D-MINIMAL EXPANSIONS OF THE REAL FIELD HAVE THE C^p ZERO SET PROPERTY

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Abstract. If $E \subseteq \mathbb{R}^n$ is closed and the structure $(\mathbb{R}, +, \cdot, E)$ is d-minimal (that is, in every structure elementarily equivalent to $(\mathbb{R}, +, \cdot, E)$, every unary definable set is a disjoint union of open intervals and finitely many discrete sets), then for each $p \in \mathbb{N}$, there exist C^p functions $f\colon \mathbb{R}^n \to \mathbb{R}$ definable in $(\mathbb{R}, +, \cdot, E)$ such that E is the zero set of f.

Throughout, p denotes a positive integer and E a closed subset of some $\mathbb{R}^n$.

We recall a result attributed to H. Whitney: There is a C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $E = Z(f) := \{\, x \in \mathbb{R}^n : f(x) = 0 \,\}$; see, *e.g.*, Krantz and Parks [12, 3.3.6] for a proof. The construction can produce f that is rather far removed from how E arose. To illustrate: If $E = \{0\} \subseteq \mathbb{R}$, then E is the zero set of the squaring function, but the zero set of the derivative of the function produced by Whitney's method has infinitely many connected components. The loose question arises: If E is well behaved in some prescribed sense, can f be chosen to be similarly well behaved? In order to make this question precise we employ a notion from mathematical logic, namely, definability in expansions of $\overline{\mathbb{R}} := (\mathbb{R}, +, \cdot, (r)_{r\in\mathbb{R}})$, the real field with constants for all real numbers; readers not familiar with this notion may consult van den Dries and Miller [3, Sections 2 and 4] for an introduction. Let $\mathfrak{R}$ denote $(\overline{\mathbb{R}}, E)$, the structure on $\overline{\mathbb{R}}$ generated by E. Unless indicated otherwise, "definable" means "definable in $\mathfrak{R}$". The question arises: Is there a definable C^∞ function $f\colon \mathbb{R}^n \to \mathbb{R}$ such that $E = Z(f)$? While visibly true for some E (say, if E is finite), it is known to be false for some very simple cases such as $E = [0, 1] \subseteq \mathbb{R}$. But often what is needed for applications is only that, for each $k \in \mathbb{N}$, there is a C^k function $f_k\colon \mathbb{R}^n \to \mathbb{R}$ such that $E = Z(f_k)$. Hence, we shall modify the question.

Given open $U \subseteq \mathbb{R}^n$, let $\mathcal{C}(U)$ denote the collection of all definable C^p functions $U \to \mathbb{R}$. Our main question: *Is there $f \in \mathcal{C}(\mathbb{R}^n)$ such that $E = Z(f)$?* If $\mathfrak{R}$ defines the set $\mathbb{Z}$ of all integers, then yes, because $(\overline{\mathbb{R}}, \mathbb{Z})$ defines all closed subsets of $\mathbb{R}^n$ (see, *e.g.*, van den Dries [2, 2.6] or Kechris [11, 37.6]), and so the result of Whitney applies. (Thus, the case that $\mathfrak{R}$ defines $\mathbb{Z}$ is of no further interest.) If $\mathfrak{R}$ is **o-minimal** (that is, every definable subset of $\mathbb{R}$ either has interior or is finite), then again yes, by [3, 4.22]. There are some situations where we know the answer under further assumptions on E alone; here are two:

Proposition A. *If $E \subseteq \mathbb{R}$ (that is, if $n = 1$), then there exist $f \in \mathcal{C}(\mathbb{R})$ such that $E = Z(f)$.*

(The proof is straightforward; see 1.1 below.)

Proposition B. *If E is a finite union of discrete sets (equivalently, countable and of finite Cantor-Bendixson rank), then there exist $f \in \mathcal{C}(\mathbb{R}^n)$ such that $E = Z(f)$.*

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(See 1.5.)

Taken in conjunction with the o-minimal case, these results suggest that our question might have a positive answer if every definable subset of $\mathbb{R}$ either has interior or is a finite union of discrete sets. We do not know if this is true, but we show that it is under a further assumption of uniformity. Following [15],* we say that $\mathfrak{R}$ is **d-minimal** (short for "discrete minimal") if for every m and definable $A \subseteq \mathbb{R}^{m+1}$ there exists $N \in \mathbb{N}$ such that for every $x \in \mathbb{R}^m$ the set $\{\, y \in \mathbb{R} : (x, y) \in A \,\}$ either has interior or is a union of N discrete sets (equivalently, by model-theoretic compactness, every unary set definable in any structure elementarily equivalent to $\mathfrak{R}$ is a disjoint union of open intervals and finitely many discrete sets). For context, history, and examples of d-minimal structures that are not o-minimal, see Friedman and Miller [6, 7], Miller and Tyne [18], and [15, 16]. Here is the main result of this paper:

Theorem A. *If $\mathfrak{R}$ is d-minimal, then there exist $f \in \mathcal{C}(\mathbb{R}^n)$ such that $E = Z(f)$.*

Corollary. *If $\mathfrak{R}$ is d-minimal and $A \subseteq \mathbb{R}^m$ is definable, then A is a finite union of sets of the form $\{\, x \in \mathbb{R}^m : f(x) = 0, g_1(x) > 0, \ldots, g_N(x) > 0 \,\}$, where $N \in \mathbb{N}$ and $f, g_1, \ldots, g_N \in \mathcal{C}(\mathbb{R}^m)$.*

Proof. By [15, Theorem 3.2] and Dougherty and Miller [1], A is a boolean combination of closed definable subsets of $\mathbb{R}^m$. Apply Theorem A. $\square$

We defer further discussion of corollaries, variants and optimality until after the proof.

1. Proofs

Given $A \subseteq \mathbb{R}^m$, let $\operatorname{int} A$ denote the interior of A and $\operatorname{cl} A$ the closure of A. (We tend to omit parentheses in circumstances where they might proliferate so long as any resulting ambiguity is resolved by context.) We put $\operatorname{fr} A = \operatorname{cl} A \setminus A$, the **frontier** of A, and $\operatorname{lc} A = A \setminus \operatorname{cl} \operatorname{fr} A$, the **locally closed points** of A (that is, the relative interior of A in $\operatorname{cl} A$). Note that $\operatorname{fr} A = \emptyset$ if and only if A is closed, and $\operatorname{fr} A$ is closed if and only if $A = \operatorname{lc} A$ if and only if A is locally closed (that is, open in its closure). We tend to write $\sim A$ instead of $\mathbb{R}^m \setminus A$ whenever m is clear from context. For $a \in \mathbb{R}^m$, we let $\mathrm{d}(x, a)$ denote the distance (in the euclidean norm) of a to A. If A is regarded as a subset of some cartesian product $X \times Y$ and $x \in X$, then A_x denotes the **fiber** of A over x, that is, $A_x = \{\, y \in Y : (x, y) \in A \,\}$.

We make no distinction between maps (or functions) and their graphs (that is, we regard functions as purely set-theoretic objects).

Let $\mathbb{I} = [0, 1]$. Given $U \subseteq \mathbb{R}^m$, put $\mathcal{C}_{\mathbb{I}}(U) = \{\, f \in \mathcal{C} : f(U) \subseteq \mathbb{I} \,\}$.

We define an auxiliary function $\hbar \in \mathcal{C}_{\mathbb{I}}(\mathbb{R})$ for use in several places by $\hbar{\restriction}\mathbb{I} = x^{2p}(3 - 2x)^p$, $\hbar{\restriction}(-\infty, 0) = 0$ and $\hbar{\restriction}(1, \infty) = 1$. Note that $\hbar$ is strictly increasing on $\mathbb{I}$.

1.1. *Proof of Proposition A.* Suppose that $\emptyset \neq E \subsetneq \mathbb{R}$. We find $f \in \mathcal{C}(\mathbb{R})$ such that $E = Z(f)$. Define $\alpha, \beta \colon \mathbb{R} \setminus E \to \mathbb{R} \cup \{\pm\infty\}$ by $\alpha(x) = \sup\bigl(E \cap (-\infty, x)\bigr)$ and $\beta(x) =$

*This is a paper in a conference proceedings that is generally unavailable electronically, but there is a version posted at https://people.math.osu.edu/miller.1987/tameness.pdf.

$\inf\bigl(E\cap(x,+\infty)\bigr)$. Define $g\in\mathcal{C}_{\mathbb{I}}(\mathbb{R})$ by $g\restriction\mathbb{I}=x^{2p}(x-1)^{2p}$ and $g\restriction{\sim}\mathbb{I}=0$, and $f\colon\mathbb{R}\to\mathbb{R}$ by

$$f(t)=\begin{cases}0, & \text{if } t\in E\\ \hbar(t-\max E), & \text{if } E\cap[t,+\infty)=\emptyset\\ \hbar(\min E-t), & \text{if } E\cap(-\infty,t]=\emptyset\\ (\beta(t)-\alpha(t))\cdot g\left(\dfrac{t-\alpha(t)}{\beta(t)-\alpha(t)}\right), & \text{otherwise.}\end{cases}$$

It is routine to check that f is as required. □

Let Φ denote the set of all $\phi\in\mathcal{C}(\mathbb{R})$ that are odd, increasing, bijective and p-flat at 0 (that is, all derivatives of ϕ of order at most p vanish at 0). Note that, given $\phi_1\in\Phi$, there exist $\phi_2\in\Phi$ and $\epsilon>0$ such that $\phi_2(1)=1$ and $\phi_2(t)\le\phi_1(t)$ for all $t\in[0,\epsilon)$.

1.2 (*cf.* [3, C.5]). *Let $f\colon(0,1]\to(0,\infty)$ be definable such that $1/f$ is locally bounded. Then there exist $\phi\in\Phi$ and $\epsilon>0$ such that $\phi(1)=1$ and $\phi(t)\le f(t)$ for all $t\in(0,\epsilon)$.*

Proof. If $\mathfrak{R}$ defines no infinite discrete closed subsets of $\mathbb{R}$, then it is o-minimal by Hieronymi [8, Lemma 3] and Miller and Speissegger [17, Theorem (b)]. Hence, the result follows from [3, 4.1 and C.5].

Let $1\in D\subseteq[1,\infty)$ be infinite, closed, discrete and definable. Put $D^{-1}=\{\,1/t:t\in D\,\}$. Define $\beta\colon\mathbb{R}^{>0}\to\mathbb{R}$ by $\beta(t)=\inf\bigl(D^{-1}\cap(t,\infty)\bigr)$ if $0<t<1$ and $\beta(t)=2$ if $t\ge1$. Define functions $\alpha,g\colon\mathbb{R}^{>0}\to\mathbb{R}$ by $\alpha(t)=\sup\bigl(D^{-1}\cap(-\infty,t]\bigr)$ and

$$g(t)=\min\bigl(t^p,\inf(\beta-\alpha)^p\restriction[t,\infty),\inf f\restriction[t,1]\bigr).$$

Since $1/f$ is locally bounded on $(0,1]$, we have $g>0$. Define ϕ piecewise by $\phi(0)=0$,

$$\phi\restriction\mathbb{R}^{>0}=g\circ\alpha\circ\alpha+\bigl(g\circ\alpha-g\circ\alpha\circ\alpha\bigr)\cdot\hbar\circ\bigl((x-\alpha)/(\beta-\alpha)\bigr);$$

and $\phi(t)=-\phi(-t)$ for $t<0$. Note that ϕ is bounded above by f, increasing, continuous, and C^p off $\{0\}$. It suffices now to show that ϕ is p-flat at 0. Let $k\in\{1,\dots,p\}$. We have $Z(\phi^{(k)})\subseteq D^{-1}$ and

$$\phi^{(k)}\restriction\mathbb{R}^{>0}\setminus D^{-1}=\bigl(g\circ\alpha-g\circ\alpha\circ\alpha\bigr)\cdot\bigl(\beta-\alpha\bigr)^{-k}\cdot\hbar^{(k)}\circ\bigl((x-\alpha)/(\beta-\alpha)\bigr).$$

Since $g\circ\alpha\circ\alpha\le\bigl(\beta-\alpha\bigr)^p$ and $\hbar^{(k)}$ is bounded, we have $\lim_{t\to0}\phi^{(k)}(t)=0$. Hence, ϕ is p-flat at 0. □

1.3 (*cf.* [3, C.8]). *Let $A\subseteq\mathbb{R}^n$ be locally closed, $g\colon A\to\mathbb{R}$ be definable and continous, and $\mathcal{F}$ be a finite set of definable locally bounded functions $A\setminus Z(g)\to\mathbb{R}$. Then there exist $\phi\in\Phi$ such that $\phi(1)=1$ and $\lim_{x\to y}\phi(g(x))f(x)=0$ for all $y\in Z(g)$ and $f\in\mathcal{F}$.*

Proof. It suffices to consider the case that $\mathcal{F}$ is a singleton $\{f\}$ such that $f\ge0$ and there exist $y\in Z(g)$ such that $\overline{\lim}_{x\to y}f(x)>0$. For $t>0$, put

$$A(t)=\{\,x\in A:|x|\le1/t\ \ \&\ \ |g(x)|=t\ \ \&\ \ \mathrm{d}(x,\mathrm{fr}(A\setminus Z(g))\ge t\,\}.$$

Each $A(t)$ is compact, the sequence $(A(t))_{t>0}$ is decreasing, and there exists $b>0$ such that $A(t)\setminus Z(f)\neq\emptyset$ for every $t\in(0,b]$. Hence, the function

$$t\mapsto1/(\sup f\restriction A(t))\colon(0,b]\to\mathbb{R}$$

is definable and increasing. By 1.2, there exists $\phi_0 \in \Phi$ such that $\phi_0(t)f(t) \leq 1$ for all $t \in (0, b]$. Put $\phi_1 = \phi_0^3$; then $|\phi_1 \circ g| = \phi_1 \circ |g|$. The result follows. $\square$

Some routine, but very useful, consequences:

1.4. *Let $U \subseteq \mathbb{R}^n$ be open and $g\colon U \to \mathbb{R}$ be definable.*

(1) *If g is continuous and C^p on $U \setminus Z(g)$, then there exist $\phi \in \Phi$ such that $\phi \circ g$ is C^p and $Z(1 - \phi \circ g) = Z(1 - g)$.*
(2) *If g is C^p, then there exist $f \in \mathcal{C}(U)$ such that $Z(f) = Z(g)$ and f is p-flat on $Z(f)$.*
(3) *If g is C^p and $Z(g) = (E \setminus \operatorname{int} E) \cap U$, then there exist $f \in \mathcal{C}(U)$ such that $Z(f) = E \cap U$.*
(4) **Gluing**. *If g is C^p and $h \in \mathcal{C}(U \setminus Z(g))$, then there exist $f \in \mathcal{C}(U)$ such that $Z(f) = Z(g) \cup Z(h)$ and $Z(f) \supseteq Z(1 - g) \cap Z(1 - h)$.*

We are now ready to establish Proposition B.

1.5. *Let $A \subseteq U \subseteq \mathbb{R}^n$ be definable such that A is a finite union of discrete sets and U is open. Then there exist $f \in \mathcal{C}_{\mathbb{I}}({\sim}\operatorname{cl}\operatorname{fr} A)$ such that $Z(f) = \operatorname{lc} A$ and $Z(1-f) \supseteq {\sim}U \setminus \operatorname{cl}\operatorname{fr} A$.*

(Proposition B is the case that A is closed and $U = \mathbb{R}^n$. The more technical statement arises from inductive needs.)

Proof. We proceed by induction on the minimal number of discrete sets that comprise A (that is, on the Cantor-Bendixson rank of A regarded as a subspace of $\mathbb{R}^n$).

Suppose that A is discrete. Define $\rho\colon A \to \mathbb{R}$ by

$$\rho(a) = \min\bigl(1, \mathrm{d}(a, A \setminus \{a\}), \mathrm{d}(a, \operatorname{fr} U)\bigr).$$

Put $V = \bigcup_{a \in A}\{\, v \in \mathbb{R}^n : 3\,\mathrm{d}(v, a) < \rho(a) \,\}$. Define $\sigma\colon V \to \mathbb{R}^n$ by letting $\sigma(v)$ be the center of the ball containing v; observe that $\sigma(v)$ is the unique $a \in A$ such that $3\,\mathrm{d}(v, a) < \rho(a)$. For $v \in V$, put $f(v) = \hbar(10\,\mathrm{d}^2(v, \sigma(v))/\rho^2(\sigma(v)))$. For $v \notin V$, put $f(v) = 1$. As A is locally closed, $\operatorname{fr} A$ is closed. It is routine to check that the restriction of f to ${\sim}\operatorname{fr} A$ is as desired.

More generally, let A_1 be the set of isolated points of A. By the preceding paragraph, there exists $f_1 \in \mathcal{C}_{\mathbb{I}}({\sim}\operatorname{fr} A_1)$ such that $Z(f_1) = A_1$ and $Z(1 - f_1) \supseteq {\sim}U \setminus \operatorname{fr} A_1$. Put $A_2 = A \setminus A_1$. Inductively, there exists $f_2 \in \mathcal{C}_{\mathbb{I}}({\sim}\operatorname{cl}\operatorname{fr} A_2)$ such that $Z(f_2) = A_2 \setminus \operatorname{cl}\operatorname{fr} A_2$ and $Z(1 - f_2) \supseteq {\sim}U \setminus \operatorname{cl}\operatorname{fr} A_2$. The result now follows by gluing. $\square$

It is natural to next consider the case that $0 < m < n$ and every fiber of E over $\mathbb{R}^m$ is a finite union of discrete sets, but we do not yet know how to deal with this level of generality, even if every fiber of E over $\mathbb{R}^m$ is discrete. Thus, we shall assume a tameness condition on $\mathfrak{R}$ and a uniform bound on the Cantor-Bendixson rank of the fibers.

As noted earlier, we are done if $\mathfrak{R}$ defines $\mathbb{Z}$. It is suspected that if $\mathfrak{R}$ does not define $\mathbb{Z}$, then every definable set either has interior or is nowhere dense (see Hieronymi and Miller [9] for some evidence). This condition holds if $\mathfrak{R}$ is d-minimal—by [15], it is enough to show that every definable subset of $\mathbb{R}$ has interior or is nowhere dense—but the converse fails (see 2.1 below). Next is a key technical lemma.

1.6. *Suppose that every definable set either has interior or is nowhere dense. Let $m \in \{0, \dots, n\}$ and π denote projection on the first m coordinates. Let $N \in \mathbb{N}$ and $A \subseteq U \subseteq \mathbb{R}^n$*

be definable such that U is open and every A_x is a union of N discrete sets ($x \in \mathbb{R}^m$). Then there exist a definable open and dense $W \subseteq \mathbb{R}^m$ and $f \in \mathcal{C}_{\mathbb{I}}(\pi^{-1}W \setminus \operatorname{cl}\operatorname{fr} A)$ such that $Z(f) = \pi^{-1}W \cap \operatorname{lc} A$ and $Z(1-f) \supseteq {\sim}U \cap \pi^{-1}W \setminus \operatorname{cl}\operatorname{fr} A$. If every A_x is discrete, then W can be taken such that $f \in \mathcal{C}_{\mathbb{I}}(\pi^{-1}W \setminus \operatorname{fr} A)$, $Z(f) = \pi^{-1}W \cap A$ and $Z(1-f) \supseteq {\sim}U \cap \pi^{-1}W \setminus \operatorname{fr} A$.

Proof. The case $m = 0$ is just 1.5, so we take $m > 0$. As $\pi A \setminus \operatorname{int} \pi A$ is nowhere dense, we reduce to the case that πA is nonempty and open. For each $d \in \mathbb{N}$, the set of $x \in \mathbb{R}^m$ such that A_x has Cantor-Bendixson rank d is definable. Thus, we reduce to the case that $N \geq 1$ and every fiber of A over πA has Cantor-Bendixson rank N. We proceed by induction.

Let $N = 1$, that is, A_x is discrete for every $x \in \pi A$. Define $\rho\colon A \to \mathbb{R}$ by

$$\rho(x, y) = \min\bigl(1, \mathrm{d}(y, A_x \setminus \{y\}), \mathrm{d}(y, \operatorname{fr} U_x)\bigr).$$

Let $\tilde{\pi}\colon \mathbb{R}^{n+1} \to \mathbb{R}^m$ be projection on the first m variables. Let C be the set of all $a \in A$ for which there exist an open box $B(a) \subseteq \mathbb{R}^{n+1}$ centered at $(a, \rho(a))$ and a C^p map $\gamma_a\colon \tilde{\pi}B(a) \to \mathbb{R}^{n-m}$ such that $\gamma_a = A \cap B'(a)$, where $B'(a)$ is the projection of $B(a)$ on the first n variables, and

$$x \mapsto \rho(x, \gamma_a(x))\colon \tilde{\pi}B(a) \to \mathbb{R}$$

is C^p. It is an exercise to see that C is definable. We now reduce to the case that $A = C$ by showing that $\pi(A \setminus C)$ has no interior (and thus is nowhere dense). Suppose to the contrary that $\pi(A \setminus C)$ has interior; then we can reduce to the case that $C = \emptyset$. As every fiber of A over $\mathbb{R}^m$ is discrete, so is every fiber of ρ over $\mathbb{R}^m$. By the Baire Category Theorem, there is an open box $B \subseteq \mathbb{R}^{n+1}$ such that the set $\{\, x \in \mathbb{R}^m : \operatorname{card}(B \cap \rho)_x = 1 \,\}$ is somewhere dense, and thus has interior. By shrinking B, we reduce to the case that there is a definable map $\varphi\colon \tilde{\pi}B \to \rho$. By [15, Theorem 3.3], we may shrink B so that φ is C^p. But then the projection of φ on the first n coordinates is contained in C, contradicting that $C = \emptyset$.

We have now reduced to the case that $A = C$. Note that A is locally closed, so $\operatorname{fr} A$ is closed. Put

$$V = \bigcup_{(x,\xi) \in A} \{\, (x, y) \in \mathbb{R}^n : 3\,\mathrm{d}(y, \xi) < \rho(x, \xi) \,\}.$$

Define $\xi\colon V \to \mathbb{R}^{n-m}$ by letting $\xi(x, y)$ be the unique $\xi \in A_x$ such that $3\,\mathrm{d}(y, \xi) < \rho(x, \xi)$. Let $(x_0, y_0) \in V$ and put $a = (x_0, \xi(x_0, y_0))$. Let $B(a)$ and γ_a be as in the definition of C. Put

$$S = \{\, (x, y) \in \tilde{\pi}B(a) \times \mathbb{R}^{n-m} : 3\,\mathrm{d}(y, \gamma_a(x)) < \rho(x, \gamma_a(x)) \,\};$$

then S is open, $(x_0, y_0) \in S \subseteq V$, and $\xi(x, y) = \gamma_a(x)$ for all $(x, y) \in S$. Hence, S is open, $\xi{\restriction}S$ is C^p, and the function

$$(x, y) \mapsto \hbar(10\,\mathrm{d}^2(y, \xi(x, y))/\rho^2(x, \xi(x, y)))\colon S \to \mathbb{R}$$

is C^p. It follows that

$$(x, y) \mapsto \hbar(10\,\mathrm{d}^2(y, \xi(x, y))/\rho^2(x, \xi(x, y)))\colon V \to \mathbb{R}$$

is definable and C^p; extend it to $f \in \mathcal{C}_{\mathbb{I}}({\sim}\pi^{-1} \operatorname{fr} \pi A \setminus \operatorname{fr} A)$ by setting $f = 1$ off V. (This concludes the proof for the case $N = 1$.)

As the union of the set of isolated points of the fibers of A over πA is definable, the rest of the induction is a routine modification of the argument for 1.5. $\square$

Let $\Pi(n,m)$ denote the collection of all coordinate projection maps

$$(x_1,\dots,x_n)\mapsto \bigl(x_{\lambda(1)},\dots,x_{\lambda(m)}\bigr):\mathbb{R}^n\to\mathbb{R}^m,$$

where $\lambda\colon \{1,\dots,m\}\to\{1,\dots,n\}$ is strictly increasing. For $A\subseteq\mathbb{R}^n$, let $\dim A$ be the supremum of all $m\in\mathbb{N}$ such that πA has interior for some $\pi\in\Pi(n,m)$.

Proof of Theorem A. Assume that $\mathfrak{R}$ is d-minimal. We must find $f\in\mathcal{C}(\mathbb{R}^n)$ such that $Z(f)=E$. It suffices to let $A\subseteq U\subseteq\mathbb{R}^n$ be definable and U be open, and find $f\in\mathcal{C}_{\mathbb{I}}(\sim\operatorname{cl}\operatorname{fr}A)$ and $g\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(f)=\operatorname{lc}A\subseteq{\sim}Z(g)\subseteq U$ (consider $A=E$ and $U=\mathbb{R}^n$). The result is trivial if $A=\emptyset$. We proceed by induction on $d=\dim A\geq 0$ and $n\geq 1$.

Suppose that $d=0$. By d-minimality, A is a finite union of discrete sets. Let f be as in 1.5. Also by d-minimality and 1.5, there exist $h\in\mathcal{C}(\mathbb{R}^n)$ such that $Z(h)=\operatorname{cl}\operatorname{fr}A$. Obtain g as desired by gluing h and $1-f$.

It suffices now by 1.4.3 to consider the case that $0<d<n$ and the result holds for all lesser values of n and d.

We first produce f (with no dependence on U). Let $\pi\in\Pi(n,d)$. By [15, Lemma 8.5], there exists $N\in\mathbb{N}$ such that for all $x\in\mathbb{R}^d$, either $\dim(A\cap\pi^{-1}x)>0$ or $A\cap\pi^{-1}x$ is a union of N discrete sets. As $d=\dim A$, the set $\{\,x\in\mathbb{R}^d:\dim(A\cap\pi^{-1}x)>0\,\}$ has no interior, and thus is nowhere dense. By 1.6, there exist dense open definable $W_\pi\subseteq\mathbb{R}^d$ and $\alpha\in\mathcal{C}_{\mathbb{I}}(\pi^{-1}W_\pi\setminus\operatorname{cl}\operatorname{fr}A)$ such that $Z(\alpha)=\operatorname{lc}A\cap\pi^{-1}W_\pi$ and $Z(1-\alpha)\supseteq{\sim}U\cap\pi^{-1}W_\pi\setminus\operatorname{cl}\operatorname{fr}A$. Inductively, there exist $\beta\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^d)$ such that $Z(\beta)={\sim}W_\pi$, hence also $\gamma\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(\gamma)={\sim}\pi^{-1}W_\pi$. By gluing α and γ there exist $f_\pi\in\mathcal{C}_{\mathbb{I}}(\sim\operatorname{cl}\operatorname{fr}A)$ such that $Z(f_\pi)=\operatorname{lc}A\cup{\sim}\pi^{-1}W_\pi$. Put

$$X=\operatorname{lc}A\cup\bigcap_{\pi\in\Pi(n,d)}{\sim}\pi^{-1}W_\pi$$

and $Y=X\setminus\operatorname{cl}A$. Observe that $\operatorname{cl}Y\subseteq\bigcap_{\pi\in\Pi(n,d)}{\sim}\pi^{-1}W_\pi$ (so $\dim Y<d$), $Y\subseteq{\sim}\operatorname{cl}A$, and Y is locally closed. Inductively, there exist $g_0\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Y\subseteq{\sim}Z(g_0)\subseteq\operatorname{cl}A$. Put $g_1=g_0{\restriction}{\sim}\operatorname{cl}\operatorname{fr}A$ and $f=(g_1+\sum_{\pi\in\Pi(n,d)}f_\pi)/(1+\operatorname{card}\Pi(n,d))$. It is routine to check that f is as desired.

We now produce $g\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $\operatorname{lc}A\subseteq{\sim}Z(g)\subseteq U$. It suffices to consider the case that A is locally closed. With data as in the preceding paragraph, ${\sim}\pi^{-1}W_\pi\cup\operatorname{fr}A$ is closed and has $\dim\leq d$. By the *result* of the preceding paragraph, there exist $\delta\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $Z(\delta)={\sim}\pi^{-1}W_\pi\cup\operatorname{fr}A$. By gluing δ and $1-\alpha$, there exist $g_\pi\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that $A\cap\pi^{-1}W_\pi\subseteq{\sim}Z(g_\pi)\subseteq U$. Inductively, there exist $g_2\in\mathcal{C}_{\mathbb{I}}(\mathbb{R}^n)$ such that

$$A\cap\bigcup_{\pi\in\Pi(n,d)}{\sim}\pi^{-1}W_\pi\subseteq{\sim}Z(g_2)\subseteq U.$$

Put $g=(g_2+\sum_{\pi\in\Pi(n,d)}g_\pi)/(1+\operatorname{card}\Pi(n,d))$ to finish. $\square$

2. Remarks

2.1. Theorem A does not settle our main question. Suppose that $E\subseteq\mathbb{R}^2$ is compact, has no isolated points, $\dim E=0$, and $\mathfrak{R}$ does not define $\mathbb{Z}$. In this generality, we do not know if there exist $f\in\mathcal{C}(\mathbb{R}^2)$ such that $Z(f)=E$ (we do not even know if there exist

C^p functions $f\colon \mathbb{R}^2 \to \mathbb{R}$ such that $E = Z(f)$ and $(\mathfrak{R}, f)$ does not define $\mathbb{Z}$). There are examples of Cantor subsets K of $\mathbb{R}$ such that the expansion of $\overline{\mathbb{R}}$ by all subsets of each K^m satisfies the "interior or nowhere dense" condition; see Friedman *et al.* [5].

Some of our proofs do suggest more general results. We give one example, beginning with a result about o-minimality.

2.2. *Let $S \subseteq \mathbb{R}^{m+n}$ be such that $(\overline{\mathbb{R}}, S)$ is o-minimal and every fiber of S over $\mathbb{R}^m$ is closed. Then there exist $F\colon \mathbb{R}^m \times \mathbb{R}^{>0} \times \mathbb{R}^n \to \mathbb{R}$ definable in $(\overline{\mathbb{R}}, S)$ such that for all $u \in \mathbb{R}^m$ and $r > 0$:*
- — *$F(u, r, \mathbb{R}^n) \subseteq \mathbb{I}$*
- — *$v \mapsto F(u, r, v)\colon \mathbb{R}^n \to \mathbb{R}$ is C^p*
- — *if $v \in \mathbb{R}^n$, then $F(u, r, v) = 0$ if and only if $v \in S_u$*
- — *if $v \in \mathbb{R}^n$ and $\mathrm{d}(v, S_u) \geq r$, then $F(u, r, v) = 1$.*

Proof. Let $u \in \mathbb{R}^m$ and $r > 0$. By [3, C.12], there is a C^p function $F_{u,r}\colon \mathbb{R}^n \to [0, 1]$ definable in $(\overline{\mathbb{R}}, S)$ such that $Z(F_{u,r}) = S_u$ and $Z(1 - F_{u,r}) = \{\, v \in \mathbb{R}^n : \mathrm{d}(v, S_u) \geq r \,\}$. Put $F(u, r, v) = F_{u,r}(v)$ for $v \in \mathbb{R}^n$. An examination of the proof of [3, C.12] (including all supporting results) yields that F is definable in $(\overline{\mathbb{R}}, S)$. □

2.3. *With all data as in 2.2, assume moreover that S is definable (in $\mathfrak{R}$). Let $A \subseteq U \subseteq \mathbb{R}^n$ be definable such that U is open and there is a definable $B \subseteq \mathbb{R}^m$ such that $A = \bigcup_{b\in B} S_b$ and $\mathrm{d}(S_b, A \setminus S_b) > 0$ for all $b \in B$. Then $\operatorname{fr} A$ is closed and there exist $f \in \mathcal{C}_{\mathbb{I}}({\sim} \operatorname{fr} A)$ such that $Z(f) = A$ and $Z(1 - f) \supseteq {\sim}U \setminus \operatorname{fr} A$.*

Proof. It is immediate from assumptions (and o-minimality) that A is locally closed, so $\operatorname{fr} A$ is closed. For $u \in \mathbb{R}^m$, put

$$\rho(u) = \min\bigl(1, \mathrm{d}(S_u, A \setminus S_u), \mathrm{d}(S_u, {\sim}U)\bigr).$$

Put $V = \bigcup_{b\in B}\{\, v \in \mathbb{R}^n : 3\,\mathrm{d}(v, S_b) < \rho(b) \,\}$. For $v \in V$, let $\sigma(v) = b$ where $S_b \in \mathbb{R}^n$ is the fiber of S such that $\mathrm{d}(v, S_b) < \rho(b)$, and put $f(v) = F(\sigma(v), \rho(\sigma(v)), v)$. For $v \notin V$, put $f(v) = 1$. The restriction of f to ${\sim}\operatorname{fr} A$ is as desired. □

If A is discrete, then we recover the conclusion of 1.5 from 2.3 by setting $S = \{(x, x) : x \in \mathbb{R}\}$ and $B = A$. While this might sound promising, a moment's thought reveals that any straightforward attempt to extend 2.3 so as to recover the full conclusion of 1.5 would seem to require a rather tedious (albeit fairly obvious) hypothesis. The assumptions of 2.3 are essentially about *the form of A* rather than a tameness property of $\mathfrak{R}$. One can imagine more results along these lines, but again with assumptions that tend to become tedious and disconnected from tameness of $\mathfrak{R}$.

2.4. As mentioned earlier, the C^∞ version of Theorem A does not hold in general, indeed, it fails if $\mathfrak{R}$ is o-minimal and does not define the function $e^x\colon \mathbb{R} \to \mathbb{R}$ (an easy consequence of results from [13, 14]). On the other hand, if e^x is definable, then the C^∞ version holds for at least some o-minimal $\mathfrak{R}$ (indeed, for most of the expansions of $(\overline{\mathbb{R}}, e^x)$ that are known to be o-minimal); see Jones [10]. By [18], the expansion of $(\overline{\mathbb{R}}, e^x)$ by the set of "towers" $\{e, e^e, e^{e^e}, \dots\}$ is d-minimal; we suspect that the C^∞ version of Theorem A holds for any closed set definable in this structure (but, as yet, it is unclear to us whether it would be the effort to prove it).

Our remaining remarks are more of model-theoretic interest. First, Theorem A is independent of parameters, that is,

2.5. *If* $(\mathbb{R},+,\cdot,E)$ *is d-minimal, then there exist* $f \in C^p(\mathbb{R}^n,\mathbb{R})$ *such that* $Z(f)=E$ *and* f *is* $\emptyset$*-definable in* $(\mathbb{R},+,\cdot,E)$*.*

This can be established by tracking parameters throughout the proof (including all supporting results from elsewhere), but here is another approach of potentially independent interest:

Proof. As every rational number is $\emptyset$-definable in $(\mathbb{R},+,\cdot\,)$, every nonempty $\emptyset$-definable (in $(\mathbb{R},+,\cdot,E)$) subset of $\mathbb{R}$ contains a $\emptyset$-definable point. An easy induction then yields that every nonempty $\emptyset$-definable set contains a $\emptyset$-definable point. By Theorem A, there exist $m \in \mathbb{N}$, $a \in \mathbb{R}^m$ and $\emptyset$-definable $f\colon \mathbb{R}^{m+n} \to \mathbb{R}$ such that $x \mapsto f(a,x)$ is C^p with zero set equal to E. The set of all $u \in \mathbb{R}^m$ such that $x \mapsto f(u,x)$ is C^p with zero set equal to E is $\emptyset$-definable and nonempty, so there is a $\emptyset$-definable $b \in \mathbb{R}^m$ such that $x \mapsto f(b,x)$ is C^p with zero set equal to E. $\square$

An example of consequences:

2.6. *If* $\mathfrak{S}$ *is a d-minimal expansion of* $(\mathbb{R},+,\cdot\,)$*, then the expansion* in the syntactic sense *of* $(\mathbb{R},<)$ *by all* C^p *functions that are* $\emptyset$*-definable in* $\mathfrak{S}$ *admits elimination of quantifiers.*

Proof. By [15, Theorem 3.2] and [1], every $\emptyset$-definable set is a boolean combination of $\emptyset$-definable closed sets. Apply 2.5. $\square$

It can be shown that Theorem A holds over arbitrary ordered fields provided that an appropriate definition of d-minimality is given:

2.7. *Let* $\mathfrak{M}$ *be an expansion of an ordered field such that, for every* $\mathfrak{M}' \equiv \mathfrak{M}$*, every unary set definable in* $\mathfrak{M}'$ *is a disjoint union of open intervals and finitely many discrete sets. Then every closed set definable in* $\mathfrak{M}$ *is the zero set of some definable* (*total*) C^p *function.*

A modification of our proof over $\mathbb{R}$ can be obtained via the emerging subject of "definably complete" expansions of ordered fields (see, *e.g.*, Fornasiero and Hieronymi [4] and its bibliography). We leave details to the interested reader.

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