure, then it is definably Baire, but not Baire in the topological sense. If $\mathbb{K}$ is the expansion of the real field with a predicate for every subset of $\mathbb{R}^n$ ($n \in \mathbb{N}$), then the two notions coincide. From now on, we will write "meager" for "definably meager", and "topologically meager" for the usual topological notion, and similarly for "residual" and "Baire".

Proposition 2.3. *Let Y be definable, and $\emptyset \neq U \subseteq Y$ be definable and open. Then, U is meager in Y iff it is meager in itself.*

Proof. Suppose U is meager in Y and let $\big(Y(t)\big)_{t\in\mathbb{K}}$ be a witness of this fact. For every $t \in \mathbb{K}$, define $X(t) := Y(t) \cap U$. Since U is open, $\operatorname{int}_U(X(t)) = \operatorname{int}_Y(Y(t)) \cap U = \emptyset$. Hence, $\big(X(t)\big)_{t\in\mathbb{K}}$ is a witness of the fact that U is meager in itself.

Viceversa, let $\big(X(t)\big)_{t\in\mathbb{K}}$ be a witness of the fact that U is meager in itself, and $Y(t) := \operatorname{cl}_Y(X(t))$. We claim that $\operatorname{int}_Y(Y(t)) = \emptyset$. In fact, $\operatorname{int}_Y(Y(t)) = \operatorname{int}_Y((\operatorname{cl}_Y(X(t)) \cap U)) = \operatorname{int}_U(Y(t)) = \operatorname{int}_U(X(t)) = \emptyset$. Hence, $\big(Y(t)\big)_{t\in\mathbb{K}}$ is a witness of the fact that U is meager in Y. □

Corollary 2.4. *Let Y be definable, and $\emptyset \neq U \subseteq Y$ be definable and open. Then,*

1. *If U is of not meager in itself, then Y is also not meager in itself.*

2. *If Y is Baire, then U is also Baire.*

Proof. For 1, note that every subset of a meager set is meager, and in particular U is meager.

Regarding 2, if Y is Baire, let $V \subseteq U$ be non-empty, definable and open in U. Since U is open in Y, V is also open in V. Hence, by 2.3, V is not meager in itself, and, again by 2.3, V is not meager in U. Therefore, U is Baire. □

Lemma 2.5. *Let $Y \subseteq \mathbb{K}^m$ be definable. The following are equivalent:*

1. *Y is Baire;*
2. *for all $X \subseteq Y$, if X is meager, then $\mathring{X} = \emptyset$;*
3. *every $x \in Y$ has a definable neighbourhood which is Baire;*
4. *every residual subset of Y is dense;*
5. *every open definable non-empty subset of Y is not meager in itself;*
6. *every meager closed definable subset of Y has empty interior.*

Proof.

($2 \Rightarrow 1$) is obvious.

($1 \Rightarrow 3$) is obvious, because Y itself is a Baire neighbourhood of each point.

($3 \Rightarrow 4$) Let $X \subseteq Y$ be meager. Suppose, for a contradiction, that U is a non-empty definable subset of X open in Y, and let $x \in U$. Let V be a definable Baire neighbourhood of x, and $W := V \cap U$. By Proposition 2.3, W is Baire, and therefore it is not meager in Y (by the same lemma), which is not possible.

($4 \Rightarrow 2$) Let $X \subseteq Y$ be meager. Hence, $Y \setminus X$ is dense, and therefore $\mathring{X} = \emptyset$.

($1 \Leftrightarrow 5$) Use Proposition 2.3.

($1 \Rightarrow 6$) Let $C \subseteq Y$ be definable, closed and meager. If $\mathring{C} \neq \emptyset$, then $\mathring{C}$ is not meager, and thus C is not meager.

($6 \Rightarrow 1$) Let $U \subseteq Y$ be open, definable and meager in Y. Then, $\overline{U}$ is also meager, because $\overline{U} = U \sqcup \operatorname{bd} U$, and $\operatorname{bd} U$ is nowhere dense. Therefore, $\overline{U}$ has empty interior, and therefore U is empty. □

Remark 2.6. $\mathbb{K}^n$ is Baire iff it is not meager in itself.

Proof. One implication is obvious.

For the other implication, assume that $\mathbb{K}^n$ is not meager in itself, and let $U \subseteq \mathbb{K}^n$ be an open definable subset. If, for a contradiction, U were meager in itself, then we could find an open non empty box $B \subseteq U$. By Proposition 2.3, B is also meager in itself. However, B is definably homeomorphic to $\mathbb{K}^n$, because $\mathbb{K}$ expands a field, contradicting the hypothesis. □

The following result is not trivial and will be proved in Section 4.

Proposition 2.7. *If $\mathbb{K}$ is Baire, then for every $m \geq 1$, $\mathbb{K}^m$ is Baire.*

3 $\mathcal{F}_\sigma$-sets

We now consider a class of sets for which it is easy to determine whether they are meager or not.

Definition 3.1. Let $X \subseteq Y \subseteq \mathbb{K}^n$, with Y definable. X is in $\mathcal{F}_\sigma$ (in Y) if X is the union of a definable increasing family of closed subsets of Y, indexed by $\mathbb{K}$. X is in $\mathcal{G}_\delta$ if its complement is an $\mathcal{F}_\sigma$.

Lemma 3.2. *Let $\mathcal{A}$ be either the family of $\mathcal{F}_\sigma$ or the family of $\mathcal{G}_\delta$ subsets of some $\mathbb{K}^n$, for $n \in \mathbb{N}$. Then, each $A \in \mathcal{A}$ is definable. Moreover, $\mathcal{A}$ is closed under finite unions, finite intersections, Cartesian products, and preimages under definable continuous functions. Besides, the following are in $\mathcal{A}$*

1. *definable closed subsets of $\mathbb{K}^n$;*
2. *definable open subsets of $\mathbb{K}^n$;*
3. *finite boolean combinations of definable open subsets of $\mathbb{K}^n$.*

The family of $\mathcal{F}_\sigma$ subsets is also closed under images under definable continuous functions.

Proof. Let A and B be in $\mathcal{F}_\sigma$. The fact that $A \cup B$ and $A \times B$ are also in $\mathcal{F}_\sigma$ is obvious.

Let $A = \bigcup_t A(t)$ and $B = \bigcup_t B(t)$, where $(A(t))_{t\in\mathbb{K}}$ and $(B(t))_{t\in\mathbb{K}}$ are two definable increasing families of closed (and, we may assume, d-compact) sets.

Then, $A \cap B = \bigcup_t (A(t) \cap B(t))$, because $(A(t))_{t\in\mathbb{K}}$ and $(B(t))_{t\in\mathbb{K}}$ are increasing families. Hence, $A \cap B$ is also in $\mathcal{F}_\sigma$.

If $f : \mathbb{K}^n \to \mathbb{K}^{n'}$ is continuous, then $f(A) = \bigcup_t f(A(t))$. For every $t \in \mathbb{K}$, $f(A(t))$ is d-compact, because $A(t)$ is d-compact, and therefore $f(A)$ is in $\mathcal{F}_\sigma$. A similar proof works for preimages.

Let $U \subseteq \mathbb{K}^n$ be open and definable, and $C := \mathbb{K} \setminus U$. For every $r \in \mathbb{K}^+$, define $U(r) := \{x \in \mathbb{K}^n : d(x, C) \geq r\}$. Note that each $U(r)$ is closed. Since U is open, $U = \bigcup_{r>0} U(r)$, and therefore U is in $\mathcal{F}_\sigma$.

If D is a finite boolean combination of open definable subsets of $\mathbb{K}^n$, then it is a finite union of sets of the form $C_i \cap U_i$, for some definable sets C_i and U_i, such that each C_i is closed and each U_i is open. Hence, D is in $\mathcal{F}_\sigma$.

The corresponding results for $\mathcal{G}_\delta$ follow immediately by considering the complements. □

If $Y \subseteq \mathbb{K}^n$ is definable, $X \subseteq Y$ is an $\mathcal{F}_\sigma$-subset of Y, and $f : Y \to Y$ is definable and continuous, it might not be true that $f(X)$ is an $\mathcal{F}_\sigma$. The point where the above proof breaks down for $Y \neq \mathbb{K}^n$ is the fact that it is not necessarily true that every $\mathcal{F}_\sigma$ set X is an increasing definable union of d-compact sets.

Notice that, by Remark 1.14, every $\mathbb{K}$-manifold is an $\mathcal{F}_\sigma$-set.

Remark 3.3. If $X \subseteq \mathbb{K}^n$ is meager, then there exists a meager $\mathcal{F}_\sigma$-set containing X.

Lemma 3.4. *Let Y be definable and Baire, and $D \subseteq Y$. Assume that D is in $\mathcal{F}_\sigma$. Then, D is meager iff $\mathring{D} = \emptyset$.*[1]

Proof. If $\mathring{D} \neq \emptyset$, then, since Y is Baire, D cannot be meager. Conversely, assume that D is not meager. If D is in $\mathcal{F}_\sigma$, then $D = \bigcup_t D(t)$, for some definable increasing family of closed subsets. Since D is not meager, at least one of the $D(t)$, say $D(t_0)$, is not meager. Hence, $\mathring{D}(t_0) \neq \emptyset$ (otherwise, $D(t_0)$ would be nowhere dense), and therefore $\mathring{D} \neq \emptyset$. □

Note that if $X \subseteq \mathbb{R}^n$ is in $\mathcal{F}_\sigma$ and of measure zero, then X is meager, but the converse is not true.

Remark 3.5. Let $X \subseteq \mathbb{K}^n$. X is an $\mathcal{F}_\sigma$ iff X is of the form $\Pi^{n+m}_n(Z)$ for some $Z \subseteq \mathbb{K}^{n+m}$ closed and definable.

Proof. The "if" direction follows from Lemma 3.2. For the other direction, let $\big(X(t)\big)_{t\in\mathbb{K}}$ be a definable increasing family of closed subsets of $\mathbb{K}^n$, such that $X = \bigcup_{t\in\mathbb{K}} X(t)$. Define $Z := \bigsqcup_{t\in\mathbb{K}}\big(X(t) \times \{t\}\big)$. □

Notice that, if $\mathbb{K}$ is o-minimal, then every X definable subset of $\mathbb{K}$ is a finite Boolean combination of definable closed sets (because X is a finite union of cells), and therefore X is an $\mathcal{F}_\sigma$.

We now give a local condition which is sufficient to prove that the image of an $\mathcal{F}_\sigma$-set under a continuous definable function is meager.

Proposition 3.6. *Let $C \subseteq \mathbb{K}^m \times \mathbb{K}^n$ be in $\mathcal{F}_\sigma$, $f : C \to \mathbb{K}^d$ be definable and continuous. Assume that for every $y \in \Pi^{m+n}_m(C)$ there exists a neighbourhood $V_y \subseteq \mathbb{K}^m$ of y, such that $f\big((V_y \times \mathbb{K}^n) \cap C\big)$ is meager. Then, $f(C)$ is meager.*

[1] This is not true for $\mathcal{G}_\delta$ sets: for instance, the set of irrational numbers in $\mathbb{R}$ is a $\mathcal{G}_\delta$ which is not meager (it is even residual), but has empty interior.

Proof. If $\mathbb{K}$ is meager in itself, then by Proposition 2.7 there is nothing to prove. Hence, we may assume that $\mathbb{K}$ is Baire.

We proceed by induction on m. The case $m = 0$ is clear, because if $m = 0$, then $V_0 = \mathbb{K}^0$.

Assume that we have already proved the conclusion for $m-1$ (and every n). We want to prove it for m. First, we consider the case when C is d-compact. W.l.o.g., $0 \in C$. For every $r > 0$ and $y \in \mathbb{K}^m$, let $T^m(y;r) \subset \mathbb{K}^m$ be the closed hypercube of side $2r$ and center y, and $S^m(y;r)$ be its boundary. Moreover, define $D(r) := f\big(C \cap (T^m(0;r) \times \mathbb{K}^n)\big)$.

Note that $f(C) = \bigcup_r D(r)$, and that each $D(r)$ is d-compact. Therefore, to prove that $f(C)$ is meager, it suffices to prove that each $D(r)$ has empty interior. Suppose, for a contradiction, that $f(C)$ is not meager, and let

$$r_0 := \inf\{r > 0 : \mathrm{int}(D(r)) \neq \emptyset\}.$$

Since the $D(r)$ are closed, $r_0 = \inf\{r > 0 : D(r) \text{ is not meager}\}$. We have that $0 < r_0$ by hypothesis, and $r_0 < +\infty$ because $f(C)$ is not meager.

Let $P := \Pi^{n+m}_m(C)$. Since P is d-compact, if $\mathbb{K} = \mathbb{R}$, we could find $y_1, \dots, y_k \in P$ such that $P \subseteq V_{y_1} \cup \dots \cup V_{y_k}$. In the general situation, we need another argument. Let $5\delta_0$ be a Lebesgue number for the open cover $\{V_y : y \in P\}$ of P (we may assume that δ_0 is small in comparison with r_0); $\delta_0 > 0$ exists by Lemma 1.11.

Note that

$$T^m(0; r_0 + \delta_0/2) \subseteq T^m(0; r_0 - \delta_0/2) \cup \bigcup_{y \in S^m(0;r_0)} T^m(y; \delta_0),$$

hence

$$D(r_0 + \delta_0/2) \subseteq D(r_0 - \delta_0/2) \cup \bigcup_{y \in S^m(0;r_0)} f\big(C \cap (T^m(y;\delta_0) \times \mathbb{K}^n)\big).$$

By definition of r_0, we know that $D(r_0 + \delta_0/2)$ is not meager, while $D(r_0 - \delta_0/2)$ is meager. Hence, to obtain a contradiction, it suffices to show that $\bigcup_{y \in S^m(0;r_0)} f\big(C \cap (T^m(y;\delta_0) \cap \mathbb{K}^n)\big)$ is meager.

Note that $S^m(0;r_0)$ is the finite union of the faces of the closed hypercube $T^m(0;r_0)$: hence, we only need to show that for each face S of $S^m(0;r_0)$ the set $D := \bigcup_{y \in S} f\big(C \cap (T^m(y;\delta_0) \times \mathbb{K}^n)\big)$ is meager. W.l.o.g., we can assume that S is the "top" face $\{y \in T^m(0;r_0) : y_m = r_0\}$ and we may identify S with $T^{m-1}(0;r_0) \times \{r_0\}$.

Define

$$C' := C \cap \bigcup_{y \in S} \big(T^m(y;\delta_0) \times \mathbb{K}^n\big),$$
$$f' := f \restriction C'.$$

Claim. C' and f' satisfy the hypothesis of the proposition, with $n' = n+1$, $m' = m-1$, and $V'_z = B(z;\delta_0)$.

C' is d-compact, and therefore it is in $\mathcal{F}_\sigma$. Let $P' \subseteq \mathbb{K}^{m-1}$ be the projection of C' onto $\mathbb{K}^{m-1}$; note that P' is d-compact. Fix $z \in P'$; by definition, there exists $t \in [r_0 - \delta_0, r_0 + \delta_0]$ such that $y := (z,t) \in P$. Notice that

$$C' \cap (V'_z \times \mathbb{K} \times \mathbb{K}^n) \subseteq C \cap (V'_z \times [r_0 - \delta_0, r_0 + \delta_0] \times \mathbb{K}^n) \subseteq C \cap (T^m(y; 2\delta_0) \times \mathbb{K}^n).$$

Since $5\delta_0$ is a Lebesgue number for the cover $\{V_y : y \in P\}$ of P, it follows that there exists $y' \in P$ such that $T^m(y;2\delta_0) \subset V_{y'}$. Putting everything together, we have that $C' \cap (V'_z \times \mathbb{K}^{n+1}) \subset C \cap (V_{y'} \times \mathbb{K}^n)$ and thus $f'\big(C' \cap (V'_z \times \mathbb{K}^{n+1})\big)$ is meager, which proves the claim.

Therefore, by inductive hypothesis, $f'(C')$ is meager. However, $D \subseteq f'(C')$, and we reached a contradiction.

We now treat the general case when C is in $\mathcal{F}_\sigma$. Note that C is an increasing union of d-compact sets $C(t)$. For each $t \in \mathbb{K}$, define $D(t) := f(C(t))$: note that each $D(t)$ is d-compact. By the d-compact case, we can conclude that each $D(t)$ is meager, and therefore nowhere dense. Thus, $D = \bigcup_t D(t)$ is meager. □

Corollary 3.7. *Let $C \subseteq \mathbb{K}^m$ be in $\mathcal{F}_\sigma$, and $f : C \to \mathbb{K}^d$ be definable and continuous. Assume that for every $x \in C$ there exists $V_x \subseteq C$ neighbourhood of x, such that $f\big(C \cap V_x\big)$ is meager. Then, $f(C)$ is meager.*

Proof. Apply the proposition to the case $n = 0$. □

Corollary 3.8. *Let $W \subseteq \mathbb{K}^m$ be a definable $\mathbb{K}$-manifold, $C \subseteq W$ be an $\mathcal{F}_\sigma$-set (in W), and $f : C \to \mathbb{K}^d$ be definable and continuous. Assume that for every $x \in C$ there exists $V_x \subseteq C$ neighbourhood of x (in C), such that $f\big(V_x\big)$ is meager. Then, $f(C)$ is meager.*

Proof. Since W is a $\mathbb{K}$-manifold, it is in $\mathcal{F}_\sigma$. Since C is an $\mathcal{F}_\sigma$-set in W, it is also an $\mathcal{F}_\sigma$-set in $\mathbb{K}^m$. Apply the previous corollary. □

Corollary 3.9. *Let $C \subseteq \mathbb{K}^m$ be an $\mathcal{F}_\sigma$. If every $x \in C$ has a neighbourhood V_x such that $C \cap V_x$ is meager, then C is meager.*

Let $\tilde{\mathbb{R}}$ be the structure on the reals numbers, with a predicate for every subset of $\mathbb{R}^n$. Proposition 3.6 and the following corollaries are trivial for $\mathbb{K} = \tilde{\mathbb{R}}$, because an $\mathcal{F}_\sigma$-subset C of $\tilde{\mathbb{R}}^n$ has a countable basis of open sets; for instance, the hypothesis in Proposition 3.6 on f and C imply that $f(C)$ is a countable union of meager sets, and hence meager.

4 The Kuratowski-Ulam Theorem

The main result of this section is the following theorem.

Theorem 4.1. *Let $D \subseteq \mathbb{K}^{m+n}$. For every $x \in \mathbb{K}^m$, let $D_x := \{y \in \mathbb{K}^n : (x,y) \in D\}$ be the corresponding section of D. Let $T := T^m(D) := \{x \in \mathbb{K}^m : D_x$ is meager in $\mathbb{K}^n\}$.*

If D is meager (in $\mathbb{K}^{m+n}$), then T is residual.

This is a definable version of Kuratowski-Ulam's Theorem [Oxtoby80, Thm. 15.1], which in turn is an analogue of Fubini's Theorem: they both imply that if D is negligible, then D_y is negligible for almost every y; in Kuratowski-Ulam's Theorem negligible means "meager", while in Fubini's Theorem negligible means "of measure zero".

It is not clear whether in the above theorem D definable implies that T is definable. Note that if $\mathbb{K}$ is o-minimal and D is definable, then T is also definable.

As a corollary, we obtain Proposition 2.7.

Proof of Proposition 2.7. By induction on m. The case $m = 1$ is our assumption on $\mathbb{K}$. Assume that we already proved that $\mathbb{K}^m$ is Baire: we want to prove that $\mathbb{K}^{m+1}$ is Baire. Suppose not; then $\mathbb{K}^{m+1}$ is meager in itself. If we apply Theorem 4.1 with $n = 1$, we obtain that either $\mathbb{K}^m$ or $\mathbb{K}$ is meager in itself, a contradiction. □

Definition 4.2. A definable function $f : Y \to \mathbb{K}$ is **lower semi-continuous** if, for every $x \in Y$, either x is an isolated point of Y, or

$$\liminf_{\substack{x' \to x \\ x' \in Y}} f(x') \geq f(x).$$

Remark 4.3. Let $C \subseteq \mathbb{K}^{n+1}$ be d-compact. For every $x \in D := \Pi^{n+1}_n(C)$, let $f(x) := \min D_x$. Then, $f : D \to \mathbb{K}$ is lower semi-continuous.

Lemma 4.4. *Let $Y \subseteq \mathbb{K}^n$ be definable, $f : Y \to \mathbb{K}$ be lower semi-continuous and definable, and $\Delta \subseteq Y$ be the set of points of discontinuity of f. Then, Δ is meager (in Y).*

Proof. By a change of variable in the co-domain, w.l.o.g. we can assume that f is bounded. For every $\varepsilon > 0$, let

$$\Delta(\varepsilon) := \{x \in Y : x \text{ is not isolated in } Y \ \& \ \limsup_{x' \to x} f(x') \geq f(x) + \varepsilon\}.$$

Since f is lower semi-continuous, we have that $\Delta = \bigcup_{\varepsilon>0} \Delta(\varepsilon)$. Hence, to prove the lemma it suffices to show that each $\Delta(\varepsilon)$ is nowhere dense. Fix $\varepsilon > 0$.

Claim 1. $\Delta(\varepsilon)$ is closed.

In fact, let $x_0 \in \overline{\Delta(\varepsilon)}$. We have to prove that $x_0 \in \Delta(\varepsilon)$. Assume not. It is clear that x_0 is not an isolated point of Y. Let

$$s := \limsup_{x \to x_0} f(x),$$
$$i := \liminf_{x \to x_0} f(x).$$

Note that s and i are in $\mathbb{K}$, because $\mathbb{K}$ is definably complete. Since $x_0 \notin \Delta(\varepsilon)$, we have that $s - \varepsilon < f(x_0)$. Hence, since f is lower semi-continuous,

$$f(x_0) \leq i \leq s < f(x_0) + \varepsilon \leq i + \varepsilon;$$

let $\rho := i + \varepsilon - s > 0$.

Therefore, if x and x' are sufficiently near x_0, then $f(x) < s + \rho/2$ (because s is the $\limsup$), and $f(x) > s - \varepsilon + \rho/2$ (because $i = s - \varepsilon + \rho$ and i is the $\liminf$), and similarly for x'. Thus,

$$|f(x) - f(x')| < (s + \rho/2) - (s - \varepsilon + \rho/2) = \varepsilon. \tag{1}$$

On the other hand, since $x_0 \in \overline{\Delta(\varepsilon)}$, we have that there exist $x \in \Delta(\varepsilon)$ near x_0. By definition of $\Delta(\varepsilon)$, there exists x' near x (and therefore near x_0) such that $f(x') \geq f(x) + \varepsilon$, contradicting (1).

Claim 2. $\operatorname{int}(\Delta(\varepsilon)) = \emptyset$.

Assume, for a contradiction that $\Delta(\varepsilon)$ contains a nonempty open subset U of Y. Let $m := \sup_{x \in U} f(x)$. Note that $m \in \mathbb{K}$, because f is bounded.

Let $x \in U$ be such that $f(x) > m - \varepsilon/4$. Since $x \in \Delta(\varepsilon)$, there exists $x' \in Y$ near x (and therefore in U) such that $f(x') > f(x) + \varepsilon/2$. Hence, $f(x') > m + \varepsilon/4 > m$, contradicting the definition of m.

The two claims imply that $\Delta(\varepsilon)$ is nowhere dense, and we are done. □

In the above lemma, if $Y = \mathbb{K} = \mathbb{R}$, we can not conclude that Δ has measure zero. In fact, let $C \subseteq \mathbb{R}$ be closed, with empty interior, and of positive measure, and f be the characteristic function of $\mathbb{R} \setminus C$. Then, $\Delta = C$, and therefore it is of positive measure.

On the other hand, it is always true that if $f : \mathbb{K}^m \to \mathbb{K}$ is definable, then Δ is in $\mathcal{F}_\sigma$. In fact, for every $\varepsilon > 0$ define

$$\Lambda(\varepsilon) := \{x \in \mathbb{K}^m : \forall \delta > 0 \ \exists y \in B(x; \delta) \ |f(y) - f(x)| \geq \varepsilon\}.$$

Note that $\Delta = \bigcup_{\varepsilon>0} \Lambda(\varepsilon)$. While $\Lambda(\varepsilon)$ might not be closed, we have that $\overline{\Lambda(\varepsilon)} \subseteq \Lambda(\varepsilon/2)$. Therefore, $\Delta = \bigcup_{\varepsilon>0} \overline{\Lambda(\varepsilon)}$.

Proof of Theorem 4.1. If $\mathbb{K}^m$ is meager in itself, then the conclusion is trivially true, because then every subset of $\mathbb{K}^m$ is meager. Hence, we can assume that $\mathbb{K}^m$ is Baire.

Case 1: $n = 1$ and D is d-compact.
Hence, D has empty interior, and each D_x is also d-compact. Therefore, by Lemma 3.4, $T = \{x \in \mathbb{K}^m : \mathring{D}_x = \emptyset\}$. Let $E := \mathbb{K}^m \setminus T$. We have to prove that E is meager.

For every $\varepsilon > 0$ let

$$X(\varepsilon) := \{(x, y) \in \mathbb{K}^m \times \mathbb{K} : B^1(y; \varepsilon) \subseteq D_x\}.$$

Let

$$E(\varepsilon) := \pi(X(\varepsilon)) = \{x \in \mathbb{K}^m : D_x \text{ contains a ball of radius } \varepsilon\}.$$

Note that $X(\varepsilon)$ is d-compact, since its complement is the projection of an open set, therefore so is $E(\varepsilon)$. Note that $E = \bigcup_{\varepsilon>0} E(\varepsilon)$; hence, to prove that E is meager, it suffices to prove that each $E(\varepsilon)$ is nowhere dense. Since each $E(\varepsilon)$ is d-compact, it suffices to prove the following claim.
Claim 1. For every $\varepsilon > 0$, $\operatorname{int}(E(\varepsilon)) = \emptyset$.

Assume, for a contradiction, that there exists a nonempty open box $U \subseteq E(\varepsilon)$. Define

$$\begin{aligned} f : U &\to \mathbb{K} \\ x &\mapsto \min\{y \in \mathbb{K} : (x, y) \in X(\varepsilon)\}. \end{aligned}$$

Note that f is lower semi-continuous and definable. By Lemma 4.4, f is continuous outside a meager set $\Delta \subseteq U$. Since $\mathbb{K}^m$ is Baire, $\Delta \neq U$, and therefore there exists $x_0 \in U$ such that f is continuous at x_0. It is now easy to show that a neighbourhood of $(x_0, f(x_0))$ is contained in D, contradicting the fact that $\mathring{D} = \emptyset$.

Case 2: $n = 1$ and D arbitrary meager subset of $\mathbb{K}^m$.
Let $\big(D(p)\big)_{p\in\mathbb{K}}$ be an increasing definable family of d-compact subsets of $\mathbb{K}^{m+1}$ with empty interior, such that $D \subseteq \bigcup_p D(p)$. For each $p \in \mathbb{K}$, let $E(p) := \{x \in \mathbb{K}^m : D(p)_x \text{ is not meager in } \mathbb{K}\}$. By what we have seen above, $E(p) = \bigcup_{\varepsilon>0} E(p, \varepsilon)$, where $(E(p, \varepsilon))_{\substack{\varepsilon\in\mathbb{K}^+\\ p\in\mathbb{K}}}$ is a definable family of subsets of $\mathbb{K}$, increasing in p and decreasing in ε, such that each $E(p, \varepsilon)$ is closed and nowhere dense. Let

$$E' := \bigcup_{\varepsilon,p} E(p, \varepsilon) = \bigcup_p E(p).$$

Claim 2. $\mathbb{K}^m \setminus T \subseteq E'$.

In fact, let $x \notin T$. Thus, D_x is not meager. However, $D_x \subseteq \bigcup_p D(p)_x$. Since $\big(D(p)_x\big)_{p\in\mathbb{K}}$ is an increasing definable family of closed subsets of $\mathbb{K}$, we obtain that there exists p_0 such that $D(p_0)_x$ has non-empty interior. Thus, $x \in E(p_0) \subseteq E'$.

Therefore, it suffices to prove that E' is meager to obtain that T is residual. However, $E' = \bigcup_{p>0} E(p, 1/p)$, and we are done.

Case 3: $n > 1$ and D arbitrary meager subset of $\mathbb{K}^m$. We argue by induction on n.
Suppose that we have already proved the conclusion for n (and for every m). We want to prove the conclusion for $n+1$. First, we will assume that D is in $\mathcal{F}_\sigma$. We want to prove that the set $T := T^m(D) := \{x \in \mathbb{K}^m : D_x \text{ is meager}\}$ is residual. Define

$$S := \mathbb{K}^{m+1} \setminus T^{m+1}(D) := \{(x, y_{n+1}) \in \mathbb{K}^m \times \mathbb{K} : D_{(x,y_{n+1})} \text{ is not meager}\},$$
$$R := T^m(S) = \{x \in \mathbb{K}^m : S_x \text{ is meager}\}.$$

By Lemma 3.4, $D_{(x,y_{n+1})}$ is meager iff its interior is empty, and therefore $S = \{(x, y_{n+1}) \in \mathbb{K}^m \times \mathbb{K} : \operatorname{int}(D_{(x,y_{n+1})}) \neq \emptyset\}$ (and in particular S is definable).

Claim 3. S is meager.

By inductive hypothesis.

Claim 4. R is residual.

By the case $n = 1$ and the previous claim.

Claim 5. $R \subseteq T$.

Fix $x \in \mathbb{K}^m$. Assume that $x \notin T$. We have to prove that $x \notin R$. Define $F := D_x \subseteq \mathbb{K}^{n+1}$. Note that F is in $\mathcal{F}_\sigma$: therefore, since $x \notin T$, $\mathring{F} \neq \emptyset$. Let $U := U_1 \times U_2$ be a non-empty open box contained in F, $U_1 \subseteq \mathbb{K}$, $U_2 \subseteq \mathbb{K}^m$. For every $y_{n+1} \in U_1$, $D_{(x,y_{n+1})} = F_{y_{n+1}} \supseteq U_2$, and therefore $(x, y_{n+1}) \in S$. Thus, $U_1 \subseteq S_x$, and $x \notin R$.

Hence, T contains a residual set, and therefore it is residual.

For D arbitrary, let $D' \subseteq \mathbb{K}^{m+n}$ be a meager $\mathcal{F}_\sigma$ containing D. By the previous case, the corresponding set $T' := T^m(D')$ is residual. Since $T' \subseteq T$, we are done. □

5 Almost open sets

In this section we will assume that $\mathbb{K}$ is definably complete and Baire.

Let $Y \subseteq \mathbb{K}^m$ be definable. We have seen that the family of meager subsets of Y is an ideal, hence it defines an equivalence relation on the family of subsets of Y, given by $X \sim X'$ iff $X \,\Delta\, X'$ is meager.

Remark 5.1. $X \sim X'$ iff there exists Z meager such that $X \,\Delta\, Z = X'$

Proof. Set $Z := X \,\Delta\, X'$. □

Definition 5.2. $X \subseteq Y$ is **almost open**, or a.o. for short, if X is equivalent to a definable open set.[2]

Lemma 5.3. *Let $Y \subseteq \mathbb{K}^m$ be definable, and A and B be a.o. subsets of Y. Then, $A \cap B$, $A \cup B$ and $Y \setminus A$ are also a.o.. Moreover, $\mathcal{F}_\sigma$ and $\mathcal{G}_\delta$ subsets of Y are a.o..*

Finally, if Y_1 and Y_2 are definable, and $A_i \subseteq Y_i$ are a.o. for $i = 1, 2$, then $A_1 \times A_2$ is a.o. in $Y_1 \times Y_2$.

Proof. It is trivial to see that $A \cap B$, $A \cup B$ and $A_1 \times A_2$ are a.o..

Let $A = U \,\Delta\, E$, where U is open and definable, and E is meager. Then, $Y \setminus A = (Y \setminus U) \,\Delta\, E$. Hence, to prove that $Y \setminus A$ is a.o. it suffices to prove that $C := Y \setminus U$ is a.o.. However, $C = \mathring{C} \cup \operatorname{bd}(C)$. Since C is closed, $\operatorname{bd}(C)$ is nowhere dense, and *a fortiori* meager, and we are done.

Let $(D(t))_{t \in \mathbb{K}}$ be a definable increasing sequence of closed subsets of Y. We have to prove that $D := \bigcup_t D(t)$ is a.o.. Let $U := \mathring{D}$ and $E := D \setminus U$. It is enough to prove that E is meager. For every t, let $E(t) := E \cap D(t)$. Note that $\mathring{D}(t) \subseteq U$; therefore, $E(t) \subseteq \operatorname{bd}(D(t))$ is nowhere dense, and we are done. □

Consequently, $X \subseteq Y$ is a.o. iff it is equivalent to a definable closed subset of Y.

Remark 5.4. Every meager set is a.o., being equivalent to the empty set. By Lemma 5.3, every residual set is a.o..

Corollary 5.5. *Let $A \subseteq Y$. The following are equivalent:*

1. *A is a.o.;*
2. *A is of the form $E \,\Delta\, F$, for some meager set E and some set F in $\mathcal{F}_\sigma$;*
3. *A is of the form $G \sqcup E$, for some G in $\mathcal{G}_\delta$ and E meager.*

Proof. Cf.[Oxtoby80, Thm. 4.4]. ($1 \Leftrightarrow 2$) and ($3 \Rightarrow 1$) are obvious. For ($1 \Rightarrow 3$), let $A = U \,\Delta\, E$ for some U open and E meager. Let Q be a meager set in $\mathcal{F}_\sigma$ containing E, and $G := U \setminus Q$. Note that G is in $\mathcal{G}_\delta$, and

$$U \,\Delta\, E = [(U \setminus Q) \,\Delta (U \cap Q)] \,\Delta (E \cap Q) = G \,\Delta [(U \,\Delta\, E) \cap Q] = G \sqcup E',$$

where $E' := (U \,\Delta\, E) \cap Q$ is meager. □

[2] Almost open sets are called sets with the "property of Baire" in [Oxtoby80].

The following is a partial converse of Theorem 4.1. Here it is important that $\mathbb{K}$ be Baire.

Lemma 5.6. *Let D be an a.o. subset of $\mathbb{K}^{m+n}$, and $T(D) := \{x \in \mathbb{K}^m : D_x$ is meager$\}$. Then, D is meager iff $T(D)$ is residual.*

Proof. The "only if" direction is Theorem 4.1. For the other direction, let U be an open set such that $E := D \,\Delta\, U$ is meager. By Theorem 4.1, $T(E)$ is residual. Moreover, since $U_x = D_x \,\Delta\, E_x$, we have $T(U) \supseteq T(D) \cap T(E)$, and therefore $T(U)$ is also residual. However, U is open and $\mathbb{K}^n$ is Baire: therefore, $T(U)$ is the complement of the projection of U on $\mathbb{K}^m$. Since U is open, $T(U)$ is closed. Therefore, $T(U)$ is closed and residual; since $\mathbb{K}^m$ is Baire, $T(U) = \mathbb{K}^m$. Thus, U is empty, and we are done. □

The hypothesis that D is a.o. in the above lemma is necessary: [Oxtoby80, Thm. 15.5] gives an example of a set $E \subseteq \mathbb{R}^2$ that is not meager, such that no three points of E are collinear.

Note that, if $D \subseteq \mathbb{K}^{m+n}$ is in $\mathcal{F}_\sigma$, then D is meager iff $\mathbb{K}^m \setminus T^m(D)$ has non-empty interior.

6 Further results and open problems

Remark 6.1. It follows from Remark 2.6 that the fact that $\mathbb{K}$ is Baire can be expressed by a recursive set of first-order sentences: that is, every $\mathbb{K}'$ elementary equivalent to $\mathbb{K}$ also satisfies the hypothesis.

Notice that an ultra-product of definably complete (resp. Baire) structures is also definably complete (resp. Baire); the same cannot be said for "o-minimal" instead of "definably complete".

Examples 6.2.

- Every expansion of $\mathbb{R}$ is definably Baire (because $\mathbb{R}$ is topologically Baire).
- Every o-minimal expansion of a field is definably Baire (the union of a definable increasing family of nowhere dense sets is finite, and hence can not coincide with the whole structure).
- Let $\mathcal{B}$ be an o-minimal expansion of a field, let $A \preccurlyeq \mathcal{B}$ be a dense substructure. Then the structure $\mathcal{B}_A$, generated by adding a unary predicate symbol for A, is definably Baire. This follows from the fact that if $X \subseteq \mathcal{B}$ is $\mathcal{B}_A$-definable, then its topological closure $\overline{X}$ is $\mathcal{B}$-definable (see [vdDries98, Theorem 4]). Hence, as above, the union of a definable increasing family of *closed* nowhere dense sets is finite.

Notice that the structures considered in all of the above examples are also definably complete.

Open problem 6.3. *It is not known to the authors if there exists a definably complete structure which is not Baire.*

Assume that $\mathbb{K}$ is definably complete and Baire.

Open problem 6.4. *Let $\big(Y(t)\big)_{t\in\mathbb{K}}$ be a definable increasing family of* meager *subsets of $\mathbb{K}^m$, and let $Y := \bigcup_t Y_t$. Is Y necessarily meager? In particular, is it necessarily $Y \neq \mathbb{K}^m$?*

Notice that, if the $Y(t)$ are closed, then Y is meager, whereas the same conclusion does not necessarily hold if the $Y(t)$ are in $\mathcal{F}_\sigma$ (actually, since every meager set is contained in a meager $\mathcal{F}_\sigma$-set, it is enough to reduce to this situation).

Moreover, the above question has positive answer if $\mathbb{K}$ is o-minimal, because then each Y_t has (o-minimal) dimension less than m, and therefore Y has dimension less than m. In fact, if $\mathbb{K}$ is o-minimal, and $Y \subseteq \mathbb{K}^m$ is definable, then Y is meager iff $\dim Y < m$; moreover, if $\big(Y(t)\big)_{0<t\in\mathbb{K}}$ is a definable family, decreasing in t, then $\bigcup_t Y(t) \subseteq \operatorname{acc}_{t\to 0} Y(t)$. Thus, the following lemma proves what we want.

Lemma 6.5. *Let $\mathbb{K}$ be an o-minimal structure, $n \leq m \in \mathbb{N}$, and $\big(Y(t)\big)_{t>0}$ be a definable family of subsets of $\mathbb{K}^m$, and $Z := \operatorname{acc}_{t\to 0} Y(t)$. If, for every $t > 0$, $\dim\big(Y(t)\big) \leq n$, then $\dim(Z) \leq n$.*

Proof. Define $W := \bigcup_{t>0} Y(t) \times \{t\} \subseteq \mathbb{K}^{m+1}$. Note that $Z = (\overline{W})_0 := \{x \in \mathbb{K}^m : (x, 0) \in \overline{W}\}$. Moreover, since $\dim Y(t) \leq n$, we have $\dim W \leq n + 1$. Since $Z \times \{0\} \subseteq \partial W := \overline{W} \setminus W$, we have $\dim Z < \dim W \leq n + 1$. □

The following is a partial result for the case of a.o. sets.

Lemma 6.6. *Let $Y \subseteq \mathbb{K}^n$ be definable and Baire, $D \subseteq Y$ be a.o. (in Y), and $(Y(t))_{t\in\mathbb{K}}$ be a definable increasing family of closed subsets of Y, such that $Y = \bigcup_t Y(t)$. Then, D is meager in Y iff each $D \cap Y(t)$ is meager (in Y).*

Proof. The "only if" direction is clear.

For the other direction, let $C \subseteq Y$ be closed, such that $E := C \,\Delta\, D$ is meager. It suffices to prove that C is meager. For every $t \in \mathbb{K}$, define

$$C(t) := C \cap Y(t),$$
$$D(t) := D \cap Y(t).$$

Then, $D(t) \,\Delta\, C(t) \subseteq E$. Therefore, since $D(t)$ and E are meager, $C(t)$ is meager and closed. Since Y is Baire, $C(t)$ is nowhere dense, and thus C is meager. □

Let $C \subseteq \mathbb{K}^n$ be meager, and $f : \mathbb{K}^n \to \mathbb{K}^n$ be definable and $\mathcal{C}^1$. We want to investigate in which circumstances $f(C)$ is meager. When $\mathbb{K} = \mathbb{R}$, Sard's Lemma implies that $f(C)$ is meager. This suggests the following definition.

Definition 6.7. Fix $d, r, m \in \mathbb{N}$. Let $V \subseteq \mathbb{K}^d$ be a $\mathbb{K}$-manifold of dimension n. Let $f : V \to \mathbb{K}^m$ be a definable C^r function and Δ_f be the set of singular points of f. If $\Sigma_f := f(\Delta_f)$ is meager in $\mathbb{K}^m$, then we say that f has the **Sard property**.

Open problem 6.8. *Does every C^r definable function $f : V \to \mathbb{K}^m$ as above (with $r > \max\{0, n-m\}$) have the Sard property?*

Remark 6.9. If $\mathbb{K}$ is o-minimal, then every C^1 definable function $f : V \to \mathbb{K}^m$ has the Sard property [Ber–Ot01, Thm. 3.5].

Lemma 6.10. *If $\mathbb{K} = \mathbb{R}$ and $f : V \to \mathbb{K}^m$ is as in the above definition, with $r > \max\{0, n-m\}$, then f has the Sard property.*

Proof. By Sard's Lemma, Σ_f has Lebesgue measure zero, and therefore it has empty interior. Since Σ_f is in $\mathcal{F}_\sigma$, it is also meager. □

Proposition 6.11. *Suppose $f : \mathbb{K}^n \to \mathbb{K}^m$ has the Sard property, and let $C \subset \mathbb{K}^n$ be meager. Then $f(C)$ is meager.*

Proof. We may assume that $C \in \mathcal{F}_\sigma$, since C is contained in a meager $\mathcal{F}_\sigma$-set. Let $\Lambda := \mathbb{K}^n \setminus \Delta_f$ be the set of regular points of f. Note that Λ is open.

By the Sard property, $f(C \cap \Delta_f)$ is meager. Hence, it suffices to show that $f(C \cap \Lambda)$ is meager. Let $x \in C \cap \Lambda$. Since x is a regular point for f, by the Implicit Function Theorem there exists a neighbourhood V of x such that f is a diffeomorphism on V; therefore, $f(C \cap V)$ is meager, and, by Corollary 3.7, $f(C \cap \Lambda)$ is meager. □

The following lemma is an analogue of the Co-area Formula.

Lemma 6.12. *Let $f : \mathbb{K}^n \to \mathbb{K}$ be a $\mathcal{C}^\infty$ definable function with the Sard property. Let Λ be the set of the regular points of f, and $C \subseteq \mathbb{K}^n$ be in $\mathcal{F}_\sigma$. For every $t \in \mathbb{K}$, let $F_t := f^{-1}(t)$, $C_t := F_t \cap C$, and $T := \{t \in \mathbb{K} : C_t \text{ is meager in } F_t\}$. Then, T is residual iff $C \cap \Lambda$ is meager.*

Proof. If $n = 0$ we have a tautology. Assume that $n \geq 1$.

Let $x \in C \cap \Lambda$. Since x is a regular point for f, there exists V open neighbourhood of x such that, up to a change of coordinates, f is the projection on the last coordinate x_n. For every $x_n \in T$, the set $C_{x_n} \cap V$ is meager in $\mathbb{K}^{n-1} \times \{x_n\}$.

Hence, if T is residual, then, by Kuratowski-Ulam's Theorem 4.1, $C \cap V$ is meager; therefore, by Corollary 3.7, $C \cap \Lambda$ is meager.

Conversely, assume that $C \cap \Lambda$ is meager; we must prove that T is residual. Since $f(\Delta_f)$ is meager, it suffices to prove that $T(C \cap \Lambda)$ is residual. Therefore, w.l.o.g. we can assume that $C \subseteq \Lambda$. Again by Kuratowski-Ulam's Theorem, the set $T(V \cap C) := \{x_n \in \mathbb{K} : C_{x_n} \cap V \text{ is meager in } \mathbb{K}^{n-1}\}$ is residual. Therefore, T is residual. □

In the following subsection we produce examples of classes of functions in definably complete and Baire structures, which have the Sard property.

6.1 The Sard property and Noetherian Differential Rings

Notation 6.13. Fix $n \in \mathbb{N} \setminus \{0\}$ and a definably connected definable open set $U \subseteq \mathbb{K}^n$. Let $C^\infty(U, \mathbb{K})$ be the ring of definable C^∞ functions from U to $\mathbb{K}$.

Definition 6.14. A ring M with the following properties

- $M \subseteq C^\infty(U, \mathbb{K})$;
- M is noetherian;
- M is closed under partial differentiation;
- $M \supseteq \mathbb{Z}[x_1, \dots, x_n]$.

is called a *Noetherian differential ring.*

If $G := (g_1, \dots, g_k) \in M^k$, we denote by $V(G)$ the set of zeroes of G, and by $V^{\text{reg}}(G)$ the set of *regular* zeroes of G.

Generalities on Noetherian differential rings of functions over definably complete structures can be found in [Servi07]. In particular, we will need the following result, which states that in a Noetherian differential ring there are no flat functions.

Proposition 6.15. *Let $M \subseteq C^\infty(U, \mathbb{K})$ be a Noetherian differential ring and let $0 \not\equiv g \in M$. Then for every $x \in U$ such that $g(x) = 0$, there exist $k \in \mathbb{N}$ and a derivative θ of order k such that $\theta g(x) \neq 0$.*

Fix a Noetherian differential ring $M \subseteq C^\infty(U, \mathbb{K})$.

Remark 6.16. For $g_1, \dots, g_k \in M$, the set $V := V^{\mathrm{reg}}(g_1, \dots, g_k)$ is in $\mathcal{F}_\sigma$; in fact consider the following closed definable subset of $U \times \mathbb{K}$:

$$C := \bigcup_{E(x)} \{(x, y) \in U \times \mathbb{K} : \bigwedge_{i=1}^{k} g_i(x) = 0 \wedge \det(E(x))y - 1 = 0\},$$

where $E(x)$ ranges over all maximal rank minors of the Jacobian matrix of $(g_1, \dots, g_k)$ in x. Now, $V = \Pi^{n+1}_n(C)$; since C is an $\mathcal{F}_\sigma$ of $\mathbb{K}^{n+1}$ and Π^{n+1}_n is continuous, V is also an $\mathcal{F}_\sigma$.

In this subsection we prove the following version of Sard's Lemma:

Theorem 6.17. *Fix $k, m \in \mathbb{N}$, $k \le n$. Let*

- *$H = (h_1, \dots, h_{n-k}) \in M^{n-k}$ and $V := V^{\mathrm{reg}}(H) \neq \emptyset$;*
- *$F = (F_1, \dots, F_m) \in M^m$ and $f := F \restriction V : V \to \mathbb{K}^m$;*
- *$\Delta_f \subseteq V$ be the set of singular points of f, and $\Sigma_f := f(\Delta_f)$ be the set of singular values of f.*

Then, Σ_f is a meager set (in $\mathbb{K}^m$).

Proof. We proceed by induction on $\dim V$ and m. If $m = 0$, there are no singular points. If $\dim V = 0$, then V is discrete. In particular, for every $a \in \Delta_f$ there exists U_a neighbourhood of a such that $\Delta_f \cap U_a = \{a\}$. Hence we can apply Corollary 3.7 and we are done.

Consider now the general case.

Claim 1. We can restrict to the case $V = \mathbb{K}^k$. By Corollary 3.7, it suffice to prove that for every $a \in \Delta_f$ there exists a neighbourhood U_a of a such that $f(U_a \cap \Delta_f)$ is meager. Fix $a \in \Delta_f$. Using the Implicit Function Theorem, it is easy to check that there is a neighbourhood U_a of a and a definable diffeomorphism $\Phi : \mathbb{K}^k \to V \cap U_a$ such that $H \circ \Phi \equiv 0$ and each $F_i \circ \Phi$ belong to a Noetherian differential ring $M' \subseteq C^\infty(\mathbb{K}^k, \mathbb{K})$ (see [Servi07] for the details). Hence Claim 1 is proved and we may assume that $f : \mathbb{K}^k \to \mathbb{K}^m$, and $f \in M \subseteq C^\infty(\mathbb{K}^k, \mathbb{K})$.

Let $X_0 := \{a \in \Delta_f : Df(a) \neq 0\}$, where Df is the Jacobian matrix of f. We first prove that $f(X_0)$ is meager.

Again by Corollary 3.7, it suffice to prove that for every $a \in X_0$ there exists a neighbourhood U_a of a such that $f(U_a \cap X_0)$ is meager.

Fix $a \in X_0$.

Claim 2. We may assume that $f(x) = (x_1, f_2(x), \dots, f_m(x))$. In fact, since $Df(a) \neq 0$, w.l.o.g. we can assume that $\partial f_1(a)/\partial x_1 \neq 0$ and $a = 0$.

Consider definable neighbourhoods O and $\tilde{O} \subset \mathbb{K}^k$ of 0, where the following map is a diffeomorphism:

$$\begin{array}{rccc} G: & O & \to & \tilde{O} \\ & x & \mapsto & (f_1(x), x_2, \dots, x_k). \end{array}$$

The ring $\tilde{M} := \{g \circ G^{-1} |\ g \in M\} \subset C^\infty(\tilde{O}, \mathbb{K})$ is clearly Noetherian and differentially closed and $\tilde{f} := f \circ G^{-1} \in \tilde{M}$. Since G is a diffeomorphism, it is enough to prove the statement for $\tilde{M}$ and $\tilde{f}$, and Claim 2 is proved.

For every $t \in \mathbb{K}$, consider the Noetherian differential ring

$$N_t := \{g_t := g(t, x_2, \dots, x_k) |\ g \in M\} \subset C^\infty(\tilde{O} \cap \mathbb{K}^{k-1}, \mathbb{K}).$$

Let $f_t : \mathbb{K}^{k-1} \to \mathbb{K}^{m-1}$ be the map $((f_2)_t, \dots, (f_m)_t)$. By inductive hypothesis, the set Σ_{f_t} is meager in $\mathbb{K}^{m-1}$. Moreover, $f(X_0 \cap \tilde{O}) \cap (\{t\} \times \mathbb{K}^{m-1}) \subseteq \{t\} \times \Sigma_{f_t}$. Hence $f(X_0 \cap \tilde{O}) \subseteq D := \{(t, y) \in \mathbb{K} \times \mathbb{K}^{k-1} |\ y \in \Sigma_{f_t}\}$. By what we have just observed, $T(D) := \{t \in \mathbb{K} : D_t \text{ is meager}\}$ is residual, because $D_t = \Sigma_{f_t}$, hence by Lemma 5.6, D is meager. It follows by Corollary 3.7 that $f(X_0)$ is meager.

Now, let $a \in \Delta_f$ such that $Df(a) = 0$, and let P be the least natural number such that there exists $i \leq m$ and a derivative θ of order P such that, if $g_\theta := \theta f_i$, then $g_\theta(a) = 0$ and $Dg_\theta(a) \neq 0$. Such a P exists by Proposition 6.15. Let $W_\theta := V^{\text{reg}}(g_\theta) \subset \mathbb{K}^k$ (notice that the inclusion is strict, hence $\dim W_\theta < k$). Then there is a definable open neighbourhood O of a such that

$$\Delta_f \cap O \subseteq \bigcup_{\text{ord}(\theta) \leq P} W_\theta.$$

Hence it is enough to prove that $f(\Delta_f \cap W_\theta)$ is meager. Let $h_\theta := f \restriction W_\theta$. By inductive hypothesis, Σ_{h_θ} is meager. Note that if $x \in W_\theta$ is a singular point for f, then x is also a singular point for h_θ; that is, $\Delta_f \cap W_\theta \subseteq \Delta_{h_\theta}$, and we are done.

□

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