On expansions of weakly o-minimal non-valuational structures by convex predicates

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ABSTRACT

We prove that if $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational structure expanding an ordered group $(M, \leq, +)$, then its expansion by a family of 'non-valuational' unary predicates remains non-valuational. The paper is based on the author's earlier work on strong cell decomposition for weakly o-minimal non-valuational expansions of ordered groups.

0 Introduction

Several examples of weakly o-minimal structures are obtained as expansions of o-minimal structures by predicates interpreted as certain convex sets [MMS]. Among these we have an expansion of a real closed field by a valuation ring and an expansion of the ordered field of real algebraic numbers by a predicate interpreted as $(-\alpha, \alpha)$, where α is a transcendental number. Structures of the first sort were investigated by L. van den Dries and A. H. Lewenberg (see [DL] and [D2]) who showed for instance that if $\mathcal{R}$ is an o-minimal expansion of a real closed field and V is a proper non-empty convex subring closed under 0-definable continuous functions, then the expansion $(\mathcal{R}, V)$ is weakly o-minimal. The phenomenon occuring in all the mentioned examples was addressed in general by Y. Baisalov and B. Poizat (see [BP]) who proved that an expansion of any o-minimal structure by a family of convex predicates has weakly o-minimal theory (so in particular it is a weakly o-minimal structure). The result of [BP] was generalized by B. Baizhanov (see [Bz]) who proved that an expansion of a model of a weakly o-minimal theory by a family of convex predicates has weakly o-minimal theory. It is worth mentioning that Baizhanov's theorem has also an easy proof when one uses the fact that weakly o-minimal theories do not have the independence property and the result of S. Shelah (see [Sh783]) concerning quantifier elimination for the theory of an expansion of a sufficiently saturated model of a theory without the independence property by all externally definable sets. The question of G. Cherlin whether an expansion of a weakly o-minimal structure (not necessarily with weakly o-minimal theory) by a family of convex predicates is also weakly o-minimal still remains an open problem.

This paper is a sequel to the study of expansions of weakly o-minimal structures by convex predicates. In [We07] I introduced weakly o-minimal non-valuational expansions of ordered groups as a natural generalization of weakly o-minimal non-valuational expansions of real closed fields considered in [MMS]. A weakly o-minimal expansion of a real closed field is said to be non-valuational iff it does not define a non-trivial valuation. Similarly, a weakly o-minimal expansion $\mathcal{M}$ of an ordered group $(M, \leq, +)$ is called non-valuational (or of non-valuational type) iff there is no proper and non-trivial subgroup of $(M, +)$ definable in $\mathcal{M}$. Being of non-valuational type is

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equivalent to several conditions which are discussed in [We07]. One of them says that the distance between the two parts of a definable cut is zero (the precise definition appears in §1).

Assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group. If $P \subseteq M$ is a finite union of convex sets, then P in a natural way determines a finite family of cuts in $(M, \leq)$. If the parts of all these cuts are 'close to' each other, then P is said to be non-valuational. Moreover, the expansion $(\mathcal{M}, P)$ is interdefinable with some expansion of $\mathcal{M}$ by a family of convex predicates. By [We07], $Th(\mathcal{M})$ is weakly o-minimal. Therefore by [Bz] also any expansion of $\mathcal{M}$ by a family of non-valuational predicates has weakly o-minimal theory. The main result of this paper is Theorem 2.11, which says that every expansion of $\mathcal{M}$ by a family of non-valuational predicates is of non-valuational type. We also show that the theory of such an expansion is weakly o-minimal without using Baizhanov's theorem (cf. Corollary 2.10).

The paper is organized as follows. In §1 we fix our notation and terminology, and recall some particularly useful results, mainly from [MMS] and [We07]. In §2 we outline the proofs of results mentioned above.

Last but not least, I would like to thank the referee whose comments allowed me to improve the quality of the paper.

1 Notation and preliminaries

Let $(M, \leq)$ be a dense linear ordering without endpoints. A set $I \subseteq M$ is called *convex* in $(M, \leq)$ iff for any $a, b \in I$ and $c \in M$ with $a \leq c \leq b$ we have that $c \in I$. If additionally $I \neq \emptyset$ and $\inf I, \sup I \in M \cup \{-\infty, +\infty\}$, then I is called an *interval* in $(M, \leq)$. A maximal convex subset of a non-empty subset of M is called a *convex component* of it. A pair $\langle C, D \rangle$ of non-empty subsets of M is called a *cut* in $(M, \leq)$ iff $C < D$, $C \cup D = M$ and D has no lowest element. A first order structure $\mathcal{M} = (M, \leq, \ldots)$ expanding $(M, \leq)$ is said to be *weakly o-minimal* iff every subset of M, definable in $\mathcal{M}$, is a finite union of convex sets. A complete first order theory is called weakly o-minimal iff all its models are weakly o-minimal. The following remark characterizes weakly o-minimal structures in terms of sets definable in them.

Remark 1.1 *Assume that $(M, \leq)$ is a dense linear ordering without endpoints and for $m \in \mathbb{N}_+$, $\mathcal{D}_m$ is a family of subsets of M^m for which the following conditions are satisfied.*

(a) If $I \subseteq M$ is an interval, then $I \in \mathcal{D}_1$.

(b) If $X \in \mathcal{D}_1$, then X is a union of finitely many convex sets.

(c) $\{\langle x, y \rangle \in M^2 : x < y\} \in \mathcal{D}_2$.

(d) $\mathcal{D}_m$ with the usual set-theoretic operations is a Boolean algebra.

(e) If $X \in \mathcal{D}_m$, then $X \times M, M \times X \in \mathcal{D}_{m+1}$.

(f) If $X \in \mathcal{D}_{m+1}$ and $\pi : M^{m+1} \longrightarrow M^m$ is the projection dropping the last coordinate, then $\pi[X] \in \mathcal{D}_m$.

(g) If $1 \leq i < j \leq m$, then $\{(x_1, \ldots, x_m) \in M^m : x_i = x_j\} \in \mathcal{D}_m$.
Then there is a weakly o-minimal structure $\mathcal{M}$ expanding $(M, \leq)$ such that for every $X \subseteq M^m$, X is definable in $\mathcal{M}$ iff $X \in \mathcal{D}_m$.

Note that if one replaces 'convex sets' in (b) with 'intervals', then (a)-(g) in Remark 1.1 imply that there is an o-minimal expansion $\mathcal{M}$ of $(M, \leq)$ such that for every $X \subseteq M^m$, X is definable in $\mathcal{M}$ iff $X \in \mathcal{D}_m$ (cf. [D1], Chapter 1).

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure. A cut $\langle C, D \rangle$ in $(M, \leq)$ is called *definable* in $\mathcal{M}$ iff the sets C, D are definable in $\mathcal{M}$. The set of all such cuts will be denoted by

$\overline{M}^{\mathcal{M}}$. The set M can be regarded as a subset of $\overline{M}^{\mathcal{M}}$ by identifying an element $a \in M$ with the cut $\langle(-\infty, a], (a, +\infty)\rangle$. After such an identification, $\overline{M}^{\mathcal{M}}$ is naturally equipped with a dense linear ordering without endpoints extending that of $(M, \leq)$, and $(M, \leq)$ is dense in $(\overline{M}^{\mathcal{M}}, \leq)$. For a definable set $X \subseteq M^m$, a function $f : X \longrightarrow \overline{M}^{\mathcal{M}}$ is said to be *definable* in $\mathcal{M}$ iff the set $\{\langle \overline{x}, y\rangle \in X \times M : f(\overline{x}) > y\}$ is definable in $\mathcal{M}$.

Now assume that $\mathcal{M} = (M, \leq, + \ldots)$ is a weakly o-minimal structure expanding an ordered group $(M, \leq, +)$. A cut $\langle C, D\rangle$ is called *non-valuational* iff $\inf\{y - x : x \in C, y \in D\} = 0$. We will say that $\mathcal{M}$ is *non-valuational* (or *of non-valuational type*) iff all cuts definable in $\mathcal{M}$ are non-valuational. If $\mathcal{M}$ is non-valuational, then $\overline{M}^{\mathcal{M}}$ can be naturally equipped with an ordered group structure extending that of $(M, \leq, +)$. The ordered groups $(M, \leq, +)$ and $(\overline{M}^{\mathcal{M}}, \leq, +)$ are divisible, abelian and torsion free. For details we refer the reader to §1 of [We07].

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure. Below, for every $m \in \mathbb{N}_+$ we inductively define *strong cells* in M^m and their *completions* in $(\overline{M}^{\mathcal{M}})^m$. The completion in $(\overline{M}^{\mathcal{M}})^m$ of a strong cell $C \subseteq M^m$ will be denoted by $\overline{C}$.

(1) A one-element subset of M is a strong $\langle 0\rangle$-cell in M and is equal to its completion.

(2) A non-empty convex open definable subset of M is a strong $\langle 1\rangle$-cell in M. If $C \subseteq M$ is a strong $\langle 1\rangle$-cell in M, then $\overline{C} := \{x \in \overline{M}^{\mathcal{M}} : (\exists a, b \in C)(a < x < b)\}$.

Assume that $m \in \mathbb{N}_+$, $i_1, \ldots, i_m \in \{0, 1\}$ and suppose that we have already defined strong $\langle i_1, \ldots, i_m\rangle$-cells in M^m and their completions in $(\overline{M}^{\mathcal{M}})^m$.

(3) If $C \subseteq M^m$ is a strong $\langle i_1, \ldots, i_m\rangle$-cell in M^m and $f : C \longrightarrow M$ is a continuous definable function which has a continuous extension $\overline{f} : \overline{C} \longrightarrow \overline{M}^{\mathcal{M}}$, then $\Gamma(f)$, the graph of f, is a strong $\langle i_1, \ldots, i_m, 0\rangle$-cell in M^{m+1}. The completion of $\Gamma(f)$ in $(\overline{M}^{\mathcal{M}})^{m+1}$ is defined as $\Gamma(\overline{f})$.

(4) If $C \subseteq M^m$ is a strong $\langle i_1, \ldots, i_m\rangle$-cell in M^m and $f, g : C \longrightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$ are continuous definable functions which have continuous extensions $\overline{f}, \overline{g} : \overline{C} \longrightarrow \overline{M}^{\mathcal{M}}$ such that $\overline{f}(\overline{x}) < \overline{g}(\overline{x})$ for $\overline{x} \in \overline{C}$, then the set

$$(f, g)_C := \{\langle \overline{a}, b\rangle \in C \times M : f(\overline{a}) < b < g(\overline{a})\}$$

is called a strong $\langle i_1, \ldots, i_m, 1\rangle$-cell in M^m. The completion of $(f, g)_C$ in $(\overline{M}^{\mathcal{M}})^{m+1}$ is defined as

$$\overline{(f, g)_C} := (\overline{f}, \overline{g})_{\overline{C}} := \{\langle \overline{a}, b\rangle \in \overline{C} \times \overline{M}^{\mathcal{M}} : \overline{f}(\overline{a}) < b < \overline{g}(\overline{a})\}.$$

(5) We say that $C \subseteq M^m$ is a strong cell in M^m iff there are $i_1, \ldots, i_m \in \{0, 1\}$ such that C is a strong $\langle i_1, \ldots, i_m\rangle$-cell in M^m.

If $C \subseteq M^m$ is a strong cell, then a definable function $f : C \longrightarrow \overline{M}^{\mathcal{M}}$ is called *strongly continuous* iff f has a continuous extension $\overline{f} : \overline{C} \longrightarrow \overline{M}^{\mathcal{M}}$. In a standard way we define the notion of *decomposition* of a definable set into strong cells partitioning a given definable set (for details we refer the reader to §2 of [We07]). We will say that $\mathcal{M}$ has the *strong cell decomposition property* iff for any $m, k \in \mathbb{N}_+$ and any definable sets $X_1, \ldots, X_k \subseteq M^m$, there is a decomposition of M^m into strong cells partitioning each of the sets $X_1, \ldots, X_k$.

Fact 1.2 *[MMS, Ar] Let $\mathcal{M} = (M, \leq, \ldots)$ be a weakly o-minimal structure and $A \subseteq M$. If $U \subseteq M$ is an infinite A-definable set and $f : U \longrightarrow \overline{M}^{\mathcal{M}}$ is an A-definable function, then there is a partition of U into A-definable sets $X, I_0, \ldots, I_m$ such that X is finite, $I_0, \ldots, I_m$ are non-empty convex open sets, and for every $i \leq m$, $f \restriction I_i$ is locally constant or locally strictly increasing or locally strictly decreasing.*

The following fact easily follows from the definition of strong cells.

Fact 1.3 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property and $A \subseteq M$. If $U \subseteq M$ is a non-empty A-definable set and $f : U \longrightarrow \overline{M}^{\mathcal{M}}$ is an A-definable function, then there is a partition of U into A-definable sets $X, I_0, \ldots, I_m$ such that X is finite, $I_0, \ldots, I_m$ are non-empty convex open sets, and for every $i \leq m$, $f \restriction I_i$ is strongly continuous and strictly monotone or constant.*

Proof. By assumption, there is a decomposition $\mathcal{C}$ of M^2 into A-definable strong cells partitioning the set $\{\langle x, y \rangle \in U \times M : y < f(x)\}$. This yields a decomposition of U into finitely many A-definable convex open sets $J_0, \ldots, J_k$ and a finite set X such that

$$\{J_0, \ldots, J_k\} \cup \{\{a\} : a \in X\} = \{\pi[C] : C \in \mathcal{C}\},$$

where $\pi : M^2 \longrightarrow M$ is the projection dropping the second coordinate. By our definition of strong cells, $f \restriction J_i$ is strongly continuous for $i \leq m$. In such a situation, weak o-minimality of $\mathcal{M}$ implies that each of the J_i's could be decomposed into finitely many A-definable convex open sets and some finite set so that on each of the open sets, f is strictly monotone or constant. This finishes the proof. ■

Theorem 1.4 *[We07] Let $\mathcal{M} = (M, \leq, +, \ldots)$ be a weakly o-minimal structure expanding an ordered group $(M, \leq, +)$. Then $\mathcal{M}$ is of non-valuational type iff $\mathcal{M}$ has the strong cell decomposition property.*

2 The main result

Throughout this section we shall work in a weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$, usually assuming that $\mathcal{M}$ has the strong cell decomposition property or is a non-valuational expansion of an ordered group. By C we will denote a non-empty, convex and non-definable (in $\mathcal{M}$) proper subset of M such that $\inf C = -\infty$. Then $\langle C, M \setminus C \rangle$ is a cut in $(M, \leq)$ (according to the terminology of [Bz] and [BVT] such cuts are called irrational). By Corollary 2.15 from [We07], we know that if $\mathcal{M}$ is a non-valuational expansion of an ordered group, then $Th(\mathcal{M})$ is weakly o-minimal. Hence, by [Bz], also $Th(\mathcal{M}, C)$ is weakly o-minimal. In a series of lemmas below we will show that if $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group and the cut $\langle C, M \setminus C \rangle$ is non-valuational, then also the expansion $(\mathcal{M}, C)$ is of non-valuational type. Moreover, we will give a direct proof of the weak o-minimality of $Th(\mathcal{M}, C)$.

For $m \in \mathbb{N}_+$ and $i \in \{1, \ldots, m+1\}$, by π_i^{m+1} we will denote the projection from M^{m+1} to M^m dropping the i-th coordinate and by $\mathcal{D}_m(\mathcal{M})$ the family of all subsets of M^m definable in $\mathcal{M}$.

Lemma 2.1 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure and $I \subseteq M$ is a convex open definable set such that $I \cap C \neq \emptyset$ and $I \setminus C \neq \emptyset$. If $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ is a definable function such that $(\forall x \in I)(f(x) > c)$ [$(\forall x \in I)(f(x) < c)$] for some $c \in M$, then there are an open interval*

$J \subseteq I$ and $\alpha > c$ [$\alpha < c$], $\alpha \in M$, such that $J \cap C \neq \emptyset$, $J \setminus C \neq \emptyset$ and $(\forall x \in J)(f(x) > \alpha)$ [$(\forall x \in J)(f(x) < \alpha)$].

Proof. Let $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ be a definable function such that $(\forall x \in I)(f(x) > c)$, where $c \in M$. By Fact 1.2 and the non-definability of C, there is an open interval $I_1 \subseteq I$ such that $I_1 \cap C \neq \emptyset$, $I_1 \setminus C \neq \emptyset$ and $f \restriction I_1$ is either strictly monotone or constant. Fix $a \in I_1 \cap C$, $b \in I_1 \setminus C$ and $\alpha \in M$ such that $c < \alpha < \min\{f(a), f(b)\}$. It is clear that $f(x) > \alpha$ whenever $x \in (a, b)$. The other case is proved in a similar way. ■

Lemma 2.2 *Let $\mathcal{M} = (M, \leq, \ldots)$ be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $I \subseteq M$ is a non-empty convex open definable set such that $I \cap C \neq \emptyset$ and $I \setminus C \neq \emptyset$, and $f, g : I \longrightarrow \overline{M}^{\mathcal{M}}$ are definable functions such that $(\forall x \in I)(f(x) < g(x))$. There are an element $a \in M$ and an open interval $J \subseteq I$ such that $J \cap C \neq \emptyset$, $J \setminus C \neq \emptyset$ and $f(x) < a < g(x)$ for $x \in J$.*

Proof. By Theorem 1.4 and Fact 1.3, without loss of generality we can assume that the functions f, g are strongly continuous. By Lemma 2.1, there are an open interval $I_1 \subseteq I$ and an element $\alpha \in M$, $\alpha > 0$, such that $I_1 \cap C \neq \emptyset$, $I_1 \setminus C \neq \emptyset$ and $(\forall x \in I_1)(g(x) - f(x) > \alpha)$. Fix $\varepsilon > 0$, $\varepsilon \in M$, such that $2\varepsilon < \alpha$. For $x_0 \in I_1$ define

$$\begin{aligned}
\delta_f(x_0) &= \min(\varepsilon, \sup\{d \in M : d > 0, (\forall x \in (x_0 - d, x_0 + d) \cap I_1)(|f(x) - f(x_0)| < \varepsilon)\}) \\
\delta_g(x_0) &= \min(\varepsilon, \sup\{d \in M : d > 0, (\forall x \in (x_0 - d, x_0 + d) \cap I_1)(|g(x) - g(x_0)| < \varepsilon)\}).
\end{aligned}$$

Again, by Lemma 2.1, there are an open interval $I_2 \subseteq I_1$ and an element $\beta \in M$, $\beta > 0$, such that $I_2 \cap C \neq \emptyset$, $I_2 \setminus C \neq \emptyset$ and $\min(\delta_f(x_0), \delta_g(x_0)) > \beta$ for $x_0 \in I_2$. Fix $c_1 \in I_2 \cap C$ and $c_2 \in I_2 \setminus C$ such that $c_2 - c_1 < \beta$. For $x_1, x_2 \in (c_1, c_2)$ we have that

$$g(x_2) - f(x_1) = (g(x_2) - g(x_1)) + (g(x_1) - f(x_1)) > -\varepsilon + 2\varepsilon = \varepsilon.$$

Consequently,

$$\inf\{g(x) : x \in (c_1, c_2)\} - \sup\{f(x) : x \in (c_1, c_2)\} \geq \varepsilon.$$

If $a \in M$ is such that $\sup\{f(x) : x \in (c_1, c_2)\} < a < \inf\{g(x) : x \in (c_1, c_2)\}$, then for $x \in (c_1, c_2)$ we have that $f(x) < a < g(x)$. ■

Lemma 2.3 *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. Assume that $X \subseteq M^2$ is a set definable in $\mathcal{M}$ such that for any $a, b \in M$, if $\langle a, b\rangle \in X$, then there are $a_1, a_2 \in M$ such that $a_1 < a < a_2$ and $(a_1, a_2) \times \{b\} \subseteq X$. The following conditions are equivalent.*
(a) There are $a_1 \in C$, $a_2 \in M \setminus C$ and $b \in M$ such that $(a_1, a_2) \times \{b\} \subseteq X$.
(b) There are $a_1 \in C$ and $a_2 \in M \setminus C$ such that $(a_1, a_2) \subseteq \pi^2_2[X]$.

Proof. The implication from (a) to (b) is obvious, so assume that (b) holds. Let $\mathcal{C}$ be a decomposition of M^2 into strong cells in M^2 partitioning the set X. There is a convex open definable set $I \subseteq M$ such that $I \cap C \neq \emptyset$, $I \setminus C \neq \emptyset$ and for every $D \in \mathcal{C}$, we have that either $I = \pi^2_2[D]$ or $\pi^2_2[D] \cap I = \emptyset$. Below we consider two cases.

Case 1. The set $(I \times M) \cap X$ has empty interior. The following claim is a consequence of Lemma 2.1 from [We06] but for the sake of completeness we give a proof in our particular situation.

Claim. For every $x \in I$, the set $\{y \in M : \langle x, y\rangle \in X\}$ is finite.

Proof of the Claim. Suppose for a contradiction that for some $a \in I$, the set $\{y \in M : \langle a, y\rangle \in X\}$ is infinite, so it contains an open interval J. For $b \in J$ define

$$f(b) = \sup\{c \in M : c > a \text{ and } \{a\} \times (b, c) \subseteq (I \times M) \cap X\}.$$

The function f assumes values greater than a in $\overline{M}^{\mathcal{M}} \cup \{+\infty\}$. By Fact 1.3, there is an open interval $J' \subseteq J$ such that $f \restriction J'$ is strongly continuous and strictly monotone or constant. It is clear that the set $\{\langle x, y\rangle : a < x < f(y), y \in J'\}$ is contained in $(I \times M) \cap X$ and contains an open box itself. This means that $(I \times M) \cap X$ has non-empty interior, a contradiction.

Using the Claim, for $x \in I$, we can define $f(x) = \min\{y \in M : \langle x, y\rangle \in X\}$. Our assumptions guarantee that f is constant (say $f(x) = b$ whenever $x \in I$), so for any $a_1 \in I \cap C$ and $a_2 \in I \setminus C$, we have that $(a_1, a_2) \times \{b\} \subseteq X$.

Case 2. There are definable strongly continuous functions $f : I \longrightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty\}$ and $g : I \longrightarrow \overline{M}^{\mathcal{M}} \cup \{+\infty\}$ such that $\overline{f}(a) < \overline{g}(a)$ for $a \in \overline{I}$ and $D := (f, g)_I \subseteq X \cap (I \times M)$. By Lemma 2.2, there are $a_1 \in I \cap C$, $a_2 \in I \setminus C$ and $b \in M$ such that $(a_1, a_2) \times \{b\} \subseteq D \subseteq X$. ■

For a weakly o-minimal structure $\mathcal{M}$ and a set $X \in \mathcal{D}_{m+1}(\mathcal{M})$ let

$$I(X, C) = \{\overline{a} \in M^m : (\exists b \in C)(\exists c \in M \setminus C)(\{\overline{a}\} \times (b, c) \subseteq X)\}.$$

Define also $\mathcal{E}_m(\mathcal{M}, C) = \{I(X, C) : X \in \mathcal{D}_{m+1}(\mathcal{M})\}$. Below we will show that if $\mathcal{M}$ is a non-valuational expansion of an ordered group, then $\mathcal{E}_m(\mathcal{M}, C)$ is exactly the family of subsets of M^m definable in $(\mathcal{M}, C)$.

Lemma 2.4 *Let $\mathcal{M}$ be a weakly o-minimal structure with the strong cell decomposition property.*
(a) If $X \in \mathcal{D}_2(\mathcal{M})$ is a strong cell, then $I(\{\langle x, y\rangle \in M^2 : \langle y, x\rangle \in X\}, C)$ is a convex set.
(b) If $Y \in \mathcal{E}_1(\mathcal{M}, C)$, then Y is a finite union of convex sets.
(c) $C \in \mathcal{E}_1(\mathcal{M}, C)$.

Proof. (a) is obvious from the definition of strong cells. (b) follows from (a) and the strong cell decomposition property of $\mathcal{M}$. For the proof of (c), note that

$$C = I(\{\langle x, y\rangle \in M^2 : y > x\}, C).$$

Here the strong cell decomposition property in not needed. ■

Lemma 2.5 *Assume that $\mathcal{M}$ is a weakly o-minimal structure and $m \in \mathbb{N}_+$.*
(a) $\mathcal{D}_m(\mathcal{M}) \subseteq \mathcal{E}_m(\mathcal{M}, C)$.
(b) $\mathcal{E}_m(\mathcal{M}, C)$ is closed under Boolean operations.
(c) If $X \in \mathcal{E}_m(\mathcal{M}, C)$, then $X \times M, M \times X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$.

Proof. (a) If $m \in \mathbb{N}_+$ and $X \in \mathcal{D}_m(\mathcal{M})$, then $X = I(X \times M, C)$.

(b) Fix $m \in \mathbb{N}_+$ and $X, Y \in \mathcal{E}_m(\mathcal{M}, C)$. There are $X_1, Y_1 \in \mathcal{D}_{m+1}(\mathcal{M})$ such that $X = I(X_1, C)$ and $Y = I(Y_1, C)$. Clearly, $X \cup Y = I(X_1 \cup Y_1, C)$, $X \cap Y = I(X_1 \cap Y_1, C)$ and $M^m \setminus X = I(M^{m+1} \setminus X_1, C)$.

(c) Let $m \in \mathbb{N}_+$ and $X \in \mathcal{E}_m(\mathcal{M}, C)$. Then $X = I(X_1, C)$ for some $X_1 \in \mathcal{D}_{m+1}(\mathcal{M})$. Hence $M \times X = I(M \times X_1, C)$ and $X \times M = I(\{\langle \overline{x}, y, z\rangle \in M^{m+1} : \langle \overline{x}, z\rangle \in X\}, C)$. ■

Lemma 2.6 *Let $\mathcal{M}$ be a weakly o-minimal expansion of an ordered group. If $X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$, then $\pi^{m+1}_{m+1}[X] \in \mathcal{E}_m(\mathcal{M}, C)$.*

Proof. Fix $m \in \mathbb{N}_+$ and $X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$. There is $X_1 \in \mathcal{D}_{m+2}(\mathcal{M})$ such that $X = I(X_1, C)$. Let

$$X_2 = \bigcup\{\{\overline{a}\} \times (b, c) : \overline{a} \in M^{m+1}, b \in C, c \in M \setminus C \text{ and } \{\overline{a}\} \times (b, c) \subseteq X_1\}.$$

Clearly, $X = I(X_2, C)$ and $X_2 \in \mathcal{D}_{m+2}(\mathcal{M})$. We claim that $\pi^{m+1}_{m+1}[X] = I(\pi^{m+2}_{m+1}[X_2], C)$.

In order to prove that $\pi^{m+1}_{m+1}[X] \subseteq I(\pi^{m+2}_{m+1}[X_2], C)$, fix $\overline{a} \in \pi^{m+1}_{m+1}[X]$. There is $b \in M$ such that $\langle \overline{a}, b\rangle \in X = I(X_2, C)$. So there are $c \in C$ and $d \in M \setminus C$ such that $\{\langle \overline{a}, b\rangle\} \times (c, d) \subseteq X_2$. Hence $\{\overline{a}\} \times (c, d) \subseteq \pi^{m+2}_{m+1}[X_2]$ and $\overline{a} \in I(\pi^{m+2}_{m+1}[X_2], C)$.

For the reverse inclusion, let $\overline{a} \in I(\pi^{m+2}_{m+1}[X_2], C)$. There are $c \in C$ and $d \in M \setminus C$ such that $\{\overline{a}\} \times (c, d) \subseteq \pi^{m+2}_{m+1}[X_2]$. Let $Z = \{\langle x, y\rangle \in M^2 : \langle \overline{a}, y, x\rangle \in X_2\}$. Clearly, $(c, d) \subseteq \pi^2_2[Z]$. By Lemma 2.3, there are $c' \in C$, $d' \in M \setminus C$ and $e \in M$ such that $(c, d) \times \{e\} \subseteq Z$. Consequently, $\{\langle \overline{a}, e\rangle\} \times (c, d) \subseteq X_2$. The latter implies that $\langle \overline{a}, e\rangle \in X$ and $\overline{a} \in \pi^{m+1}_{m+1}[X]$. ■

Lemmas 2.4, 2.5 and 2.6 imply that if $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group, then

(a) all intervals in $(M, \leq)$ belong to $\mathcal{E}_1(\mathcal{M}, C)$;
(b) every set belonging to $\mathcal{E}_1(\mathcal{M}, C)$ is a union of finitely many convex sets;
(c) $\{\langle x, y\rangle \in M^2 : x < y\} \in \mathcal{E}_2(\mathcal{M}, C)$;
(d) for every $m \in \mathbb{N}_+$, $\emptyset, M^m \in \mathcal{E}_m(\mathcal{M}, C)$ and $(\mathcal{E}(\mathcal{M}, C), \cap, \cup, ^c)$ is a Boolean algebra;
(e) if $X \in \mathcal{E}_m(\mathcal{M}, C)$, then $X \times M, M \times X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$;
(f) if $1 \leq i < j \leq m$, then $\{\langle x_1, \ldots, x_m\rangle \in M^m : x_i = x_j\} \in \mathcal{E}_m(\mathcal{M}, C)$;
(g) if $X \in \mathcal{E}_{m+1}(\mathcal{M}, C)$ and $i \in \{1, \ldots, m\}$, then $\pi^{m+1}_i[X] \in \mathcal{E}_m(\mathcal{M}, C)$;
(h) $C \in \mathcal{E}_1(\mathcal{M}, C)$.

Therefore, by Remark 1.1, there is a weakly o-minimal structure $\mathcal{M}'$ expanding $\mathcal{M}$ such that a set $X \subseteq M^m$ is definable in $\mathcal{M}'$ iff $X \in \mathcal{E}_m(\mathcal{M}, C)$. In the following lemma, $\mathcal{D}_m(\mathcal{M}, C)$ denotes the family of all subsets of M^m which are definable in the structure $(\mathcal{M}, C)$.

Lemma 2.7 *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group. For every $m \in \mathbb{N}_+$, $\mathcal{E}_m(\mathcal{M}, \mathcal{C}) = \mathcal{D}_m(\mathcal{M}, C)$.*

Proof. The inclusion $\subseteq$ is obvious. That $\mathcal{D}_m(\mathcal{M}, C) \subseteq \mathcal{E}_m(\mathcal{M}, C)$ follows easily by induction from the above remark. ■

Corollary 2.8 *If $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group, then $(\mathcal{M}, C)$ is weakly o-minimal.*

Lemma 2.9 *Let $\mathcal{M} = (M, \leq, +, \ldots)$ be a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$. Assume that I is a non-empty convex open definable (in $\mathcal{M}$) set with $I \cap C \neq \emptyset$ and $I \setminus C \neq \emptyset$. Let $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ be a definable strongly continuous and strictly monotone function.*

(a) If f is strictly increasing, then

$$\langle \{a \in M : (\exists c \in C)(f(c) > a)\}, \{a \in M : (\exists d \in M \setminus C)(f(d) < a)\}\rangle$$

is a non-definable and non-valuational cut in $(M, \leq, +)$.

(b) If f is strictly decreasing, then

$$\langle\{a\in M:(\exists d\in M\setminus C)(f(d)>a)\},\{a\in M:(\exists c\in C)(f(c)<a)\}\rangle$$

is a non-definable and non-valuational cut in $(M,\leq,+)$.

Proof. As both cases are similar, we will only prove (a). Assume that f is strictly increasing and let

$$C'=\{a\in M:(\exists c\in C)(f(c)>a)\}\text{ and }D'=\{a\in M:(\exists d\in M\setminus C)(f(d)<a)\}.$$

It is clear that C' and D' are both convex definable sets with $\inf C'=-\infty$ and $\sup D'=+\infty$. The sets C',D' are disjoint since otherwise we would have $f(d)<a<f(c)$ for some $a\in M$, $c\in C$ and $d\in M\setminus C$.

To show that $C'\cup D'=M$, suppose for a contradiction that there exists an element $a\in M\setminus(C'\cup D')$. This means that $f(c)\leq a\leq f(d)$ whenever $c\in C$ and $d\in M\setminus C$. Note that if there was a $c\in C$ with $f(c)=a$, then there would be also a $c'\in C$ with $c'>c$ and $f(c')>f(c)=a$, a contradiction. So $(\forall c\in C)(f(c)<a)$ and similarly $(\forall d\in M\setminus C)(f(d)>a)$. Now,

$$C=\{x\in M:x\leq\inf I\}\cup\{x\in I:f(x)<a\},$$

which means that C is definable in $\mathcal{M}$, a contradiction. In this way we have shown that $\langle C',D'\rangle$ is a cut in $(M,\leq)$. Its non-definability is a consequence of the non-definability of C.

In order to complete the proof, suppose for a contradiction that

$$\inf\{z-y:y\in C',z\in D'\}=\inf\{f(d)-f(c):c\in C,d\in M\setminus C\}>0$$

and fix $\varepsilon>0$, $\varepsilon\in M$, such that $\varepsilon<\inf\{f(d)-f(c):c\in C,d\in M\setminus C\}$. So clearly $f(x_2)-f(x_1)>\varepsilon$ whenever $x_1\in I\cap C$ and $x_2\in I\setminus C$. For $x_0\in I$ define

$$\delta(x_0)=\min\{\varepsilon,\sup\{d\in M:d>0\text{ and }|f(x)-f(x_0)|<\varepsilon\text{ for }x\in(x_0-d,x_0+d)\cap I\}\}.$$

By Lemma 2.1, there are an open interval $J\subseteq I$ and an element $\alpha>0$, $\alpha\in M$, such that $J\cap C\neq\emptyset$, $J\setminus C\neq\emptyset$ and $(\forall x_0\in J)(\delta(x_0)>\alpha)$. Since the cut $\langle C,M\setminus C\rangle$ is non-valuational, we can choose $x_1\in J\cap C$ and $x_2\in J\setminus C$ so that $x_2-x_1<\alpha$. In such a situation we have that $\delta(x_1)>\alpha$ and $|f(x_2)-f(x_1)|<\varepsilon$, which contradicts our choice of ε. ■

Corollary 2.10 *Let $\mathcal{M}$ be a weakly o-minimal non-valuational expansion of an ordered group.*
(a) The structure $(\mathcal{M},C)$ is of non-valuational type.
(b) $Th(\mathcal{M},C)$ is weakly o-minimal.

Proof. We already know by Corollary 2.8 that $(\mathcal{M},C)$ is weakly o-minimal. To demonstrate that $(\mathcal{M},C)$ is of non-valuational type, fix $\langle D,D'\rangle$, a cut in $(M,\leq)$ which is definable in $(\mathcal{M},C)$. We have to show that $\langle D,D'\rangle$ is non-valuational. As there is nothing to do in case $D\in\mathcal{D}_1(\mathcal{M})$, suppose that D is not definable in $\mathcal{M}$. By Lemma 2.7, there exists a set $X\in\mathcal{D}_2(\mathcal{M})$ such that $D=I(X,C)$. Denote by X' the union of all sets of the form $\{a\}\times(b,c)$, where $a,b,c\in M$, $b<c$ and $\{a\}\times(b,c)\subseteq X$. Obviously, X' is definable in $\mathcal{M}$ and $I(X',C)=D$. Fix $\mathcal{C}$, a decomposition of M^2 into strong cells partitioning X' and let $I_0,\ldots,I_n$ be an enumeration of all convex sets of the form $\pi^2_2[Y]$ where $Y\in\mathcal{C}$ and $Y\subseteq X'$ such that I_i precedes I_j whenever $i<j\leq n$. Without loss of generality we can assume that the functions appearing in definitions of cells from $\mathcal{C}$ are strictly monotone or constant. Non-definability of D guarantees that there is $k\leq n$ such that

I_k is a convex open set intersecting both D and $M \setminus D$. There are strongly continuous functions $f_0, g_0, \ldots, f_m, g_m : I_k \longrightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$ such that

(a) each of $f_0, g_0, \ldots, f_m, g_m$ is either strictly monotone or constant;

(b) $\overline{f}_i(x) < \underline{g}_i(x)$ for $i \leq m$ and $x \in \overline{I}_k$;

(c) $\overline{g}_i(x) < \underline{f}_{i+1}(x)$ for $i < m$ and $x \in \overline{I}_k$;

(d) $X' \cap (I_k \times M) = (f_0, g_0)_{I_k} \cup \ldots \cup (f_m, g_m)_{I_k}$.

There is a unique $i \leq m$ such that $\sup I((f_i, g_i)_{I_k}, C) = \sup D \cap I_k$ and exactly one of the following two conditions holds.

(1) f_i is strictly increasing and $D \cap I_k = I((f_i, +\infty)_{I_k}, C)$.

(2) g_i is strictly decreasing and $D \cap I_k = I((-\infty, g_i)_{I_k}, C)$.

Since the reasoning is similar in both cases, we will only consider (1). To simplify notation let $f := f_i$. Fix $a \in C$ such that $a > \inf\{f(x) : x \in I_k\}$ and $b \in M \setminus C$ such that $b < \sup\{f(x) : x \in I_k\}$, and let $J = (a, b)$. For $y \in J$ define

$$h(y) = \sup\{x \in M : \langle x, y\rangle \in (f, +\infty)_{I_k}\}.$$

The function h is strictly increasing and strongly continuous. As

$$D = \{x \in M : (\exists x' \in C)(h(x') > x)\},$$

by Lemma 2.9, the cut $\langle D, D'\rangle$ is non-valuational.

The weak o-minimality of $Th(\mathcal{M}, C)$ now follows from the fact that $(\mathcal{M}, C)$ is a weakly o-minimal non-valuational structure and from Corollary 2.15 from [We07]. ■

Before formulating the main result, we will introduce so called unary non-valuational predicates. Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group and let $X \subseteq M$ be a finite union of convex sets. For $a \in X$ denote by $R(a, X)$ the convex component of X containing a. Similarly, for $a \in M \setminus X$, let $R(a, X)$ be the convex component of $M \setminus X$ containing a. For $a \in M$ define

$$D(a, X) = \bigcup_{\alpha \in R(a,X)} (\alpha, +\infty).$$

Clearly, if $D(a, X) \neq M$, then $\langle M \setminus D(a, X), D(a, X)\rangle$ is a cut in $(M, \leq)$. A cut $\langle C, D\rangle$ in $(M, \leq)$ is said to be determined by X if it is of the form $\langle M \setminus D(a, X), D(a, X)\rangle$ for some $a \in M$.

We say that a set $X \subseteq M$ is a unary non-valuational predicate iff X is a union of finitely many convex sets and all cuts determined by X are non-valuational.

Theorem 2.11 *Assume that $\mathcal{M}$ is a weakly o-minimal non-valuational expansion of an ordered group and $\mathcal{N}$ is an expansion of $\mathcal{M}$ by a family of non-valuational unary predicates. Then $\mathcal{N}$ is of non-valuational type.*

Proof. There is a family of convex open sets $C_i \subsetneq M$, $i \in I$, such that

- for every $i \in I$, C_i is not definable in $\mathcal{M}$ and $\inf C_i = -\infty$;
- the structures $\mathcal{N}$ and $(\mathcal{M}, C_i : i \in I)$ have the same definable sets.

Without loss of generality we can assume that I is finite, in which case the theorem follows easily by induction on $|I|$ from Corollary 2.10(b). ■

Theorem 2.11 actually shows that for weakly o-minimal expansions of ordered groups, the property of a structure having the strong cell decomposition is preserved under expansions by families of unary non-valuational predicates. Having in mind that non-valuational predicates are those which determine non-valuational cuts, one can speak of valuational/non-valuational cuts in an arbitrary weakly o-minimal structure with the strong cell decomposition property, not necessarily expanding an ordered group. More precisely, if $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property, then a cut $\langle C, D \rangle$ in $(M, \leq)$ could be called non-valuational if the structure $(\mathcal{M}, C)$ has the strong cell decomposition property. This gives us notions of "being close" and "being far" for the parts of a cut $\langle C, D \rangle$ in $(M, \leq)$ and it would be interesting to further investigate this topic, probably relating it to the canonical o-minimal extension of a weakly o-minimal structure with the strong cell decomposition property constructed in [We07].

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