Weakly o-minimal non-valuational structures

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ABSTRACT

A weakly o-minimal structure $\mathcal{M} = (M, \leq, +, \ldots)$ expanding an ordered group $(M, \leq, +)$ is called non-valuational iff for every cut $\langle C, D \rangle$ of $(M, \leq)$ definable in $\mathcal{M}$, we have that $\inf\{y - x : x \in C, y \in D\} = 0$. The study of non-valuational weakly o-minimal expansions of real closed fields carried out in [MMS] suggests that this class is very close to the class of o-minimal expansions of real closed fields. Here we further develop this analogy. We establish an o-minimal style cell decomposition for weakly o-minimal non-valuational expansions of ordered groups. For structures enjoying such a strong cell decomposition we construct a canonical o-minimal extension. Finally, we make attempts towards generalizing the o-minimal Euler chararacteristic to the class of sets definable in weakly o-minimal structures with the strong cell decomposition property.

0 Introduction

A good measure of complexity of a weakly o-minimal structure is its depth, a concept studied in [Ve] and [BVT]. A weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$, where $\leq$ denotes a dense linear ordering without endpoints, is said to be of depth 0 iff for every infinite definable set $U \subseteq M$ and every definable function $f : U \longrightarrow \overline{M}$, where $\overline{M}$ denotes the set of all cuts of $(M, \leq)$ definable in $\mathcal{M}$, there is a partition of U into a finite set X and convex open definable sets $I_0, \ldots, I_k$ such that for every $i \leq k$, $f \restriction I_i$ is constant or strictly monotone and continuous. In particular, every o-minimal structure has depth 0. Without formally stating the definition of depth, we will only point out that (a) expansions of o-minimal structures by convex predicates are of depth 0 or 1, in particular real closed valued fields have depth 1; (b) the depth of a model of a weakly o-minimal theory is always finite. Generally speaking, the lower is the depth of the structure, the easier are the definable sets to understand.

In this paper we exclusively deal with weakly o-minimal structures of depth 0. The property of a weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$ being of depth 0 is equivalent to the condition that every equivalence relation on M definable in $\mathcal{M}$ has only finitely many infinite classes (see Lemma 1.4). In the presence of the ordered group structure, depth 0 is characterized by each of the following: (a) the absence of non-trivial proper definable subgroups, and (b) the fact that for any cut $\langle C, D \rangle$ definable in $\mathcal{M}$, $\inf\{y - x : x \in C, y \in D\} = 0$. A weakly o-minimal expansion of an ordered group in which the latter holds is after [MMS] said to be of *non-valuational type.* This is because a weakly o-minimal expansion of a real closed field is of depth 0 iff the underlying field has no nontrival convex definable valuations.

The principal goal of this paper is to develop essential model theory for weakly o-minimal non-valuational expansions of ordered groups. Our most important results are contained in §2 and §3. In §2 we introduce a notion of a strong cell and prove an o-minimal style strong cell decomposition theorem. This in two ways improves a similar result from [MMS]. Firstly, we work with expansions

[1]This research was supported by a Marie Curie Intra-European Fellowships within the 6th European Community Framework Programme. Contract number: MEIF-CT-2003-501326.

of ordered groups instead of real closed fields. Secondly, we use more sophisticated strong cells, i.e. strong cells in our sense are strong cells in the sense of [MMS], but not conversely. In §3, given a weakly o-minimal structure $\mathcal{M}$ with the strong cell decomposition property we construct its canonical o-minimal extension $\overline{\mathcal{M}}$.

In §4, for sets definable in weakly o-minimal structues with the strong cell decomposition property, we propose a weak variant of Euler characteristic, whose values lie in the ring $\mathbb{Z}[\frac{1}{2}]$.

Throughout the paper we often refer to [MMS] and [We] for basic results concerning weak o-minimality.

1 Notation and preliminaries

Consider a dense linear ordering $(M, \leq)$ without endpoints. A subset I of M is said to be *convex* in $(M, \leq)$ iff for any $a, b \in I$ and $c \in M$ with $a \leq c \leq b$, we have that $c \in I$. If additionally $I \neq \emptyset$ and $\inf I, \sup I \in M \cup \{-\infty, +\infty\}$, then I is called an *interval* in $(M, \leq)$. A maximal convex subset of a non-empty subset of M is called its *convex component.* A pair $\langle C, D\rangle$ of non-empty subsets of M is called a *cut* in $(M, \leq)$ iff $C < D$ and $C \cup D = M$. A cut $\langle C, D\rangle$ for which $\sup C \in M$ will be usually identified with the element $\sup C \in M$. A first order structure $\mathcal{M} = (M, \leq, \ldots)$ expanding $(M, \leq)$ is *o-minimal* [*weakly o-minimal*] iff every subset of M, definable in $\mathcal{M}$, is a finite union of intervals [respectively: convex sets] in $(M, \leq)$. Weak o-minimality unlike o-minimality in general is not preserved under elementary equivalence. We say that a complete first order theory is *weakly o-minimal* iff all its models are weakly o-minimal. Clearly, an L-structure $\mathcal{M} = (M, \leq, \ldots)$ has weakly o-minimal L-theory iff for every formula $\varphi(\overline{x}, y) \in L(M)$, there is $n \in \mathbb{N}_+$ such that for every $\overline{a} \in M^{|\overline{x}|}$, the set $\varphi(\overline{a}, M)$ has at most n convex components.

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure. A cut $\langle C, D\rangle$ in $(M, \leq)$ is called definable in $\mathcal{M}$ iff the sets C, D are definable in $\mathcal{M}$. The set of all cuts $\langle C, D\rangle$ definable in $\mathcal{M}$ and such that D has no lowest element will be denoted by $\overline{M}$. The set M can be regarded as a subset of $\overline{M}$ by identifying an element $a \in M$ with the cut $\langle(-\infty, a], (a, +\infty)\rangle$. After such an identification, $\overline{M}$ is naturally equipped with a dense linear ordering extending $(M, \leq)$: $\langle C_1, D_1\rangle \leq \langle C_2, D_2\rangle$ iff $C_1 \subseteq C_2$. Clearly, $(M, \leq)$ is a dense substructure of $(\overline{M}, \leq)$.

The topological dimension of an infinite definable set $X \subseteq M^m$ is defined by the following condition: $\dim(X) \geq k$ iff there is a projection $\pi : M^m \longrightarrow M^k$ such that $\pi[X]$ contains an open box. Non-empty finite sets have dimension 0 whereas $\dim(\emptyset) = -\infty$. Theorem 4.2 from [MMS] together with [Ar] imply that if $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure and $X, Y \subseteq M^m$ are sets definable in $\mathcal{M}$, then $\dim(X \cup Y) = \max(\dim(X), \dim(Y))$. A definable set $Y \subseteq M^m$ is said to be large in X iff $\dim(X \setminus Y) < \dim(X)$. In §2 we will use the following fact.

Fact 1.1 *[We, Lemma 2.1] Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $m \in \mathbb{N}_+$, $\overline{a} \in M^m$, $I \subseteq M$ is an open interval, and $X \subseteq M^{m+1}$ is a set definable in $\mathcal{M}$ such that for every $b \in I$, there is an open box B containing $\overline{a}$ and contained in $\{\overline{x} \in M^m : \langle \overline{x}, b\rangle \in X\}$. Then* $\dim(X) = m + 1$.

The projection from M^m onto M^{m-k} dropping coordinates $i_1, \ldots, i_k$ will be denoted by $\pi^m_{i_1,\ldots,i_k}$. The projection from M^m onto M^k [from $\overline{M}^m$ onto $\overline{M}^k$] dropping all coordinates except $i_1, \ldots, i_k$ will be denoted by $\varrho^m_{i_1,\ldots,i_k}$ [$\overline{\varrho}^m_{i_1,\ldots,i_k}$ respectively].

If $m \in \mathbb{N}_+$ and $X \subseteq M^m$ is a non-empty definable [over $A \subseteq M$] set, then a function $f : X \longrightarrow \overline{M}$ is said to be definable [over A] iff the set $\{\langle \overline{x}, y\rangle \in X \times M : f(\overline{x}) > y\}$ is definable [over A]. A function $f : X \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ is said to be definable iff f is a definable function from X to $\overline{M}$ or $(\forall \overline{x} \in X)(f(\overline{x}) = -\infty)$, or $(\forall \overline{x} \in X)(f(\overline{x}) = +\infty)$.

The following lemma will be referred to as the (local) monotonicity theorem. Recall that if $I \subseteq M$ is an open interval and $f : I \longrightarrow \overline{M}$, then f is said to be locally increasing on I iff for any element $a \in I$, there is an open interval $J \subseteq I$ containing a such that $f \restriction J$ is strictly increasing. In a similar manner we define locally constant and locally strictly decreasing functions.

Lemma 1.2 *([MMS, Theorem 3.3], [Ar]) Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure and $A \subseteq M$. If $U \subseteq M$ is an infinite A-definable set and $f : U \longrightarrow \overline{M}$ [respectively: $f : U \longrightarrow M$] is an A-definable function, then there is a partition of U into A-definable sets $X, I_0, \ldots, I_m$ such that X is finite, $I_0, \ldots, I_m$ are non-empty convex open sets, and for every $i \leq m$, $f \restriction I_i$ is locally constant or locally strictly increasing [and continuous], or locally strictly decreasing [and continuous].*

Lemma 1.3 *If $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $I \subseteq M$ is a non-empty convex open definable set and $f : I \longrightarrow \overline{M}$ is a definable function, then the limits $\lim_{x \to (\inf I)^+} f(x)$ and $\lim_{x \to (\sup I)^-} f(x)$ exist in $\overline{M} \cup \{-\infty, +\infty\}$.*

Proof. We will only prove the existence of $\lim_{x \to (\inf I)^+} f(x)$. For $c \in M$ define

$$I_1(c) = \{x \in I : f(x) < c\}, I_2(c) = \{x \in I : f(x) = c\}, I_3(c) = \{x \in I : f(x) > c\}.$$

Clearly, $\langle I_1(c), I_2(c), I_3(c)\rangle$ is a partition of I into definable sets. For $i \in \{1, 2, 3\}$ define $X_i = \{c \in M : I_i(c) \text{ is coinitial with } I\}$. Again, $\langle X_1, X_2, X_3\rangle$ is a partition of M, $X_1 > X_2 > X_3$ and $|X_2| \leq 1$. Note that

- if $X_1 = \emptyset$, then $X_3 = M$ and $\lim_{x \to (\inf I)^+} f(x) = +\infty$;
- if $X_3 = \emptyset$, then $X_1 = M$ and $\lim_{x \to (\inf I)^+} f(x) = -\infty$;
- if $X_1 \neq \emptyset$ and $X_3 \neq \emptyset$, then $\lim_{x \to (\inf I)^+} f(x) = \inf X_1 = \sup X_3 \in \overline{M}$;
- if $X_2 = \{b\}$, $b \in M$, then $\lim_{x \to (\inf I)^+} f(x) = b$.

This finishes the proof. ∎

We say that a weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$ has *strong monotonicity* iff for every $A \subseteq M$, every infinite A-definable set $I \subseteq M$ and every A-definable function $f : I \longrightarrow \overline{M}$, there is a partition of I into a finite set X and convex open A-definable sets $I_0, \ldots, I_k$ such that for every $i \leq k$, one of the following conditions holds.

- $f \restriction I_i$ is constant;
- $f \restriction I_i$ is strictly increasing and for any $a, b \in I_i$ with $a < b$ and any $c, d \in M$ with $f(a) < c < d < f(b)$, there is $x \in (a, b)$ with $c < f(x) < d$; in particular, $f \restriction I_i$ is continuous;
- $f \restriction I_i$ is strictly decreasing and for any $a, b \in I_i$ with $a < b$ and any $c, d \in M$ with $f(a) > c > d > f(b)$, there is $x \in (a, b)$ with $c > f(x) > d$; in particular, $f \restriction I_i$ is continuous;

It is easy to see that a weakly o-minimal structure has strong monotonicity iff its depth equals 0. Note that the notion of strong monotonicity defined above differs from that introduced in [Ve]. If $\mathcal{M}$ has the strong monotonicity, then for any open interval $I \subseteq M$ and any definable function $f : I \longrightarrow \overline{M}$, there is an open interval $I' \subseteq I$ such that the function $f \upharpoonright I'$ is continuous. Hence, by [We, Theorem 4.2], the topological dimension for sets definable in $\mathcal{M}$ has the usual addition property. Moreover, if $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}$ is a definable function, then there is an open box $B' \subseteq B$ such that $f \upharpoonright B'$ is continuous. By [We, Corollary 4.3], the definable closure has the exchange property in $\mathcal{M}$.

Lemma 1.4 *If $\mathcal{M} = (M, \leq, \dots)$ is a weakly o-minimal structure, then the following conditions are equivalent.*

(a) $\mathcal{M}$ has the strong monotonicity.

(b) Every equivalence relation on M definable in $\mathcal{M}$ has only finitely many infinite classes.

Proof. (a)$\Longrightarrow$(b). Assume that $\mathcal{M}$ has the strong monotonicity and E is an equivalence relation on M definable in $\mathcal{M}$. For $a \in M$, denote by $C(a)$ the convex component of $E(a, M)$ containing a. In case $\sup C(a) < +\infty$, define $f(a) \in \overline{M}$ as the supremum of $C(a)$. Otherwise put $f(a) = a$. The function f is definable in $\mathcal{M}$ and constant on $C(a)$ whenever $a \in M$ and $\sup C(a) < +\infty$. Now, the strong monotonicity of $\mathcal{M}$ implies that E has only finitely many infinite equivalence classes.

(b)$\Longrightarrow$(a). Assume that $A \subseteq M$, $I \subseteq M$ is an infinite set and $f : I \longrightarrow \overline{M}$ is an A-definable function. By the local monotonicity theorem, there is a partition of I into a finite set and infinite convex open A-definable sets $I_0, \dots, I_k$ such that for any $i \leq k$, $f \upharpoonright I_i$ is locally constant or locally stictly increasing, or locally strictly decreasing. Fix $i \leq k$ such that $f \upharpoonright I_i$ is, say, locally strictly increasing. There is an A-definable equivalence relation E on I_i whose classes are the maximal convex subsets of I_i on which f is strictly increasing. (b) implies that E has only finitely many infinite classes, which are necessarily A-definable. The same argument applies if $f \upharpoonright I_i$ is locally strictly decreasing or locally constant. In this way we obtain a partition of I into a finite set and A-definable open convex sets $J_0, \dots, J_l$ such that for every $i \leq l$, $f \upharpoonright J_i$ is constant or strictly monotone. Fix $i \leq l$ such that $f \upharpoonright J_i$ is strictly monotone, and define an equivelence relation on M as follows: $a \sim b$ iff $a = b$, or $a \neq b$ and for every $x \in J_i$ we have that $f(x) < \min(a, b)$ or $f(x) > \max(a, b)$. Clearly, every equivalence class of $\sim$ is a convex set. By the assumption, $\sim$ has only finitely many infinite classes. The remaining classes are singletons. Hence there is a partition of J_i into a finite set X_i and convex open A-definable sets $J^i_0, \dots, J^i_{k_i}$ such that for any $j \leq k_i$ and $b \in M$ with $\inf\{f(x) : x \in J^i_j\} < b < \sup\{f(x) : x \in J^i_j\}$ we have that $b/{\sim} = \{b\}$. From this the strong monotoncity of $\mathcal{M}$ follows. ■

Assume that $\mathcal{M} = (M, \leq, +, \dots)$ is a weakly o-minimal structure expanding an ordered group $(M, \leq, +)$. Then $(M, \leq, +)$ is divisible and abelian (see [MMS, Theorem 5.1]). A cut $\langle C, D\rangle$ in $(M, \leq)$ is called *non-valuational* iff $\inf\{y - x : x \in C \text{ and } y \in D\} = 0$. The structure $\mathcal{M}$ is called *non-valuational* (or *of non-valuational type*) iff all cuts in $(M, \leq)$ definable in $\mathcal{M}$ are non-valuational. Otherwise $\mathcal{M}$ is said to be *valuational* (or *of valuational type*). If $\mathcal{M}$ is non-valuational and $\mathcal{N} \equiv \mathcal{M}$, then also $\mathcal{N}$ is non-valuational.

Lemma 1.5 *Assume that $\mathcal{M} = (M, \leq, + \dots)$ is a weakly o-minimal expansion of an ordered group $(M, \leq, +)$. The following conditions are equivalent.*

(a) $\mathcal{M}$ is of non-valuational type.

(b) Every subgroup of $(M, \leq, +)$ definable in $\mathcal{M}$ is either trivial or equal M.

(c) $\mathcal{M}$ has the strong monotonicity.

(d) Every equivalence relation on M, definable in $\mathcal{M}$, has only finitely many infinite classes.

Proof. The equivalence of (c) and (d) has been established in Lemma 1.4.

(a)$\Longrightarrow$(b). Suppose for a contradiction that $(M, \leq, +)$ has a proper non-trivial subgroup H which is definable in $\mathcal{M}$. By Lemma 5.2 from [MMS], H is convex. Let $\varepsilon \in H \cap (0, +\infty)$. Then for any $x \in H$ and $y > H$, we have that $y - x > \varepsilon$, which means that $\mathcal{M}$ is of valuational type.

(b)$\Longrightarrow$(d). Suppose that there is a definable equivalence relation E on M with infinitely many infinite classes. Let E' be the equivalence relation on M whose classes are convex components of E-classes. Each E-class has finitely many convex components, therefore the equivalence relation E' has infinitely many infinite classes, and these classes are convex. (b) implies that if B is an infinite E'-class with $\inf B, \sup B \in \overline{M}$, then the set $I_B := \{b_1 - b_2 : b_1, b_2 \in B\}$ is not a subgroup of $(M, +)$. Then $J_B := \{b \in I_B : b + b \in I_B\}$ is a proper subset of I_B, and the set defined by a formula

$$\varphi(x) \equiv \forall y (E'(x, y) \wedge y < x \longrightarrow E'(x + x - y, x))$$

does not contain B but has a non-empty intersection with B. Hence $\varphi(M)$ is not a union of finitely many convex sets, which contradicts weak o-minimality of $\mathcal{M}$.

(d)$\Longrightarrow$(a). Suppose that $\mathcal{M}$ has a definable valuational cut $\langle C, D\rangle$. There is a positive $\varepsilon \in M$, such that $d - c > \varepsilon$ whenever $c \in C$ and $d \in D$. Let

$$\begin{aligned} G_0 &= \{g \in M : g \geq 0 \text{ and } (\forall c \in C)(g + c \in C)\}; \\ G &= G_0 \cup \{-g : g \in G_0\}. \end{aligned}$$

Then G is a convex subgroup of $(M, +)$. Since $\varepsilon \in G$ and $C \neq M$, the group $(G, +)$ is not trivial and the index $[M : G]$ is infinite. Thus the equivalence relation $E(x, y) \equiv x - y \in G$ has infinitely many classes. ■

Now assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$. For every cut $\langle C, D\rangle$ definable in $\mathcal{M}$ we have that $C - D := \{x - y : x \in C \text{ and } y \in D\} = (-\infty, 0)$ and $D - C = (0, +\infty)$. The set $\overline{M}$ can be naturally equipped with an operation of addition (to be denoted by $+$):

$$\langle C_1, D_1\rangle + \langle C_2, D_2\rangle = \langle C_1 + C_2, D_1 + D_2\rangle.$$

It is easy to see that $(\overline{M}, \leq, +)$ is an ordered divisible abelian group and (after suitable identification of elements), $(M, \leq, +)$ is a dense subgroup of it.

If $\mathcal{M} = (R, \leq, +, \cdot, \ldots)$ is a weakly o-minimal structure expanding an ordered field $(R, \leq, +, \cdot)$, then by Theorem 5.3 from [MMS], $(R, \leq, +, \cdot)$ is real closed. The structure $\mathcal{M}$ regarded as an expansion of the ordered group $(R, \leq, +)$ is of non-valuational type iff there are no non-trivial valuations of the field $(R, \leq, +, \cdot)$ definable in $\mathcal{M}$ (see [MMS, Theorem 6.3]). In $\overline{R}$, apart of the operation of addition, we can define an operation of multiplication in the following way.

$$\langle C_1, C_2\rangle \cdot \langle D_1, D_2\rangle = \langle E_1, E_2\rangle,$$

where

$$E_2 = C_2 \cdot D_2,\ E_1 = R \setminus E_2$$

if $C_2 \subseteq (0, +\infty)$ or $D_2 \subseteq (0, +\infty)$, and

$$E_2 = \mathrm{int}(C_1 \cdot D_1), E_1 = R \setminus E_2$$

in case $C_2 \cap D_2 \cap (-\infty, 0) \neq \emptyset$. It is not difficult to check that $(\overline{R}, \leq, +, \cdot)$ is a real closed field and $(R, \leq, +, \cdot)$ is a dense subfield of it.

For a tuple $\overline{a} = \langle a_1, \ldots, a_m \rangle \in M^m$ we define $\|\overline{a}\| = \sum_{i=1}^{m} |a_i|$. Distance between two non-empty definable sets $X, Y \subseteq M^m$ is a non-negative element of $\overline{M}$ defined as $\mathrm{dist}(X, Y) = \inf\{\|\overline{x} - \overline{y}\| : \overline{x} \in X, \overline{y} \in Y\}$.

2 The strong cell decomposition property

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure. Below, for every $m \in \mathbb{N}_+$ we inductively introduce strong cells in M^m and their completions in $\overline{M}^m$. The completion in $\overline{M}^m$ of a strong cell $C \subseteq M^m$ will be denoted by $\overline{C}$. We also introduce the strong cell decomposition property, a notion which will be used throughout the rest of the paper.

(1) A one-element subset of M is a strong $\langle 0 \rangle$-cell in M. If $C \subseteq M$ is a strong $\langle 0 \rangle$-cell, then $\overline{C} := C$.

(2) A non-empty convex open definable subset of M is a strong $\langle 1 \rangle$-cell in M. If $C \subseteq M$ is a strong $\langle 1 \rangle$-cell in M, then $\overline{C} := \{x \in \overline{M} : (\exists a, b \in C)(a < x < b)\}$.

Assume that $m \in \mathbb{N}_+$, $i_1, \ldots, i_m \in \{0, 1\}$ and suppose that we have already defined strong $\langle i_1, \ldots, i_m \rangle$-cells in M^m and their completions in $\overline{M}^m$.

(3) If $C \subseteq M^m$ is a strong $\langle i_1, \ldots, i_m \rangle$-cell in M^m and $f : C \longrightarrow M$ is a continuous definable function which has a (necessarily unique) continuous extension $\overline{f} : \overline{C} \longrightarrow \overline{M}$, then $\Gamma(f)$ is a strong $\langle i_1, \ldots, i_m, 0 \rangle$-cell in M^{m+1}. The completion of $\Gamma(f)$ in $\overline{M}^{m+1}$ is defined as $\Gamma(\overline{f})$.

(4) If $C \subseteq M^m$ is a strong $\langle i_1, \ldots, i_m \rangle$-cell in M^m and $f, g : C \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ are continuous definable functions which have (necessarily unique) continuous extensions $\overline{f}, \overline{g} : \overline{C} \longrightarrow \overline{M}$ such that $\overline{f}(\overline{x}) < \overline{g}(\overline{x})$ for $\overline{x} \in \overline{C}$, then the set

$$(f, g)_C := \{\langle \overline{a}, b \rangle \in C \times M : f(\overline{a}) < b < g(\overline{a})\}$$

is called a strong $\langle i_1, \ldots, i_m, 1 \rangle$-cell in M^m. The completion of $(f, g)_C$ in $\overline{M}^{m+1}$ is defined as

$$\overline{(f, g)_C} := (\overline{f}, \overline{g})_{\overline{C}} := \{\langle \overline{a}, b \rangle \in \overline{C} \times \overline{M} : \overline{f}(\overline{a}) < b < \overline{g}(\overline{a})\}.$$

(5) We say that $C \subseteq M^m$ is a strong cell in M^m iff there are $i_1, \ldots, i_m \in \{0, 1\}$ such that C is a strong $\langle i_1, \ldots, i_m \rangle$-cell in M^m.

A strong cell $C \subseteq M^m$, $m \geq 2$, is called a *refined strong cell* iff each of the boundary functions appearing in its definition assumes values in one of the following sets: $\{-\infty\}$, $\{+\infty\}$, M, $\overline{M} \setminus M$. Refined strong cells in M coincide with strong cells in M.

Definition 2.1 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure and $C \subseteq M^m$ is a strong cell. We say that a definable function $f : C \longrightarrow \overline{M}$ is strongly continuous iff f has a (necessarily unique) continuous extension $\overline{f} : \overline{C} \longrightarrow \overline{M}$. A function which is identically equal $-\infty$ or $+\infty$, and whose domain is a strong cell will be also called strongly continuous.*

Note that according to the above definition, all functions appearing in the definition of a strong cell are strongly continuous. The following trivial example illustrates the difference between cells, strong cells and refined strong cells.

Example. Fix a positive irrational number α and consider a first-order structure $\mathcal{M} = (\mathbb{Q}, \leq, +, P)$, where $(\mathbb{Q}, \leq, +)$ is the ordered group of rationals and P denotes a unary predicate interpreted as the set of all rationals greater than α. By [BP], the structure $\mathcal{M}$ has a weakly o-minimal theory. Obviously, it is of non-valuational type. Define unary funtions f, g and h_1 as follows:

$$f(x) = 0,\ g(x) = \alpha,\ \text{and } h_1(x) = |x - \alpha| + \alpha$$

for $x \in \mathbb{Q}$. Let also

$$h_2(x) = \begin{cases} \alpha & \text{if } x < \alpha \\ 2\alpha & \text{otherwise.} \end{cases}$$

The functions f, g, h_1 and h_2 are all definable in $\mathcal{M}$ and continuous in the order topology. Moreover, f, g and h_1 are strongly continuous while h_2 is not. Note that

- $(f, g)_{\mathbb{Q}}$ is a refined strong cell in $\mathbb{Q}^2$;
- $(f, h_1)_{\mathbb{Q}}$ is a strong cell but not a refined strong cell;
- $(g, h_1)_{\mathbb{Q}}$ and $(f, h_2)_{\mathbb{Q}}$ are cells but not strong cells.

Definition 2.2 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure. Below we inductively define the notion of strong cell decomposition (or decomposition into strong cells in M^m) of a non-empty definable set $X \subseteq M^m$, $m \in \mathbb{N}_+$.*

(a) If $X \subseteq M$ is a non-empty definable set and $\mathcal{D} = \{C_0, \ldots, C_k\}$ is a partition of X into strong cells in M, then $\mathcal{D}$ is a decomposition of X into strong cells in M.

(b) Assume that $m \in \mathbb{N}_+$, $X \subseteq M^{m+1}$ is a non-empty definable set and $\mathcal{D} = \{C_0, \ldots, C_k\}$ is a partition of X into strong cells in M^{m+1}. We say that $\mathcal{D}$ is a decomposition of X into strong cells in M^{m+1} iff $\{\pi^{m+1}_{m+1}[C_0], \ldots, \pi^{m+1}_{m+1}[C_k]\}$ is a decomposition of $\pi^{m+1}_{m+1}[X]$ into strong cells in M^m.

Definition 2.3 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $m \in \mathbb{N}_+$, $X, Y \subseteq M^m$ are definable sets, $X \neq \emptyset$ and $\mathcal{D}$ is a decomposition of X into strong cells in M^m. We say that $\mathcal{D}$ partitions Y iff for every strong cell $C \in \mathcal{D}$, either $C \subseteq Y$ or $C \cap Y = \emptyset$.*

Definition 2.4 *A weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$ is said to have the strong cell decomposition proerty iff for any $m, k \in \mathbb{N}_+$ and any definable sets $X_1, \ldots, X_k \subseteq M^m$, there is a decomposition of M^m into strong cells which partitions each of the sets $X_1, \ldots, X_k$.*

In an obvious way we can introduce the notion of a decomposition of a definable set into refined strong cells [partitioning a given finite family of definable sets]. The proof of the following fact is a routine exercise.

Fact 2.5 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property and $m \in \mathbb{N}_+$.*

$(a)_m$ If $k \in \mathbb{N}_+$ and $X_1, \ldots, X_k \subseteq M^m$ are definable sets, then there is a decomposition of M^m into refined strong cells partitioning each of the sets $X_1, \ldots, X_k$.

$(b)_m$ If $X \subseteq M^m$ is a non-empty definable set and $f : X \longrightarrow \overline{M}$ is a definable function, then there is a decomposition $\mathcal{D}$ of X into refined strong cells in M^m such that for every $D \in \mathcal{D}$, we have that $f \upharpoonright D$ is strongly continuous and

$$(\forall \overline{x} \in D)(f(\overline{x}) \in M) \text{ or } (\forall \overline{x} \in D)(f(\overline{x}) \in \overline{M} \setminus M).$$

Lemma 2.6 *If $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property, then*
(a) $Th(\mathcal{M})$ is weakly o-minimal;
(b) $\mathcal{M}$ has the strong monotonicity;
(c) (in case $\mathcal{M}$ expands an ordered group) $\mathcal{M}$ is of non-valuational type.

Proof. (a) Fix $m \in \mathbb{N}_+$, a definable set $X \subseteq M^{m+1}$ and a strong cell decomposition $\mathcal{D}$ of M^m partitioning X. For $\overline{a} \in M^m$, the number of convex components of $\{b \in M : \langle \overline{a}, b \rangle \in X\}$ does not exceed the cardinality of $\mathcal{D}$. Hence $Th(\mathcal{M})$ is weakly o-minimal.

(b) Assume that $I \subseteq M$ is an infinite definable set and $f : I \longrightarrow \overline{M}$ is a definable function. By the monotonicity theorem, the set I can be partitioned as $I = X \cup I_0 \cup \ldots \cup I_m$, where X is finite and $I_0, \ldots, I_m$ are infinite open convex definable sets such that for any $i \leq m$, $f \restriction I_i$ is locally constant or locally strictly increasing, or locally strictly decreasing. Fix $i \leq m$ and suppose that $f \restriction I_i$ is locally strictly increasing. By assumption, the set $\{\langle x, y \rangle \in I_i \times M : f(x) > y\}$ is a union of finitely many strong cells in M^2. Since the boundary functions appearing in the definition of a strong cell extend to continuous functions on completions of their domains, the set I_i can be decomposed as $I_i = Y_i \cup J_0^i \cup \ldots \cup J_{k_i}^i$, where Y_i is finite and $J_0^i, \ldots, J_{k_i}^i$ are open convex definable sets, and for any $j \leq k_i$, the function $f \restriction I_i$ is strictly increasing and strongly continuous.

(c) follows from (b) and Lemma 1.5. ■

Note that the strong cell decomposition property of a weakly o-minimal structure $\mathcal{M}$ is not implied by any of the (independent) conditions (a), (b) of Lemma 2.6. If $\mathcal{M} = (R, \leq, +, \cdot, V)$ is a real closed valued field, then by [BP] or [Bz], $Th(\mathcal{M})$ is weakly o-minimal. However, $\mathcal{M}$ lacks the strong monotonicity, and consequently, the strong cell decomposition property. The structure defined in section 2.5 of [MMS] has the strong monotonicity but not the strong cell decomposition property, and its theory is not weakly o-minimal. It turns out that in general even the conjunction of (a) and (b) is insufficient for the strong cell decomposition property, as illustrated by the following example. Let $\mathcal{M} = (M, \leq, P)$, where $M := \mathbb{N} \times \mathbb{Q}$ is ordered lexicographically by $\leq$, and $\mathcal{M} \models P((m, q), (n, r))$ iff $m = n$ and $r - q \in \{0, 1\}$. It is not difficult to see that $Th(\mathcal{M})$ is weakly o-minimal. For $n \in \mathbb{N}$, let $R_n = \{n\} \times \mathbb{Q}$. By [Bz], the structure $\mathcal{N} := (\mathcal{M}, R_n : n \in \mathbb{N})$ has a weakly o-minimal theory. Moreover, $\mathcal{N}$ has the strong monotonicity, but $P(M) \subseteq M^2$ cannot be decomposed into finitely many strong cells in M^2. Indeed, let $f(\langle m, q \rangle) = \langle m, q \rangle$ and $g(\langle m, q \rangle) = \langle m, q + 1 \rangle$ for $\langle m, q \rangle \in M$. Both functions are definable in $\mathcal{N}$ and strongly continuous. However, for any natural m we have that $\lim\limits_{q \longrightarrow +\infty} f(\langle m, q \rangle) = \lim\limits_{q \longrightarrow +\infty} g(\langle m, q \rangle)$.

The remaining part of this section is devoted to showing that for a weakly o-minimal structure expanding an ordered group, strong monotonicity implies the strong cell decomposition property. In other words we will prove (see Theorem 2.15) that weakly o-minimal non-valuational expansions of ordered groups enjoy the strong cell decomposition property.

It has to be noted that sets definable in weakly o-minimal non-valuational expansions of ordered groups are much more difficult to handle than sets definable in o-minimal structures. Strong cells in general are not definably connected. When working with definable functions, we often have to pay attention to their extensions to completions of strong cells. Let $f, g : M \longrightarrow \overline{M}$ be two definable and strongly continuous functions such that $f(x) < g(x)$ whenever $x \in M$ and $\lim\limits_{x \to b} f(x) = \lim\limits_{x \to b} g(x)$ for some $b \in \overline{M} \setminus M$. Of course, $(f, g)_M$ is not a strong cell. Let $\varphi(x, y)$ be a formula saying that $g(x) - f(x) > y > 0$. If $(f, g)_{(-\infty, b)}$ and $(f, g)_{(b, +\infty)}$ are strong cells, then for any $d > 0$, some interval I with $\inf I < b < \sup I$ has an empty intersection with $\varphi(M, d)$. The following Lemma

2.7(a) guarantees that the number of such b's is finite, so we can partition M into finitely many convex open sets J with the property that $(f,g)_J$ is a strong cell.

Also, it is not clear whether the intersection of a definable family of closed bounded sets must be nonempty. The following series of lemmas together with condition (b)$_m$ in Theorem 2.15 were designed to deal with difficulties of this sort.

Lemma 2.7 *Assume that $\mathcal{M} = (M, \leq, +, \dots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$ and $A \subseteq M$.*

(a) If $I \subseteq M$ is a non-empty open A-definable set and $\varphi(x,y)$ is an $L(A)$-formula such that

- *if $d > 0$, then $\varphi(M,d)$ is an open subset of I,*
- *$\varphi(M,d_2) \subseteq \varphi(M,d_1)$ whenever $0 < d_1 \leq d_2$,*
- *$\bigcup_{d>0} \varphi(M,d) = I$,*

then there is a partition of I into convex open A-definable sets $I_0, \dots, I_s$ such that for every $i \leq s$, if $I' \subseteq I_i$ is an open interval with $\inf I_i < \inf I' < \sup I' < \sup I_i$, then $I' \subseteq \varphi(M,d)$ for some $d > 0$.

(b) Assume that $I \subseteq M$ is a non-empty convex open definable set and $E(x,y,z)$ is an $L(M)$-formula such that for every $a \in I$, the formula $E(x,y,a)$ defines an equivalence relation E^a on M with finitely many classes each of which is a convex set. Then there is $n \in \mathbb{N}_+$ such that E^a has at most n equivalence classes as a varies over I.

Proof. (a) Assume that $I \subseteq M$ and $\varphi(x,y) \in L(A)$ satisfy the assumptions of the lemma. Let E be an equivalence relation on I such that for any $a,b \in I$, $\mathcal{M} \models E(a,b)$ iff for some $d > 0$, the elements a,b belong to the same convex component of $\varphi(M,d)$. All equivalence classes of E are convex open sets. By Lemma 1.5, E has only finitely many equivalence classes: $I_0, \dots, I_s$. These are necessarily A-definable. Clearly, $\langle I_0, \dots, I_s \rangle$ is a partition of I satisfying our demands.

(b) Fix an $L(M)$-formula $E(x,y,z)$ satisfying the assumptions of (b). It is easy to see that there is an $L(M)$-formula $\varphi(y_1,y_2,x)$ such that for any $a, b_1, b_2 \in M$, we have that $\mathcal{M} \models \varphi(b_1,b_2,a)$ iff the classes $[b_1]_{E^a}$, $[b_2]_{E^a}$ are both infinite and $\inf[b_1]_{E^a} < b_1 < \sup[b_1]_{E^a} = \inf[b_2]_{E^a} < b_2 < \sup[b_2]_{E^a}$.

Also, for every non-empty convex open definable set $J \subseteq M$, there is an $L(M)$-formula $\psi_J(y_1,y_2,x,z)$ such that for any $a, b_1, b_2 \in M$ and $d > 0$, we have that $\mathcal{M} \models \psi_J(b_1,b_2,a,d)$ iff the following conditions are satisfied

- $a \in J$ and $(a-d, a+d) \cap J \subseteq \varphi(b_1,b_2,M)$;
- the function $f : (a-d,a+d) \cap J \longrightarrow \overline{M}$ defined by $f(c) = \sup[b_1]_{E^c} = \inf[b_2]_{E^c}$ is constant or strictly monotone and strongly continuous on $(a-d,a+d) \cap J$.

Claim. The interior of the (A-definable) set

$$C := \{a \in I : \mathcal{M} \models (\forall y_1,y_2)(\varphi(y_1,y_2,a) \longrightarrow (\exists z > 0)\psi_I(y_1,y_2,a,z))\}.$$

is cofinite in I.

Proof of the claim. To establish the claim, it is enough to show that the set $I \setminus C$ is finite. Suppose that $I \setminus C$ contains an open interval J. For $a \in J$ define

$$\begin{aligned} Y(a) &= \{\langle b_1,b_2\rangle \in M^2 : \mathcal{M} \models \varphi(b_1,b_2,a) \wedge \neg(\exists z>0)\psi_J(b_1,b_2,a,z) \text{ and} \\ &\qquad \mathcal{M} \models (\forall y_1,y_2 \leq b_1)(\varphi(y_1,y_2,a) \longrightarrow (\exists z>0)\psi_J(y_1,y_2,a,z))\}, \text{ and} \\ f(a) &= \sup\{b_1 \in M : (\exists b_2 > b_1)(\langle b_1,b_2\rangle \in Y(a))\}. \end{aligned}$$

Clearly, f is a definable function from J to $\overline{M}$. By the strong monotonicity, there is an open interval $J_1 \subseteq J$ such that $f \restriction J_1$ is strongly continuous. For $a \in J_1$, denote by $g(a)$ the supremum of an infinite equivalence class of E^a whose infimum is $f(a)$, and by $h(a)$ the infimum of an infinite equivalence class of E^a whose supremum is $f(a)$. By the strong monotonicity there are $a_1, a_2 \in J_1$ and $b_1, b_2 \in M$ such that $a_1 < a_2$ and $h(a) < b_1 < f(a) < b_2 < g(a)$ whenever $a \in (a_1, a_2)$. But then for every $a \in (a_1, a_2)$, there is $d > 0$ such that $\mathcal{M} \models \psi_{J_1}(b_1, b_2, a, d)$. This contradicts our definition of f and finishes the proof of the Claim.

Now, let J be a convex component of $\mathrm{int}(C)$. Then for every $a \in J$,

$$\mathcal{M} \models (\forall y_1, y_2)[\varphi(y_1, y_2, a) \longrightarrow (\exists z > 0)\psi_J(y_1, y_2, a, z)].$$

So for every $a \in J$, there is $d > 0$, $d \in M$, such that

$$\mathcal{M} \models (\forall y_1, y_2)[\varphi(y_1, y_2, a) \longrightarrow (\exists z_1, z_2)(\psi_J(z_1, z_2, a, d) \wedge E(y_1, z_1, a) \wedge E(y_2, z_2, a))].$$

Let

$$u(x, y) \equiv x \in J \wedge (\forall y_1, y_2)[\varphi(y_1, y_2, x) \longrightarrow (\exists z_1, z_2)(\psi_J(z_1, z_2, x, y) \wedge E(y_1, z_1, a) \wedge E(y_2, z_2, a))].$$

Clearly, the convex open A-definable set J and the formula $u(x, y)$ satisfy the assumptions of (a), so there is a partition of J into convex open definable sets $J_0, \ldots, J_s$ such that for any $i \leq s$ and any open interval $J' \subseteq J_i$, if $\inf J_i < \inf J' < \sup J' < \sup J_i$, then $J' \subseteq u(M, d)$ for some $d > 0$. For a given $i \leq s$, the number of equivalence classes of E^a is constant as a varies over J_i. This finishes the proof. ■

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box, $f : B \longrightarrow \overline{M}$ is a definable function, $i \in \{1, \ldots, m\}$ and $\overline{a} = \langle a_1, \ldots, a_m \rangle \in B$. In the definitions below we will use the following notation. By $B_i(\overline{a})$ we will denote the set of tuples $\langle b_1, \ldots, b_m \rangle \in B$ such that $b_j = a_j$ whenever $j \neq i$. Also, by $f_{i,\overline{a}}$ we will denote the function from $\varrho_i^m[B]$ into $\overline{M}$ sending $c \in \varrho_i^m[B]$ to $f(\overline{b})$, where $\overline{b} \in B_i(\overline{a})$ is the unique tuple such that $\varrho_i^m(\overline{b}) = c$. Note that if $B_i(\overline{a}) = B_i(\overline{a}')$, then the functions $f_{i,\overline{a}}$, $f_{i,\overline{a}'}$ coincide. Also, if $m = 1$ and $a \in B$, then $f_{1,a} = f$.

Definition 2.8 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box in M^m, $f : B \longrightarrow \overline{M}$ is a definable function, and $i \in \{1, \ldots, m\}$. We say that*
(a) f is i-constant iff for every $\overline{a} \in B$, the function $f_{i,\overline{a}}$ is constant;
(b) f is i-strictly increasing iff for every $\overline{a} \in B$, the function $f_{i,\overline{a}}$ is strictly increasing;
(c) f is i-strictly decreasing iff for every $\overline{a} \in B$, the function $f_{i,\overline{a}}$ is strictly decreasing.

Definition 2.9 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}$ is a definable function. We say that f is coordinate strongly continuous iff for every $i \in \{1, \ldots, m\}$, f is either i-constant or i-strictly increasing and strongly continuous, or i-strictly decreasing and strongly continuous.*

The proof of the following lemma is similar to the proof of Lemma 1.10 from [We].

Lemma 2.10 *Assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}$ is a definable function. Then there is an open box $B' \subseteq B$ such that $f \restriction B'$ is coordinate strongly continuous.*

Proof. For $m = 1$ the lemma is a consequence of the strong monotonicity of $\mathcal{M}$. Let $m \geq 1$. Fix a definable function $f : B \times I \longrightarrow \overline{M}$, where $B \subseteq M^m$ is an open box and $I \subseteq M$ is an open interval. For $\overline{a} \in B$ and $b \in I$, let $g_{\overline{a}}(b) = f(\overline{a}, b)$ and denote by $I(\overline{a})$ the maximal convex open subset of I for which $\inf I(\overline{a}) = \inf I$ and $g_{\overline{a}} \restriction I(\overline{a})$ is constant or strictly monotone and strongly continuous. For $\overline{a} \in B$, let $h(\overline{a}) = \sup I(\overline{a})$. Clearly $h : B \longrightarrow \overline{M}$ is a definable function, so by [Ar] and [MMS, Theorem 4.3], there are an open box $B_0 \subseteq B$ and an open interval $I_0 \subseteq I$ such that $\sup I_0 < h(\overline{a})$ whenever $\overline{a} \in B_0$. Let

$$
\begin{aligned}
X_1 &= \{\overline{a} \in B_0 : g_{\overline{a}} \restriction I_0 \text{ is constant}\}; \\
X_2 &= \{\overline{a} \in B_0 : g_{\overline{a}} \restriction I_0 \text{ is strictly increasing and strongly continuous }\}; \\
X_3 &= \{\overline{a} \in B_0 : g_{\overline{a}} \restriction I_0 \text{ is strictly decreasing and strongly continuous }\}.
\end{aligned}
$$

As $B_0 = X_1 \cup X_2 \cup X_3$, at least one of the sets X_1, X_2, X_3 contains an open box B_1. Suppose for example that $B_1 \subseteq X_1$. It is clear that $f \restriction B_1 \times I_0$ is $(m+1)$-constant. Repeating the above procedure for the remaining coordinates, we obtain an open box $B' \subseteq B_1 \times I_0$ such that $f \restriction B'$ is coordinate strongly continuous. ■

Definition 2.11 *Assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal expansion of an ordered group $(M, \leq, +)$, $m \geq 2$, $B \subseteq M^m$ is an open box, $f : B \longrightarrow \overline{M}$ is a definable function, $i, j \in \{1, \ldots, m\}$ and $i \neq j$.*

(a) f is $\langle i, j\rangle$-constant iff for any $\overline{a}, \overline{b} \in B$ with $\pi^m_i(\overline{a}) = \pi^m_i(\overline{b})$, the function $f_{j,\overline{b}} - f_{j,\overline{a}} : \varrho^m_j[B] \longrightarrow \overline{M}$ is constant.

(b) f is $\langle i, j\rangle$-strictly increasing [decreasing] iff for any $\overline{a}, \overline{b} \in B$ with $\pi^m_i(\overline{a}) = \pi^m_i(\overline{b})$ and $\varrho^m_i(\overline{a}) < \varrho^m_i(\overline{b})$, the function $f_{j,\overline{b}} - f_{j,\overline{a}} : \varrho^m_j[B] \longrightarrow \overline{M}$ is strictly increasing [decreasing].

Definition 2.12 *Assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal expansion of an ordered group $(M, \leq, +)$.*

(a) If $I \subseteq M$ is an open interval, then a function $f : I \longrightarrow \overline{M}$ is called monotonically strongly continuous iff f is constant or strictly monotone and strongly continuous.

(b) If $m \geq 2$ and $B \subseteq M^m$ is an open box, then a function $f : B \longrightarrow \overline{M}$ is called monotonically strongly continuous iff f is coordinate strongly continuous and for any $i, j \in \{1, \ldots, m\}$, $i \neq j$, f is either $\langle i, j\rangle$-constant or $\langle i, j\rangle$-strongly increasing, or $\langle i, j\rangle$-strongly decreasing.

Lemma 2.13 *Assume that $\mathcal{M} = (M, \leq, + \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, $I, J \subseteq M$ are open intervals and $f : I \times J \longrightarrow \overline{M}$ is a definable function. There are open intervals $I' \subseteq I$ and $J' \subseteq J$ such that $f \restriction I' \times J'$ is monotonically strongly continuous.*

Proof. By Lemma 2.10, without loss of generality we can assume that f is coordinate strongly continuous. Below, we will show how to find open intervals $I' \subseteq I$ and $J' \subseteq J$ such that the function $f \restriction I' \times J'$ is either $\langle 1, 2\rangle$-constant or $\langle 1, 2\rangle$-strictly increasing, or $\langle 1, 2\rangle$-strictly decreasing.

For $a_1, a_2 \in I$ define a function $\alpha_{a_1,a_2} : J \longrightarrow \overline{M}$ as follows: $\alpha_{a_1,a_2}(y) = f(a_2, y) - f(a_1, y)$. There is an $L(M)$-formula $E(x, y, z, t)$ such that for any $a_1, a_2, b, c \in M$, $\mathcal{M} \models E(b, c, a_1, a_2)$ iff $\inf I < a_1 < a_2 < \sup I$ and one of the following conditions is satisfied.

- $b, c < J$;
- $b, c > J$;

- $b = c \in J$;
- $b \neq c$ and there is an open interval $U \subseteq J$ containing b, c such that $\alpha_{a_1,a_2} \upharpoonright U$ is constant or strictly monotone.

Clearly, for any $a_1, a_2 \in I$ with $a_1 < a_2$, the formula $E(x, y, a_1, a_2)$ defines an equivalence relation E^{a_1,a_2} on M, which has finitely many equivalence classes, each of which constitutes a convex set.

By Lemma 2.7(b), for every $a_1 \in I$, there is $n(a_1) \in \mathbb{N}_+$ such that the equivalence relation E^{a_1,a_2} has at most $n(a_1)$ equivalence classes whenever $a_2 \in I$ and $a_2 > a_1$. For $a_1, a_2 \in I$ with $a_1 < a_2$ and for $i \in \{1, \ldots, n(a_1)\}$ define $g_i(a_1, a_2)$ as the supremum of the i-th equivalence class of E^{a_1,a_2} if E^{a_1,a_2} has at least i equivalence classes, and $+\infty$ in case E^{a_1,a_2} has less than i equivalence classes.

Fix $a_1 \in I$. Let $b_0 < \ldots < b_k \in \overline{M}$ be all distinct limits of the form $\lim\limits_{a_2 \longrightarrow a_1^+} g_i(a_1, a_2)$, $i \in \{1, \ldots, n(a_1)\}$ (existence of these is guaranteed by Lemma 1.3). Note that if $j < k$, $(b_j, b_{j+1}) \subseteq J$ and $\varepsilon > 0$, then there are $b, c \in (b_j, b_{j+1})$, $i < n(a_1)$ and $a \in (a_1, \sup I)$ such that $0 < b_{i+1} - c + b - b_i < \varepsilon$, and $g_i(a_1, a_2) < b < c < g_{i+1}(a_1, a_2)$ whenever $a_2 \in (a_1, a)$. This justifies our next step.

There is an $L(M)$-formula $F(x, y, z)$ such that for $a_1, b, c \in M$ we have that $\mathcal{M} \models F(b, c, a_1)$ iff $a_1 \in I$ and one of the following conditions holds.

- $b, c < J$;
- $b, c > J$;
- $b = c \in J$;
- $b, c \in J$, $b \neq c$ and there is $a > a_1$, $a \in I$, such that for every $a_2 \in (a_1, a)$, we have that $\mathcal{M} \models E(b, c, a_1, a_2)$.

Clearly, for every $a_1 \in I$, the formula $F(x, y, a_1)$ defines an equivalence relation F^{a_1} on M which has at most $2n(a_1)$ equivalence classes (some limits of the form $\lim\limits_{a_2 \longrightarrow a_1^+} g_i(a_1, a_2)$ might belong to M). By Lemma 2.7(b), the number of equivalence classes of F^a is bounded as a varies over I. There are an open interval $I_1 \subseteq I$ and definable functions $h_1, h_2 : I_1 \longrightarrow \overline{J} \cup \{\inf J, \sup J\}$ such that for every $a_1 \in I_1$, $h_1(a_1) < h_2(a_1)$ and $(h_1(a_1), h_2(a_1))$ is an equivalence class of F^{a_1}. By Lemma 1.5, without loss of generality we can assume that h_1, h_2 are strongly continuous. Then it is easy to see that there are open intervals $I_2 \subseteq I_1$ and $J_2 \subseteq J$ such that $h_1(a) < \inf J_2 < \sup J_2 < h_2(a)$ whenever $a \in I_2$. Let

$$\begin{aligned} X_1 &= \{a_1 \in I_2 : (\exists a > a_1)(\forall a_2 \in (a_1, a))(\alpha_{a_1,a_2} \upharpoonright J_2 \text{ is constant})\}; \\ X_2 &= \{a_1 \in I_2 : (\exists a > a_1)(\forall a_2 \in (a_1, a))(\alpha_{a_1,a_2} \upharpoonright J_2 \text{ is strictly increasing})\}; \\ X_3 &= \{a_1 \in I_2 : (\exists a > a_1)(\forall a_2 \in (a_1, a))(\alpha_{a_1,a_2} \upharpoonright J_2 \text{ is strictly decreasing})\}. \end{aligned}$$

Clearly, $I_2 = X_1 \cup X_2 \cup X_3$, so at least one of the sets X_1, X_2, X_3 contains an open interval. Say for instance that X_1 contains an open interval I_3. For $a_1 \in I_3$ define

$$u(a_1) = \sup\{a > a_1 : (\forall a_2 \in (a_1, a))(\alpha_{a_1,a_2} \upharpoonright J_2 \text{ is constant})\} - a_1.$$

It is easy to see that there are an open interval $I_4 \subseteq I_3$ and $d > 0$ such that $(\forall a_1 \in I_4)(u(a_1) > d)$. This implies that $(\forall a_1 \in I_4)(\forall a_2 \in (a_1, a_1 + d))(\alpha_{a_1,a_2} \upharpoonright J_2 \text{ is constant})$. Let $I' \subseteq I_4$ be an open interval of length at most d and let $J' = J_2$. Then $(\forall a_1, a_2 \in I')(\alpha_{a_1,a_2} \upharpoonright J' \text{ is constant})$. ∎

Lemma 2.14 *Assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}$ is a definable function.*

(a) There is an open box $B' \subseteq B$ such that $f \restriction B'$ is monotonically strongly continuous.

(b) If f is monotonically strongly continuous, then f is strongly continuous.

Proof. (a) For $m \in \mathbb{N}_+$ we will inductively prove the following statement.

$(*)_m$ If $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}$ is a definable function, then there is an open box $B' \subseteq B$ such that $f \restriction B'$ is monotonically strongly continuous.

Note that $(*)_1$ is a consequence of the strong monotonicity while $(*)_2$ follows from Lemma 2.13. So suppose that $m \geq 2$ and $(*)_m$ holds. Assume that $B \subseteq M^m$ is an open box, $I \subseteq M$ is an open interval and $f : B \times I \longrightarrow \overline{M}$ is a definable function. By Lemma 2.10, there are an open box $B_0 \subseteq B$ and an open interval $I_0 \subseteq I$ such that $f \restriction B_0 \times I_0$ is coordinate strongly continuous. For $\overline{a} \in B_0$ and $b \in I_0$ define $g_b(\overline{a}) = f(\overline{a}, b)$ and

$$\begin{aligned} U_b = \{\overline{a} \in B_0 :\ & \text{there is an open box } B_0' \subseteq B_0 \text{ containing } \overline{a} \\ & \text{such that } g_b \restriction B_0' \text{ is monotonically strongly continuous}\}. \end{aligned}$$

By $(*)_m$, U_b is large in B_0 whenever $b \in I_0$, so the set $\bigcup_{b \in I_0} U_b \times \{b\}$ must be large in $B_0 \times I_0$. Otherwise, there would exist an open box $B_0' \subseteq B_0$ and an open interval $I_0' \subseteq I_0$ such that $B_0' \times I_0' \subseteq (B_0 \times I_0) \setminus \bigcup_{b \in I_0} U_b \times \{b\}$. The latter would mean that U_b is not large in B_0 for $b \in I_0'$.

Since the set $\bigcup_{b \in I_0} U_b \times \{b\}$ is large in $B_0 \times I_0$, there are an open box $B_1 \subseteq B_0$ and an open interval $I_1 \subseteq I_0$ such that $B_1 \times I_1 \subseteq \bigcup_{b \in I_0} U_b \times \{b\}$. Fix $\overline{a} \in B_1$, and for $b \in I_1$ define

$$\begin{aligned} V_b = \bigcup \{C \subseteq B_1 :\ & C \text{ is an open box whose center is } \overline{a} \text{ and such that} \\ & g_b \restriction C \text{ is monotonically strongly continuous}\}. \end{aligned}$$

Let $V = \bigcup_{b \in I_1} V_b \times \{b\}$. By Fact 1.1, $\dim(V) = m + 1$, so there are an open box $B_2 \subseteq B_1$ and an open interval $I_2 \subseteq I_1$ such that $B_2 \times I_2 \subseteq V$. Our choice of B_2 and I_2 guarantees that for every $b \in I_2$, the function $g_b \restriction B_2$ is monotonically strongly continuous. For $b \in I_2$ and $i \in \{1, \ldots, m\}$ define

$$\alpha(b, i) = \begin{cases} -1 & \text{if} \quad g_b \restriction B_2 \text{ is } i\text{-strictly decreasing and strongly continuous} \\ 0 & \text{if} \quad g_b \restriction B_2 \text{ is } i\text{-constant} \\ 1 & \text{if} \quad g_b \restriction B_2 \text{ is } i\text{-strictly increasing and strongly continuous.} \end{cases}$$

Also, for $b \in I_2$ and $i, j \in \{1, \ldots, m\}$, $i \neq j$ define

$$\beta(b, i, j) = \begin{cases} -1 & \text{if} \quad g_b \restriction B_2 \text{ is } \langle i, j \rangle\text{-strictly decreasing} \\ 0 & \text{if} \quad g_b \restriction B_2 \text{ is } \langle i, j \rangle\text{-constant} \\ 1 & \text{if} \quad g_b \restriction B_2 \text{ is } \langle i, j \rangle\text{-strictly increasing.} \end{cases}$$

Now, for $b \in I_2$ define

$$\begin{aligned} \gamma(b) &= \langle \alpha(b, i) : 1 \leq i \leq m \rangle \text{ and} \\ \delta(b) &= \langle \beta(b, i, j) : i, j \in \{1, \ldots, m\}, i \neq j \rangle. \end{aligned}$$

There is an open interval $I_3 \subseteq I_2$ such that $\langle \gamma(b), \delta(b) \rangle$ is constant as b varies over I_3. Note that

- for every $i \in \{1, \ldots, m\}$, $f \restriction B_2 \times I_3$ is either i-constant or i-strictly increasing and strongly continuous, or i-strictly decreasing and strongly continuous;
- for any $i, j \in \{1, \ldots, m\}$, $i \neq j$, $f \restriction B_2 \times I_3$ is either $\langle i, j\rangle$-constant or $\langle i, j\rangle$-strictly incsreasing, or $\langle i, j\rangle$-strictly decreasing.

Repeating the above argument we can find an open box $B' \subseteq B_2 \times I_3$ such that for any distinct $i_1, \ldots, i_m \in \{1, \ldots, m+1\}$, the following conditions are satisfied.

- For every $i \in \{i_1, \ldots, i_m\}$, $f \restriction B'$ is either i-constant or i-strictly increasing and strongly continuous, or i-strictly decreasing and strongly continuous.
- For any $i, j \in \{i_1, \ldots, i_m\}$, $i \neq j$, $f \restriction B'$ is either $\langle i, j\rangle$-constant or $\langle i, j\rangle$-strictly incsreasing, or $\langle i, j\rangle$-strictly decreasing.

Clearly, $f \restriction B'$ is monotonically strongly continuous.

(b) For $m \in \mathbb{N}_+$ we will inductively prove the following condition $(*)_m$.

$(*)_m$ If $B \subseteq M^m$ is an open box, $\overline{a} \in \overline{B}$, $f : B \longrightarrow \overline{M}$ is a monotonically strongly continuous definable function and $\varepsilon > 0$, $\varepsilon \in M$, then there is an open box $C \subseteq B$ such that $\overline{a} \in \overline{C}$ and for any tuples $\overline{c}, \overline{d} \in C$ which differ on at most one coordinate, we have that $|f(\overline{c}) - f(\overline{d})| < \varepsilon$.

The condition $(*)_1$ being obvious, suppose that $(*)_m$ holds. Assume that $B \subseteq M^m$ is an open box, $I \subseteq M$ is an open interval, $\langle \overline{a}, b\rangle \in \overline{B} \times \overline{I}$, $f : B \times I \longrightarrow \overline{M}$ is a monotonically strongly continuous definable function, and $\varepsilon > 0$. Fix $b_1, b_2 \in I$ such that $b_1 < b < b_2$ and consider functions $g, h : B \longrightarrow \overline{M}$ defined as follows:

$$g(\overline{x}) = f(\overline{x}, b_1) \text{ and } h(\overline{x}) = f(\overline{x}, b_2) \text{ for } \overline{x} \in B.$$

By $(*)_m$, there is an open box $C \subseteq B$ such that $\overline{a} \in \overline{C}$, $\mathrm{cl}(C) \subseteq B$ (here $\mathrm{cl}(C)$ denotes the closure of C in M^m in the usual topology of M^m) and for any tuples $\overline{a}_1, \overline{a}_2 \in C$ which differ on at most one coordinate we have that

$$|g(\overline{a}_1) - g(\overline{a}_2)| < \varepsilon \text{ and } |h(\overline{a}_1) - h(\overline{a}_2)| < \varepsilon.$$

There are $b_3, b_4 \in M$ such that $b_1 < b_3 < b < b_4 < b_2$ and if $\overline{v}$ is a vertex of the box C, then

$$|f(\overline{v}, b_4) - f(\overline{v}, b_3)| < \varepsilon.$$

In the following two claims we will show that for any $\overline{c}, \overline{d} \in C \times (b_3, b_4)$ which differ on at most one coordinate, we have that $|f(\overline{c}) - f(\overline{d})| < \varepsilon$.

Claim 1. If $\overline{u} \in C$, then $|f(\overline{u}, b_4) - f(\overline{u}, b_3)| < \varepsilon$.

Proof of Claim 1. Fix a tuple $\langle u_1, \ldots, u_m\rangle \in C$. Denote by $\langle a_1^{i_1}, \ldots, a_m^{i_m}\rangle$, where $i_1, \ldots, i_m \in \{0, 1\}$ and $a_j^0 < a_j^1$ for $j \in \{1, \ldots, m\}$, the vertices of C. Also for $\eta : \{1, \ldots, m\} \longrightarrow \{0, 1\}$ and $i \leq m$ define tuples $\overline{w}(\eta, i)$ as follows:

$$\begin{aligned} \overline{w}(\eta, 0) &= \langle a_1^{\eta(1)}, \ldots, a_m^{\eta(m)}\rangle; \\ \overline{w}(\eta, i) &= \langle a_1^{\eta(1)}, \ldots, a_{m-i}^{\eta(m-i)}, u_{m+i-1}, \ldots, u_m\rangle \ (1 \leq i \leq m). \end{aligned}$$

To prove Claim 1, we will inductively on $i \leq m$ show that

$(\Delta)_i$ if $\eta : \{1,\ldots,m\} \longrightarrow \{0,1\}$, then $|f(\overline{w}(\eta,i),b_4) - f(\overline{w}(\eta,i),b_3)| < \varepsilon$.

The condition $(\Delta)_0$ is a consequence of our choice of b_3 and b_4. Let $0 \le i < m$, $\eta : \{1,\ldots,m\} \longrightarrow \{0,1\}$, and suppose that $(\Delta)_i$ holds. Let $\vartheta_0, \vartheta_1 : \{1,\ldots,m\} \longrightarrow \{0,1\}$ be functions defined by the conditions:

$$\vartheta_0(m-i) = 0,\ \vartheta_1(m-i) = 1 \text{ and } \vartheta_0(j) = \vartheta_1(j) = \eta(j) \text{ for } j \neq m-i.$$

Since f is $\langle m+1, m-i\rangle$-constant or $\langle m+1, m-i\rangle$-srictly incressing, or $\langle m+1, m-i\rangle$-strictly decreasing by $(\Delta)_i$, we have that

$$|f(\overline{w}(\eta,i+1),b_4) - f(\overline{w}(\eta,i+1),b_3)| \le$$
$$\max(|f(\overline{w}(\vartheta_0,i),b_4) - f(\overline{w}(\vartheta_0,i),b_3)|, |f(\overline{w}(\vartheta_1,i),b_4) - f(\overline{w}(\vartheta_1,i),b_3)|) < \varepsilon.$$

This proves Claim 1.

Claim 2. If $\overline{c},\overline{d} \in C \times (b_3,b_4)$ differ on at most one coordinate, then $|f(\overline{c}) - f(\overline{d})| < \varepsilon$.

Proof of Claim 2. Assume first that $\overline{c},\overline{d} \in C \times (b_3,b_4)$ do not differ on the first m coordinates. There are $\overline{c}' \in C \times \{b_3\}$ and $\overline{d}' \in C \times \{b_4\}$ such that $\overline{c},\overline{d},\overline{c}',\overline{d}'$ do not differ on the first m coordinates. Since f is coordinate strongly continuous, by Claim 1 we have that

$$|f(\overline{c}) - f(\overline{d})| \le |f(\overline{c}') - f(\overline{d}')| < \varepsilon.$$

Now fix $i \in \{1,\ldots,m\}$ and assume that $\overline{c},\overline{d} \in C \times (b_3,b_4)$ do not differ on coordinates $j \neq i$. There are $\overline{c}',\overline{d}' \in C \times \{b_1\}$ and $\overline{c}'',\overline{d}'' \in C \times \{b_2\}$ such that

- $\overline{c},\overline{c}',\overline{c}''$ do not differ on the first m coordinates;
- $\overline{d},\overline{d}',\overline{d}''$ do not differ on the first m coordinates.

Then the tuples $\overline{c}',\overline{d}'$ do not differ on coordinates $j \neq i$, and similarly $\overline{c}'',\overline{d}''$ do not differ on coordinates $j \neq i$. Our assumptions guarantee that

$$|f(\overline{c}) - f(\overline{d})| \le \max(|f(\overline{c}') - f(\overline{d}')|, |f(\overline{c}'') - f(\overline{d}'')|) < \varepsilon.$$

This proves Claim 2 and finishes the proof of $(*)_{m+1}$.

Now, for the proof of (b), consider an open box $B \subseteq M^m$, $m \in \mathbb{N}_+$, and a monotonically strongly continuous definable function $f : B \longrightarrow \overline{M}$. Let $\overline{a} \in \overline{B}$ and $\varepsilon > 0$, $\varepsilon \in M$. By $(*)_m$, there is an open box $C \subseteq B$ such that $\overline{a} \in \overline{C}$ and for any tuples $\overline{c},\overline{d} \in C$ which differ on at most one coordinate we have that $|f(\overline{c}) - f(\overline{d})| < \frac{\varepsilon}{m}$. From this one can easily infer that for any tuples $\overline{c},\overline{d} \in C$, we have that $|f(\overline{c}) - f(\overline{d})| < \varepsilon$. As this works for arbitrary $\varepsilon > 0$, the limit $\lim\limits_{\overline{x} \longrightarrow \overline{a}} f(\overline{x})$ exists in $\overline{B}$. Moreover, if $\overline{a} \in B$, then $f(\overline{a}) = \lim\limits_{\overline{x} \longrightarrow \overline{a}} f(\overline{x})$. Hence f is strongly continuous. ■

Assume that $\mathcal{M} = (M, \le, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \le, +)$, $m \in \mathbb{N}_+$ and $C = C_{m+1} \subseteq M^{m+1}$ is a strong open cell. By definition, for every $i \in \{1,\ldots,m\}$, there are strong open cells $C_i \subseteq M^i$ and strongly continuous definable functions $f_i : C_i \longrightarrow \overline{M} \cup \{-\infty\}$ and $g_i : C_i \longrightarrow \overline{M} \cup \{+\infty\}$ such that $\overline{f}_i(\overline{a}) < \overline{g}_i(\overline{a})$ for $\overline{a} \in \overline{C}_i$, and

$C_{i+1} = (f_i, g_i)_{C_i}$. For $\varepsilon > 0$ such that $2\varepsilon < \sup C_1 - \inf C_1$, we define ε-approximations of the cells $C_1, \ldots, C_{m+1}$ as follows ($1 \leq i \leq m$):

$$\begin{aligned} C_1(\varepsilon) &= \{x \in C_1 : \mathrm{dist}(\{x\}, M \setminus C_1) > \varepsilon\}; \\ h_i(\varepsilon) &= \min(\varepsilon, \sup\{z \in M : (f_i + 2z, g_i - 2z)_{C_i(\varepsilon)} \text{ is a strong cell}\}) = \\ &\quad \min(\varepsilon, \sup\{z \in M : \inf_{\overline{x} \in C_i(\varepsilon)} (f_i(\overline{x}) - g_i(\overline{x}) - 4z) > 0\}); \\ C_{i+1}(\varepsilon) &= (f_i + h_i(\varepsilon), g_i - h_i(\varepsilon))_{C_i(\varepsilon)}. \end{aligned}$$

Note that $C_i(\varepsilon)$ is a strong cell whenever $i \in \{1, \ldots, m+1\}$ and $0 < 2\varepsilon < \sup C_1 - \inf C_1$. Moreover,

- $\varepsilon \geq \mathrm{dist}(C_i(\varepsilon), M^i \setminus C_i) > 0$;
- if $\varepsilon' \in (0, \varepsilon)$, then $C_i(\varepsilon) \subseteq C_i(\varepsilon')$;
- $\bigcup_{\varepsilon > 0} C_i(\varepsilon) = C_i$.

Theorem 2.15 *Assume that $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, $A \subseteq M$ and $m \in \mathbb{N}_+$.*

(a)$_m$ If $k \in \mathbb{N}_+$ and $X_1, \ldots, X_k \subseteq M^m$ are A-definable sets, then there is a decomposition of M^m into A-definable strong cells partitioning each of the sets $X_1, \ldots, X_k$.

(b)$_m$ Assume that $U \subseteq M^m$ is a non-empty open A-definable set and $\varphi(\overline{x}, y)$, where $|\overline{x}| = m$, is an $L(A)$-formula such that

- *$\varphi(M, d)$ is an open subset of U whenever $d > 0$;*
- *$\varphi(M, d_2) \subseteq \varphi(M, d_1)$ whenever $0 < d_1 \leq d_2$;*
- *$\bigcup_{d > 0} \varphi(M, d) = U$.*

Then there is a decomposition $\mathcal{D}$ of U into strong cells in M^m such that for every open cell $D \in \mathcal{D}$, if $B \subseteq D$ is an open box with $\mathrm{dist}(B, M^m \setminus D) > 0$, then $B \subseteq \varphi(M, d)$ for some $d > 0$.

(c)$_m$ If $X \subseteq M^m$ is a non-empty A-definable set and $f : X \longrightarrow \overline{M}$ is an A-definable function, then there is a decomposition of X into A-definable strong cells such that for every $D \in \mathcal{D}$, $f \restriction D$ is strongly continuous.

(d)$_m$ If $X \subseteq M^m$ is a non-empty definable set and $E(x, y, \overline{z})$ is an $L(M)$-formula such that $|\overline{z}| = m$ and for every $\overline{a} \in X$, $E(x, y, \overline{a})$ defines an equivalence relation on M with finitely many classes, then there is $n \in \mathbb{N}_+$ such that for every $\overline{a} \in X$, the equivalence relation defined by $E(x, y, \overline{a})$ has at most n equivalence classes.

(e)$_m$ If $X \subseteq M^{m+1}$ is a definable set, then there is a positive integer n such that for any $\overline{a} \in \pi^{m+1}_{m+1}[X]$, the set $\{b \in M : \langle \overline{a}, b \rangle \in X\}$ has at most n convex components.

Proof. (a)$_1$ is obvious by the weak o-minimality of $\mathcal{M}$. (b)$_1$, (c)$_1$ and (d)$_1$ are consequences of Lemma 2.7 and the strong monotonicity of $\mathcal{M}$. Let $m \in \mathbb{N}_+$ and suppose that the conditions (a)$_m$–(d)$_m$ hold.

Proof of (d)$_m$ $\Longrightarrow$(e)$_m$. Assume that $X \subseteq M^{m+1}$ is a definable set. For $\overline{a} \in M^m$ denote by $R(\overline{a})$ the set of all elements $b \in M$ for which $\langle \overline{a}, b \rangle \in X$, and define the following formula.

$$\begin{aligned} E(x, y, \overline{z}) \equiv\ & [x \leq y \wedge ([x, y] \subseteq R(\overline{z}) \vee [x, y] \cap R(\overline{z}) = \emptyset)] \vee \\ & [y \leq x \wedge ([y, x] \subseteq R(\overline{z}) \vee [y, x] \cap R(\overline{z}) = \emptyset)]. \end{aligned}$$

Clearly, for every $\overline{a} \in M^m$, $E(x, y, \overline{a})$ defines an equivalence relation $E^{\overline{a}}$ on M, whose number of equivalence classes is not lower than the number of convex components of $R(\overline{a})$. By (d)$_m$, there exists $n \in \mathbb{N}_+$ such that for every $\overline{a} \in M^m$, $E^{\overline{a}}$ has at most n equivalence classes. Hence, for every $\overline{a} \in M^m$, the set $R(\overline{a})$ has at most n convex components.

Proof of (a)$_{m+1}$. Assume that $X_1, \ldots, X_k \subseteq M^{m+1}$ are definable sets. By (e)$_m$, there is $n \in \mathbb{N}_+$ such that for any $\overline{a} \in M^m$ and $\eta : \{1, \ldots, k\} \longrightarrow \{0, 1\}$, the set $\{b \in M : \langle \overline{a}, b\rangle \in X_1^{\eta(1)} \cap \ldots \cap X_k^{\eta(k)}\}$ has at most n convex components (here X_i^0 denotes X_i whereas $X_i^1 = M^m \setminus X_i$). For $\overline{a} \in M^m$ denote by $\langle I_0(\overline{a}), \ldots, I_{s(\overline{a})}(\overline{a})\rangle$ the unique partition of M into convex sets such that

- $I_0(\overline{a}) < \ldots < I_{s(\overline{a})}(\overline{a})$;
- for every $i \leq s(\overline{a})$, there is $\eta : \{1, \ldots, k\} \longrightarrow \{0, 1\}$ such that $I_i(\overline{a})$ is a convex component of $\{b \in M : \langle \overline{a}, b\rangle \in X_1^{\eta(1)} \cap \ldots \cap X_k^{\eta(k)}\}$.

Clearly, $s(\overline{a}) \leq n \cdot 2^k$ whenever $\overline{a} \in M^m$. For $\overline{a} \in M^m$ and $i \leq s(\overline{a})$ define:

$$\alpha_i(\overline{a}) = \begin{cases} 0 & \text{if } \inf I_i(\overline{a}) = -\infty, \\ 1 & \text{if } \inf I_i(\overline{a}) \in I_i(\overline{a}), \\ 2 & \text{if } \inf I_i(\overline{a}) \in M \setminus I_i(\overline{a}), \\ 3 & \text{if } \inf I_i(\overline{a}) \in \overline{M} \setminus M; \end{cases} \quad \beta_i(\overline{a}) = \begin{cases} 1 & \text{if } \sup I_i(\overline{a}) \in I_i(\overline{a}), \\ 2 & \text{if } \sup I_i(\overline{a}) \in M \setminus I_i(\overline{a}), \\ 3 & \text{if } \sup I_i(\overline{a}) \in \overline{M} \setminus M, \\ 4 & \text{if } \sup I_i(\overline{a}) = +\infty; \end{cases}$$

$$\gamma_i(\overline{a}) = \eta \text{ iff } I_i(\overline{a}) \subseteq \{b \in M : \langle \overline{a}, b\rangle \in X_1^{\eta(1)} \cap \ldots \cap X_k^{\eta(k)}\}.$$

For $\overline{a} \in M^m$, let $\theta(\overline{a}) = \{\langle i, \alpha_i(\overline{a}), \beta_i(\overline{a}), \gamma_i(\overline{a})\rangle : i \leq s(\overline{a})\}$ and $\{\theta_0, \ldots, \theta_l\} = \{\theta(\overline{a}) : \overline{a} \in M^m\}$. By (a)$_m$, there is a decomposition $\mathcal{D}_0$ of M^m into strong cells in M^m partitioning each of the sets $\{\overline{a} \in M^m : \theta(\overline{a}) = \theta_i\}$, $i \leq l$. To finish the proof, by (a)$_m$, it is enough to show that for every $D \in \mathcal{D}_0$, there is a decomposition of $D \times M$ into strong cells in M^{m+1} which partitions each of the sets $X_1 \cap (D \times M), \ldots, X_k \cap (D \times M)$.

Fix $D \in \mathcal{D}_0$ and let $s_D = s(\overline{a})$ for $\overline{a} \in D$. By (a)$_m$, our assertion is obvious in case $\dim(D) < m$, so assume that D is open. For $i \leq s_D$ define functions $f_i, g_i : D \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ as follows:

$$f_i(\overline{a}) = \inf I_i(\overline{a}),\ \ g_i(\overline{a}) = \sup I_i(\overline{a}),\ \ \overline{a} \in D.$$

Clearly, $f_0(\overline{a}) = -\infty$, $g_{s_D}(\overline{a}) = +\infty$ and $g_i(\overline{a}) = f_{i+1}(\overline{a})$ whenever $\overline{a} \in D$ and $0 < i < s_D$. Note that for every $i \leq s_D$, the set $\{\overline{a} \in D : f_i(\overline{a}) = g_i(\overline{a})\}$ is either empty or equal to D. Let $h_0, \ldots, h_r$ denote all distinct functions appearing in $\{f_i, g_i : i \leq s_D\}$, enumerated so that $-\infty = h_0(\overline{a}) < \ldots < h_r(\overline{a}) = +\infty$ for $\overline{a} \in D$. By (c)$_m$, there is a decomposition $\mathcal{C}_D$ of D into strong cells such that on each cell from $\mathcal{C}_D$, the functions $h_0, \ldots, h_r$ are strongly continuous. Again, by (a)$_m$, without loss of generality we can assume that $\mathcal{C}_D = \{D\}$.

There is an $L(A)$-formula $\varphi(\overline{x}, z)$ such that for any $\overline{a} \in M^m$ and $d > 0$, $\mathcal{M} \models \varphi(\overline{a}, d)$ iff $\overline{a} \in D$ and $h_{i+1}(\overline{a}) - h_i(\overline{a}) > d$ for $0 \leq i < r$. Note that

- $\varphi(M, d)$ is an open definable subset of D;
- $\varphi(M, d_1) \subseteq \varphi(M, d_2)$ for $0 < d_2 \leq d_1$
- $\bigcup_{d>0} \varphi(M, d) = D$.

By (b)$_m$, there is a decomposition $\mathcal{D}_1$ of D into strong cells in M^m such that for every open cell $C \in \mathcal{D}_1$, if $B \subseteq C$ is an open box such that $\mathrm{dist}(B, M^m \setminus C) > 0$, then $B \subseteq \varphi(M, d)$ for some $d > 0$. This implies that if $i < s_D$ and C is an open cell in $\mathcal{D}_1$, then $\overline{h}_i(\overline{x}) < \overline{h}_{i+1}(\overline{x})$ for all $\overline{x} \in \overline{C}$ and $(h_i, h_{i+1})_C$ is a strong open cell in M^{m+1}. Now, using (a)$_m$ we can easily find a decomposition of M^{m+1} into strong cells in M^{m+1} satisfying our demands.

Proof of (b)$_{m+1}$. Assume that $U \subseteq M^{m+1}$ is a non-empty open A-definable set and $\varphi(\overline{x}, y)$, where $|\overline{x}| = m+1$, is an $L(A)$-formula satisfying the requirements of (b)$_{m+1}$. By (a)$_{m+1}$, there is a decomposition $\mathcal{D}_0$ of U into A-definable strong cells. In what follows we will inductively find cell decompositions $\mathcal{D}_1, \ldots, \mathcal{D}_{m+1}$ of U into strong cells in M^{m+1} such that $\mathcal{D}_{i+1}$ refines $\mathcal{D}_i$ whenever $i \leq m$, and for every $i \in \{1, \ldots, m+1\}$, if C is an open cell in $\mathcal{D}_i$ and $0 < 2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$, then the following conditions are satisfied.

- $(*)_1$ (for $i = 1$)$(\forall \overline{a} \in \pi_{m+1}^{m+1}[C(\varepsilon)])(\exists d > 0)[\{b \in M : \langle \overline{a}, b\rangle \in C(\varepsilon)\} \subseteq \varphi(\overline{a}, M, d)]$;
- $(*)_i$ $(\forall \overline{a} \in \varrho_{1,\ldots,m+1-i}^{m+1}[C(\varepsilon)])(\exists d > 0)[\{b \in M : \langle \overline{a}, b\rangle \in \varrho_{1,\ldots,m+2-i}^{m+1}[C(\varepsilon) \setminus \varphi(M, d)]\} = \emptyset]$ in case $1 < i \leq m$;
- $(*)_{m+1}$ (for $i = m+1$)$(\exists d > 0)(\varrho_1^{m+1}[C(\varepsilon) \setminus \varphi(M, d)] = \emptyset)$.

This is clearly sufficient as for any open cell $C \in \mathcal{D}_{m+1}$ and any open box $B \subseteq C$ with $\mathrm{dist}(B, M^{m+1} \setminus C) > 0$, there are $\varepsilon > 0$ and $d > 0$ for which $B \subseteq C(\varepsilon) \subseteq \varphi(M, d)$. Our construction will consist in three steps.

Step 1. Let $C \in \mathcal{D}_0$ be an open strong cell. For $\overline{a} \in \pi_{m+1}^{m+1}[C]$ and $d > 0$ define $R(\overline{a}, d) = \{c \in M : \langle \overline{a}, c\rangle \in \varphi(M, d)\}$. Note that for every $\overline{a} \in \pi_{m+1}^{m+1}[C]$,

- $\bigcup_{d>0} R(\overline{a}, d) = \{c \in M : \langle \overline{a}, c\rangle \in C\}$;
- (by (e)$_1$) there is $n(\overline{a}) \in \mathbb{N}_+$ such that $R(\overline{a}, d)$ has at most $n(\overline{a})$ convex components as d varies over $(0, +\infty)$.

There is an $L(A)$-formula $E(x, y, \overline{z})$ such that for any $\overline{a} \in \pi_{m+1}^{m+1}[C]$ and $b_1, b_2 \in M$, $\mathcal{M} \models E(b_1, b_2, \overline{a})$ iff one of the following conditions is satisfied.

- $b_1, b_2 < \{c \in M : \langle \overline{a}, c\rangle \in C\}$;
- $b_1, b_2 > \{c \in M : \langle \overline{a}, c\rangle \in C\}$;
- $\langle \overline{a}, b_1\rangle, \langle \overline{a}, b_2\rangle \in C$ and there is $d > 0$ such that b_1, b_2 belong to the same convex component of $R(\overline{a}, d)$.

Clearly, for every $\overline{a} \in \pi_{m+1}^{m+1}[C]$, the formula $E(x, y, \overline{a})$ defines an equivalence relation $E^{\overline{a}}$ on M whose equivalence classes are convex infinite sets. For every $\overline{a} \in \pi_{m+1}^{m+1}[C]$, the equivalence relation $E^{\overline{a}}$ has only finitely many equivalence classes. By (d)$_m$, there is $n \in \mathbb{N}_+$ such that $E^{\overline{a}}$ has at most n equivalence classes as $\overline{a}$ varies over $\pi_{m+1}^{m+1}[C]$. The ordering of the structure determines a natural ordering of equivalence classes of $E^{\overline{a}}$, $\overline{a} \in \pi_{m+1}^{m+1}[C]$. For $i \in \{1, \ldots, n\}$ define

$$X_i = \{\langle \overline{a}, c\rangle \in \pi_{m+1}^{m+1}[C] \times M : c \text{ is in the } i\text{-th equivalence class of } E^{\overline{a}}\}.$$

By (a)$_{m+1}$, there is a decomposition $\mathcal{D}_0^C$ of of C into strong cells in M^{m+1} partitioning each of the sets $X_1, \ldots, X_n$. By (a)$_{m+1}$ there is a cell decomposition $\mathcal{D}_1$ of U partitioning each of the cells from $\mathcal{D}_0^C$, where C is an open cell from $\mathcal{D}_0$.

We claim that for open cells from $\mathcal{D}_1$, condition $(*)_1$ holds. Let C be an open cell from $\mathcal{D}_1$ and let $\varepsilon > 0$ be an element of M such that $0 < 2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$. For $\overline{a} \in \pi_{m+1}^{m+1}[C(\varepsilon)]$ and $d > 0$, the set $\{b \in M : \langle \overline{a}, b\rangle \in C \cap \varphi(M, d)\}$ is convex and $\bigcup_{d>0} \{b \in M : \langle \overline{a}, b\rangle \in C \cap \varphi(M, d)\} = \{b \in M : \langle \overline{a}, b\rangle \in C\}$. Hence there is $d > 0$ for which $\{b \in M : \langle \overline{a}, b\rangle \in C(\varepsilon)\} \subseteq \varphi(\overline{a}, M, d)$. This completes the proof of $(*)_1$.

Step $i + 1$ $(1 \leq i < m)$. Suppose we have already constructed strong cell decompositions $\mathcal{D}_1, \ldots, \mathcal{D}_i$, $i < m$, of U satisfying our demands. Let C be an open cell in $\mathcal{D}_i$. For $d > 0$, $\varepsilon > 0$ with $2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$ and $\overline{a} \in \varrho_{1,\ldots,m-i}^{m+1}[C(\varepsilon)]$ define

$$R(\overline{a}, d, \varepsilon) = \{c \in M : \langle \overline{a}, c\rangle \in \varrho_{1,\ldots,m+1-i}^{m+1}[C(\varepsilon)] \setminus \varrho_{1,\ldots,m+1-i}^{m+1}[C(\varepsilon) \setminus \varphi(M, d)]\}.$$

Our inductive assumption about $\mathcal{D}_i$ guarantees that for every $\varepsilon > 0$ with $2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$ we have that

$$\bigcap_{d>0} \varrho_{1,\ldots,m+1-i}^{m+1}[C(\varepsilon) \setminus \varphi(M, d)] = \emptyset.$$

Consequently for $\overline{a} \in \varrho_{1,\ldots,m-i}^{m+1}[C(\varepsilon)]$,

$$\bigcup_{d>0} R(\overline{a}, d, \varepsilon) = \{c \in M : \langle \overline{a}, c\rangle \in \varrho_{1,\ldots,m+1-i}^{m+1}[C(\varepsilon)]\}.$$

By (e)$_1$, for any $\overline{a} \in \varrho_{1,\ldots,m-i}^{m+1}[C(\varepsilon)]$ and any $\varepsilon > 0$ with $2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$, there is $n(\overline{a}, \varepsilon) \in \mathbb{N}_+$ such that the set $R(\overline{a}, d, \varepsilon)$ has at most $n(\overline{a}, \varepsilon)$ convex components as d varies over $(0, +\infty)$. There is an $L(A)$-formula $E(x, y, \overline{z}, t)$ such that for any $\overline{a} \in \varrho_{1,\ldots,m-i}^{m+1}[C(\varepsilon)]$, any $b_1, b_2 \in M$, and any $\varepsilon > 0$ with $2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$, $\mathcal{M} \models E(b_1, b_2, \overline{a}, \varepsilon)$ iff one of the following conditions is satisfied.

- $b_1, b_2 < \{c \in M : \langle \overline{a}, c\rangle \in \varrho_{1,\ldots,m+2-i}^{m+1}[C(\varepsilon)]\}$;
- $b_1, b_2 > \{c \in M : \langle \overline{a}, c\rangle \in \varrho_{1,\ldots,m+2-i}^{m+1}[C(\varepsilon)]\}$;
- $\langle \overline{a}, b_1\rangle, \langle \overline{a}, b_2\rangle \in \varrho_{1,\ldots,m+1-i}^{m+1}[C(\varepsilon)]$ and there is $d > 0$ such that b_1, b_2 belong to the same convex component of $R(\overline{a}, d, \varepsilon)$.

For any $\overline{a} \in \varrho_{1,\ldots,m-i}^{m+1}[C(\varepsilon)]$ and any $\varepsilon > 0$ with $2\varepsilon < \sup \varrho_1^{m+1}[C] - \inf \varrho_1^{m+1}[C]$, the formula $E(x, y, \overline{a}, \varepsilon)$ defines an equivalence relation $E^{\overline{a},\varepsilon}$ on M with finitely many equivalence classes each of which is a convex set. By Lemma 2.7(b), the number of equivalence classes of $E^{\overline{a},\varepsilon}$ is bounded by some $n(\overline{a}) \in \mathbb{N}_+$ as ε varies over $(0, +\infty)$. There is an $L(A)$-formula $F(x, y, \overline{z})$ such that for any $\overline{a} \in M^{m-i}$ and $b_1, b_2 \in M$ we have that $\mathcal{M} \models F(b_1, b_2, \overline{a})$ iff one of the following conditions holds.

- $b_1, b_2 < \{c \in M : \langle \overline{a}, c\rangle \in \varrho_{1,\ldots,m+1-i}^{m}[C]\}$;
- $b_1, b_2 > \{c \in M : \langle \overline{a}, c\rangle \in \varrho_{1,\ldots,m+1-i}^{m}[C]\}$;
- $b_1 = b_2$;
- $\langle \overline{a}, b_1\rangle, \langle \overline{a}, b_2\rangle \in \varrho_{1,\ldots,m+1-i}^{m+1}[C]$, $b_1 \neq b_2$, and there are $\varepsilon_0 > 0$ and an open interval $J \subseteq M$ such that for any $c_1, c_2 \in J$ and any $\varepsilon \in (0, \varepsilon_0)$, we have that $\mathcal{M} \models E(c_1, c_2, \overline{a}, \varepsilon)$.

Clearly, for every $\overline{a} \in \varrho^{m+1}_{1,\dots,m-i}[C]$, the formula $F(x,y,\overline{a})$ defines an equivalence relation on M with finitely many equivalence classes. Moreover, the equivalence classes of $F^{\overline{a}}$ are convex sets. By (d)$_{m-i}$, there is $n \in \mathbb{N}_+$ such that for every $\overline{a} \in \varrho^{m+1}_{1,\dots,m-i}[C]$, the equivalence relation $F^{\overline{a}}$ has at most n equivalence classes. For $j \in \{1,\dots,n\}$ define

$$X_j = \{\langle \overline{a}, c\rangle \in \varrho^{m+1}_{1,\dots,m+1-i} \times M : c \text{ is in the } i\text{-th equivalence class of } F^{\overline{a}}\}.$$

Let $\mathcal{C}$ be a decomposition of M^{m+1-i} into strong cells partitioning each of the sets $X_1,\dots,X_n$. There is a decomposition $\mathcal{D}^C_i$ of C such that $\varrho^{m+1}_{1,\dots,m+1-i}[\mathcal{D}^C_i]$ partitions all cells from $\mathcal{C}_C$. Now by (a)$_{m+1}$, there is a decompositon of U into strong open cells partitioning each of the cells from $\mathcal{D}^C_i$, where C is an open cell from $\mathcal{D}_i$.

Step m+1. Suppose that we have already constructed $\mathcal{D}_1,\dots,\mathcal{D}_m$ satisfying our demands. Fix $C \in \mathcal{D}_m$ and $\varepsilon > 0$ with $2\varepsilon < \sup \varrho_1[C] - \inf \varrho_1[C]$. Our construction guarantees that

$$\bigcap_{d>0} \varrho^{m+1}_1[C(\varepsilon) \setminus \varphi(M,d)] = \emptyset.$$

For $d > 0$ and $\varepsilon > 0$ with $2\varepsilon < \sup \varrho^{m+1}_1[C] - \inf \varrho_1[C]$ define

$$R(d,\varepsilon) = \varrho^{m+1}_1[C(\varepsilon)] \setminus \varrho^{m+1}_1[C(\varepsilon) \setminus \varphi(M,d)].$$

For every $\varepsilon > 0$ with $2\varepsilon < \sup \varrho^{m+1}_1[C] - \inf \varrho^{m+1}_1[C]$ we have that $\bigcup_{d>0} R(d,\varepsilon) = \varrho^{m+1}_1[C(\varepsilon)]$. Let $E(x,y,t)$ be an $L(A)$-formula such that for any $b_1, b_2 \in M$ and any $\varepsilon > 0$ with $2\varepsilon < \sup \varrho^{m+1}_1[C] - \inf \varrho^{m+1}_1[C]$, we have that $\mathcal{M} \models E(b_1,b_2,\varepsilon)$ iff one of the following conditions holds.

- $b_1, b_2 < \varrho^{m+1}_1[C(\varepsilon)]$;
- $b_1, b_2 > \varrho^{m+1}_1[C(\varepsilon)]$;
- $b_1, b_2 \in \varrho^{m+1}_1[C(\varepsilon)]$ and b_1, b_2 are in the same convex component of $R(d,\varepsilon)$ for some $d > 0$.

Clearly, for every $\varepsilon > 0$ with $2\varepsilon < \sup \varrho^{m+1}_1[C] - \inf \varrho^{m+1}_1[C]$, the formula $E(x,y,\varepsilon)$ defines an equivalence relation E^ε on M with finitely many equivalence classes. Repeating an appropriate argument from Step 2 we define an equivalence relation F on M which has finitely many equivalence classes all of which are convex sets whose boundary points are limits of functions determined by boundary points of the equivalence classes of $E(x,y,\varepsilon)$ as ε tends to 0. Let $\mathcal{C}_C$ be a decomposition of $\varrho^{m+1}_1[C]$ into strong cells partitioning all the equivalence classes of E. There is a decomposition $\mathcal{D}^C_m$ of C such that $\varrho^{m+1}_1[\mathcal{D}^C_m]$ partitions all cells from $\mathcal{C}_C$. Now by (a)$_{m+1}$, there is a decompositon of U into strong open cells partitioning each of the cells from $\mathcal{D}^C_m$, where C is an open cell from $\mathcal{D}_m$.

Proof of (c)$_{m+1}$. Assume that $X \subseteq M^{m+1}$ is a non-empty A-definable set and $f : X \longrightarrow \overline{M}$ is an A-definable function. By (a)$_{m+1}$, without loss of generality we can assume that X is a strong cell in M^{m+1}. Below we consider two cases.

Case 1. $\dim(X) \leq m$. There is a projection $\pi : M^{m+1} \longrightarrow M^m$ dropping one coordinate such that $\pi[C]$ is a strong cell in M^m. Let $g : \pi[C] \longrightarrow C$ be the map defined by $g(\pi(\overline{a})) = \overline{a}$, $\overline{a} \in C$. By (c)$_m$, there is a decomposition $\mathcal{D}_0$ of $\pi[C]$ into strong A-definable cells in M^m such that the function $f \circ g$ is strongly continuous on every strong cell from this decomposition. Let

$\mathcal{D}_1 = \{g[D] : D \in \mathcal{D}_0\}$. Clearly, $\mathcal{D}_1$ is a decomposition of D into strong cells in M^{m+1}, and for every $C \in \mathcal{D}_0$, we have that $f \restriction D$ is strongly continuous.

Case 2. $\dim(X) = m + 1$. There is an $L(A)$-formula $\varphi(\overline{x}, y)$, $|\overline{x}| = m + 1$, such that for any $\overline{a} \in M^{m+1}$ and $d \in M$, $\mathcal{M} \models \varphi(\overline{a}, d)$ iff $\overline{a} \in X$, $d > 0$ and there is an open box $B \subseteq X$ containing $\overline{a}$ such that the edges of B are of length d and $f \restriction B$ is monotonically strongly continuous. By Lemma 2.14 , the set $X' := \bigcup_{d>0} \varphi(M, d)$ is large in X. By (a)$_{m+1}$, there is a decomposition $\mathcal{D}$ of X into strong cells in M^{m+1} partitioning X'. By (a)$_{m+1}$, (c)$_m$ and the argument from Case 1, without loss of generality we can assume that f restricted to any cell from $\mathcal{D}$ of dimension $\leq m$ is strongly continuous.

Fix an open cell $C \in \mathcal{D}$. To finish the proof, by (a)$_{m+1}$, it is enough to find a decomposition of C into strong cells such that f is strongly continuous on each cell from this decomposition. Let $\psi(\overline{x}, y) \equiv \varphi(\overline{x}, y) \wedge \overline{x} \in C$. Note that

- $\psi(M, d)$ is an open subset of C for every $d > 0$;
- $\bigcup_{d>0} \psi(M, d) = C$;
- $\psi(M, d_1) \subseteq \psi(M, d_2)$ whenever $0 < d_2 \leq d_1$.

By (b)$_{m+1}$, there is a decomposition $\mathcal{D}'$ of C into strong cells in M^{m+1} such that for any open cell $C' \in \mathcal{D}'$ and any open box $B' \subseteq C'$ with $\mathrm{dist}(B, M^{m+1} \setminus C') > 0$, there is $d > 0$ such that $B \subseteq \psi(M, d)$. This means that for every $\overline{a} \in \overline{C}'$, there in an open box $B \subseteq C'$ such that $\overline{a} \in \overline{B}$ and $B \subseteq \psi(M, d)$ for some $d > 0$. But then $f \restriction B$ is strongly continuous. Consequently, $f \restriction C'$ is strongly continuous. Again, by (a)$_{m+1}$ and (c)$_m$, there is a deecomposition $\mathcal{D}''$ of C into strong cells in M^{m+1} refining $\mathcal{D}'$ and such that for every $D \in \mathcal{D}''$, $f \restriction D$ is strongly continuous.

Proof of (d)$_{m+1}$. Assume that $X \subseteq M^{m+1}$ is a non-empty definable set, $E(x, y, \overline{z})$ is an $L(M)$-formula such that $|\overline{z}| = m+1$ and for every $\overline{a} \in X$, $E(x, y, \overline{a})$ defines on M an equivalence relation $E^{\overline{a}}$ with finitely many classes. Without loss of generality we can assume that each equivalence class of $E^{\overline{a}}$ is convex whenever $\overline{a} \in X$. By (a)$_{m+1}$ and (d)$_m$, we can also assume that X is an open strong cell.

There is an $L(M)$-formula $\varphi(t_1, t_2, \overline{z})$ such that for any $\overline{a} \in X$ and $b_1, b_2 \in M$, $\mathcal{M} \models \varphi(b_1, b_2, \overline{a})$ iff the classes $[b_1]_{E^{\overline{a}}}$, $[b_2]_{E^{\overline{a}}}$ are both infinite and

$$\inf[b_1]_{E^{\overline{a}}} < b_1 < \sup[b_1]_{E^{\overline{a}}} = \inf[b_2]_{E^{\overline{a}}} < b_2 < \sup[b_2]_{E^{\overline{a}}}.$$

For $\overline{a} \in M^{m+1}$ and $d > 0$ denote by $B(\overline{a}, d)$ the open box whose center is $\overline{a}$ and whose edges are of length d. There is an $L(M)$-formula $\psi(y_1, y_2, \overline{x}, z)$ such that for any $\overline{a} \in X$, $b_1, b_2 \in M$ and $d \in M$, we have that $\mathcal{M} \models \psi(b_1, b_2, \overline{a}, d)$ iff the following conditions are satisfied.

- $B(\overline{a}, d) \subseteq X \cap \varphi(b_1, b_2, M)$;
- the function $f : B(\overline{a}, d) \longrightarrow \overline{M}$ defined by $f(\overline{c}) = \sup\{d \in (b_1, b_2) : \inf E(d_1, M, \overline{c}) \leq b_1\}$ is monotonically strongly continuous on $B(\overline{a}, d)$.

Let

$$C = \{\overline{a} \in X : \mathcal{M} \models (\forall y_1, y_2)(\varphi(y_1, y_2, \overline{a}) \longrightarrow (\exists z > 0)\psi(y_1, y_2, \overline{a}, z))\}.$$

We claim that $\dim(X \setminus \mathrm{int}(C)) \leq m$. For the latter, it suffices to show that $\dim(X \setminus C) \leq m$. Indeed, if $\dim(X \setminus C) \leq m$ and $\dim(X \setminus \mathrm{int}(C)) = m + 1$, then $\mathrm{int}(C \setminus \mathrm{int}(C)) \neq \emptyset$, which is impossible.

Suppose for a contradiction that the set $X \setminus C$ contains an open box B. For $\overline{a} \in B$ define

$$\begin{aligned} Y(\overline{a}) &= \{\langle b_1, b_2\rangle \in M^2 : \mathcal{M} \models \varphi(b_1, b_2, \overline{a}) \wedge \neg(\exists z > 0)\psi(b_1, b_2, \overline{a}, z) \text{ and} \\ &\quad \mathcal{M} \models (\forall y_1, y_2 \leq b_1)(\varphi(y_1, y_2, \overline{a}) \longrightarrow (\exists z > 0)\psi(y_1, y_2, \overline{a}, z))\}, \text{ and} \\ f(\overline{a}) &= \sup\{b_1 \in M : (\exists b_2 > b_1)(\langle b_1, b_2\rangle \in Y(\overline{a}))\}. \end{aligned}$$

Clearly, f is a definable function from B to $\overline{M}$. By Lemma 2.14, there is an open box $B_1 \subseteq B$ such that $f \restriction B_1$ is strongly continuous. For $\overline{a} \in B_1$, denote by $g(\overline{a})$ the supremum of the infinite equivalence class of $E^{\overline{a}}$ whose infinmum is $f(\overline{a})$, and by $h(\overline{a})$ the infimum of the infinite equivalence class of $E^{\overline{a}}$ whose supremum is $f(\overline{a})$. It is easy to see that there is an open box B_2 and $b_1, b_2 \in M$ such that $h(\overline{a}) < b_1 < f(\overline{a}) < b_2 < g(\overline{a})$ whenever $\overline{a} \in B_2$. But then for every $a \in B_2$, there is $d > 0$ such that $\mathcal{M} \models \psi(b_1, b_2, \overline{a}, d)$. This contradicts our definition of f.

By $(a)_{m+1}$ there is a decomposition of X into strong cells in M^{m+1} partitioning C. Fix an open cell X_0 from this decomposition. By $(d)_m$, we will be done if we prove that the number of equivalence classes of $E^{\overline{a}}$ is bounded by some $n \in \mathbb{N}_+$ as $\overline{a}$ varies over X_0. Note that for every $\overline{a} \in X_0$,

$$\mathcal{M} \models (\forall y_1, y_2)(\varphi(y_1, y_2, \overline{a}) \longrightarrow (\exists z > 0)\psi(y_1, y_2, \overline{a}, z)).$$

So for every $\overline{a} \in X_0$, there is $d > 0$ such that

$$\mathcal{M} \models (\forall y_1, y_2)[\varphi(y_1, y_2, \overline{a}) \longrightarrow (\exists z_1, z_2)(\psi(z_1, z_2, \overline{a}, d) \wedge E(y_1, z_1, \overline{a}) \wedge E(y_2, z_2, \overline{a})].$$

Let

$$u(\overline{x}, y) \equiv \overline{x} \in X_0 \wedge (\forall y_1, y_2)[\varphi(y_1, y_2, \overline{x}) \longrightarrow (\exists z_1, z_2)(\psi(z_1, z_2, \overline{x}, y) \wedge (y_1, y_2) \cap (z_1, z_2) \neq \emptyset)]\}$$

Clearly, the formula $u(\overline{x}, y)$ satisfies the hypothesis of $(b)_{m+1}$, so there is a decomposition $\mathcal{D}$ of X_0 into strong cells in M^{m+1} such that for every open cell $D \in \mathcal{D}$, if $B \subseteq D$ is an open box with $\text{dist}(B, M^m \setminus D) > 0$, then $B \subseteq u(M, d)$ for some $d > 0$. Note that the number of equivalence classes of $E^{\overline{a}}$ is constant as $\overline{a}$ varies over an open cell from $\mathcal{D}$. By $(a)_{m+1}$ and $(d)_m$ this finishes the proof. ■

Corollary 2.16 *If $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, then $\mathcal{M}$ has the strong cell decomposition property and $Th(\mathcal{M})$ is weakly o-minimal.*

Corollary 2.17 *If $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, $m \in \mathbb{N}_+$ and $X_1, \ldots, X_k$ are sets definable in $\mathcal{M}$, then there is a decomposition of M^m into refined strong cells definable in $\mathcal{M}$ which partitions each of the sets $X_1, \ldots, X_k$.*

3 Canonical o-minimal extension

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property. Below, for any $m \in \mathbb{N}_+$ and $i_1, \ldots, i_m \in \{0, 1\}$ we introduce $\langle i_1, \ldots, i_m\rangle$-cells in $\overline{M}^m$ and so called elementary functions whose domains are $\langle i_1, \ldots, i_m\rangle$-cells in $\overline{M}^m$.

(1) A one-element subset of $\overline{M}^m$ is called a $\langle 0, \ldots, 0\rangle$-cell in $\overline{M}^m$, where $\langle 0, \ldots, 0\rangle$ is a sequence of zeros of length m.

(2) If C is a refined strong $\langle 1\rangle$-cell in M, then $\overline{C}$ is called a $\langle 1\rangle$-cell in $\overline{M}$. Note that $\overline{\varrho}_1^1[\overline{C}] \cap M = \overline{C} \cap M = C$ is an open refined strong cell in M.

(3) If $C = \{\overline{a}\} \subseteq \overline{M}^m$ and I is a $\langle 1\rangle$-cell in $\overline{M}$, then $C \times I$ is a $\langle 0, \ldots, 0, 1\rangle$-cell in $\overline{M}^{m+1}$. Clearly, $\varrho_{m+1}^{m+1}[C \times I] \cap M = I \cap M$ is an open refined strong cell in M.

Assume that $i_1, \ldots, i_m \in \{0, 1\}$, $i_1 + \ldots + i_m > 0$ and suppose that we have already defined $\langle i_1, \ldots, i_m\rangle$-cells in $\overline{M}^m$. Let $\{j_1, \ldots, j_k\} = \{j \in \{1, \ldots, m\} : i_j = 1\}$ and suppose we know that if $C \subseteq \overline{M}^m$ is an $\langle i_1, \ldots, i_m\rangle$-cell in $\overline{M}^m$, then $\varrho^m_{j_1,\ldots,j_k}[C] \cap M^k$ is an open refined strong cell in M^k.

(4) Let C be an $\langle i_1, \ldots, i_m\rangle$-cell in $\overline{M}^m$ and consider $D = \overline{\varrho}^m_{j_1,\ldots,j_k}[C] \cap M^k$, an open refined strong cell in M^k. If f is a strongly continuous definable function from D to M or a strongly continuous definable function from D to $\overline{M} \setminus M$, then $\Gamma(\overline{f} \circ (\overline{\varrho}^m_{j_1,\ldots,j_k} \restriction C))$ is an $\langle i_1, \ldots, i_m, 0\rangle$-cell in $\overline{M}^{m+1}$. Note that $\overline{\varrho}^{m+1}_{j_1,\ldots,j_k}[\Gamma(\overline{f} \circ (\overline{\varrho}^m_{j_1,\ldots,j_k} \restriction C))] \cap M^k = D$ is an open refined strong cell in M^k.

(5) Let C be an $\langle i_1, \ldots, i_m\rangle$-cell in $\overline{M}^m$ and consider $D = \overline{\varrho}^m_{j_1,\ldots,j_k}[C] \cap M^k$, an open refined strong cell in M^k. If $f, g : D \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ are strongly continuous definable functions such that
   - all values of f lie in one of the sets: $\{-\infty\}$, M, $\overline{M} \setminus M$,
   - all values of g lie in one of the sets: M, $\overline{M} \setminus M$, $\{+\infty\}$, and
   - $\overline{f}(\overline{x}) < \overline{g}(\overline{x})$ for $\overline{x} \in \overline{D}$,

   then the set
   $$(\overline{f} \circ \overline{\varrho}^m_{j_1,\ldots,j_k}, \overline{g} \circ \overline{\varrho}^m_{j_1,\ldots,j_k})_C := \{\langle \overline{a}, b\rangle \in C \times \overline{M} : (\overline{f} \circ \overline{\varrho}^m_{j_1,\ldots,j_k})(\overline{a}) < b < (\overline{g} \circ \overline{\varrho}^m_{j_1,\ldots,j_k})(\overline{a})\}$$
   is called an $\langle i_1, \ldots, i_m, 1\rangle$-cell in $\overline{M}^{m+1}$. Note that $\overline{\varrho}^{m+1}_{j_1,\ldots,j_k,m+1}[(\overline{f} \circ \overline{\varrho}^m_{j_1,\ldots,j_k}, \overline{g} \circ \overline{\varrho}^m_{j_1,\ldots,j_k})_C] \cap M^k = (f, g)_D$ is an open refined strong cell in M^{k+1}.

In a standard way we introduce the notion of cell decomposition of a subset of $\overline{M}^m$ into cells in $\overline{M}^m$ [paritioning a given set].

Remark 3.1 *If $m > 1$, $X, Y \subseteq M^m$ and $\mathcal{D}$ is a decomposition of X into cells in $\overline{M}^m$ partitioning Y, then $\{\pi[D] : D \in \mathcal{D}\}$ partitions $\pi[Y]$.*

Definition 3.2 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property and C is a cell in $\overline{M}^m$. A function $f : C \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ is called elementary iff one of the following conditions holds.*
(a) $f(\overline{x}) = +\infty$ for all $\overline{x} \in C$;
(b) $f(\overline{x}) = -\infty$ for all $\overline{x} \in C$;
(c) $f : C \longrightarrow \overline{M}$ and $\Gamma(f)$ is a cell in $\overline{M}^{m+1}$.

In the following lemma and its proof, for $k \in \mathbb{N}_+$ we will denote by
$$\psi^k_0(x_1, \ldots, x_k), \ldots, \psi^k_{s_k}(x_1, \ldots, x_k)$$
the quantifier-free formulas in the language $\{\leq\}$ isolating all complete types over $\emptyset$ in the theory of dense linear orderings without endpoints.

Lemma 3.3 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property and $m \in \mathbb{N}_+$.*

(a)$_m$ If $k \in \mathbb{N}_+$ and $C_1, \ldots, C_k$ are cells in $\overline{M}^m$, then there is a decomposition of $\overline{M}^m$ into cells in $\overline{M}^m$ which partitions each of the cells $C_1, \ldots, C_k$.

(b)$_m$ If C is a cell in $\overline{M}^m$, $k \in \mathbb{N}_+$ and $f_1, \ldots, f_k : C \longrightarrow \overline{M}$ are elementary functions, then there is a decomposition of C into cells in $\overline{M}^m$ partitioning each of the sets $\{\overline{x} \in C : \psi_j^k(f_1(\overline{x}), \ldots, f_k(\overline{x}))\}$, $j \leq s_k$.

Proof. We use induction on m. (a)$_1$ is obvious. For the proof of (b)$_1$, assume that C is a cell in $\overline{M}$ and $f_1, \ldots, f_k : C \longrightarrow \overline{M}$ are elementary functions. The assertion of (b)$_1$ is trivial in case C is a singleton, so assume that $C = \overline{I}$, where $I \subseteq M$ is an open convex and definable (in $\mathcal{M}$) set. There are definable functions $g_1, \ldots g_k : I \longrightarrow \overline{M}$ such that for every $i \in \{1, \ldots, k\}$, we have that

- $f_i = \overline{g}_i$;
- $(\forall x \in I)(g_i(x) \in M)$ or $(\forall x \in I)(g_i(x) \in \overline{M} \setminus M)$.

By the assumption there is a decomposition $\mathcal{D}$ of $I \times M$ into refined strong cells in M^2 which partitions each of the sets $\{\langle x, y\rangle \in I \times M : \psi_j^{k+1}(y, g_1(x), \ldots, g_k(x))\}$, $j \leq s_{k+1}$. Let $\pi : M^2 \longrightarrow M$ be the projection dropping the second coordinate. By Remark 3.1, for any $X, Y \in \mathcal{D}$, either $\overline{\pi[X]} = \overline{\pi[Y]}$ or $\overline{\pi[X]} \cap \overline{\pi[Y]} = \emptyset$. Moreover, the set $C \setminus \bigcup_{X \in \mathcal{D}} \overline{\pi[X]}$ is finite. If $a_1, \ldots, a_l$ are all its elements, then $\{\overline{\pi[X]} : X \in \mathcal{D}\} \cup \{a_1, \ldots, a_l\}$ is a decomposition of C into cells in $\overline{M}$ satisfying our demands.

For the rest of the proof fix $m \in \mathbb{N}_+$ and suppose that the conditions (a)$_m$ and (b)$_m$ are true.

Proof of (a)$_{m+1}$. Assume that $C_1, \ldots, C_k$ are cells in $\overline{M}^{m+1}$ and let $\pi : \overline{M}^{m+1} \longrightarrow \overline{M}^m$ be the projection dropping the last coordinate. Then $\pi[C_1], \ldots, \pi[C_k]$ are cells in $\overline{M}^m$. By (a)$_m$ there is a decomposition $\mathcal{D}$ of $\overline{M}^m$ into cells in $\overline{M}^m$ which partitions each of the cells $\pi[C_1], \ldots, \pi[C_k]$. For a cell $D \in \mathcal{D}$ denote by J_D the set of all $j \in \{1, \ldots, m\}$ such that $D \subseteq \pi[C_j]$. Fix a cell $D \in \mathcal{D}$ with $J_D \neq \emptyset$. For every $j \in J_D$ one of the following conditions holds.

- There is an elementary function $f : D \longrightarrow \overline{M}$ such that $(D \times \overline{M}) \cap C_j = \Gamma(f)$.
- There are elementary functions $f_1, f_2 : D \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ such that $f_1(\overline{x}) < f_2(\overline{x})$ for $\overline{x} \in D$ and $(D \times \overline{M}) \cap C_j = (f_1, f_2)_D$.

Let $f_1^D, \ldots, f_l^D$ be all elementary functions from D to $\overline{M}$ appearing in the above representations. By (b)$_m$, there is a decomposition $\mathcal{D}_D$ of D which partitions each of the sets $\{\overline{x} \in D : \psi_i^l(f_1(\overline{x}), \ldots, f_l(\overline{x}))\}$, $l \leq s_l$. In case for every $j \in J_D$ we have that $D \times \overline{M} \subseteq C_j$ we take $\mathcal{D}_D = \{D\}$. By (a)$_m$, there is a decomposition $\mathcal{D}_1$ of $\overline{M}^m$ into cells in $\overline{M}^m$ which partitions every cell from $\bigcup_{D \in \mathcal{D}} \mathcal{D}_D$. For $D \in \mathcal{D}_1$ let $g_0^D < \ldots < g_{l_D}^D$ be all elementary functions from D to $\overline{M} \cup \{-\infty, +\infty\}$ appearing in the representations of all cells of the form $(D \times \overline{M}) \cap C_j$ together with $-\infty$ and $+\infty$. Let

$$\mathcal{E} = \{(g_i^D, g_{i+1}^D)_D : i < l_D \text{ and } D \in \mathcal{D}_1\} \cup \{\Gamma(g_i^D) : 1 \leq i < l_D \text{ and } D \in \mathcal{D}_1\}.$$

Clearly, $\mathcal{E}$ is a cell decomposition of $\overline{M}^{m+1}$ partitioning each of the cells $C_1, \ldots, C_k$.

Proof of (b)$_{m+1}$. Assume that C is an $\langle i_1, \ldots, i_{m+1}\rangle$-cell in $\overline{M}^{m+1}$, $k \in \mathbb{N}_+$ and $f_1, \ldots, f_k : C \longrightarrow \overline{M}$ are elementary functions. Below we consider two cases.

Case 1. There is $j \in \{1, \ldots, m+1\}$ such that $i_j = 0$. Let $\pi : \overline{M}^{m+1} \longrightarrow \overline{M}^m$ denote the projection dropping the j-th coordinate. There are elementary functions $g_1, \ldots, g_k : \pi[C] \longrightarrow \overline{M}$ such that $f_1 = g_1 \circ \pi, \ldots, f_k = g_k \circ \pi$. (b)$_m$ implies that there is a cell decomposition $\mathcal{D}_1$ of $\pi[C]$ which partitions the sets $\{\overline{z} \in \pi[C] : \psi_l^k(g_1(\overline{z}), \ldots, g_k(\overline{z}))\}$, $l \leq s_k$. Let $\mathcal{D} = \{\pi^{-1}[D] \cap C : D \in \mathcal{D}_1\}$. $\mathcal{D}$ is a cell decomposition of C satisfying our demands.

Case 2. $i_1 = \ldots = i_{m+1} = 1$. There are D, a refined strong open cell in M^{m+1} and functions $g_1, \ldots, g_k : D \longrightarrow \overline{M}$, definable in $\mathcal{M}$, such that $\overline{D} = C$ and for every $i \in \{1, \ldots, k\}$ we have that

- $f_i = \overline{g}_i$;
- $(\forall x \in I)(g_i(x) \in M)$ or $(\forall x \in I)(g_i(x) \in \overline{M} \setminus M)$.

By the assumption there is $\mathcal{C}$, a decomposition of M^{m+2} into refined strong cells in M^{m+2} which partitions each of the sets $\{\langle \overline{x}, y\rangle \in D \times M : \psi_l^{k+1}(g_1(\overline{x}), \ldots, g_k(\overline{x}), y)\}$, $l \leq s_{k+1}$. By (a)$_{m+1}$, there is a decomposition $\mathcal{D}$ of $\overline{M}^{m+1}$ into cells in $\overline{M}^{m+1}$ which partitions each of the cells in $\{\pi[\overline{D}] : D \in \mathcal{C}\} \cup \{C\}$. Note that if $E \in \mathcal{D}$ and $E \subseteq C \setminus \bigcup\{\pi[\overline{D}] : D \in \mathcal{C}\}$, then E is not open in $\overline{M}^{m+1}$. In such a situation, by an argument given in Case 1, there is a decomposition $\mathcal{D}_E$ of E partitioning each of the sets $\{\overline{x} \in E : \psi_j^k(f_1(\overline{x}), \ldots, f_k(\overline{x}))\}$, $j \leq s_k$. Again, by (a)$_{m+1}$, there is a decomposition of $\overline{M}^{m+1}$ which partitions each of the cells in $\{\pi[\overline{D}] : D \in \mathcal{C}\} \cup \{C\} \cup \bigcup_{E \in \mathcal{D}} \mathcal{D}_E$. This provides a decomposition of C satisfying our demands. ■

Now, we are in a position to construct a canonical o-minimal extension $\overline{\mathcal{M}}$ of a weakly o-minimal structure $\mathcal{M}$ with strong cell decomposition.

For any $m \in \mathbb{N}_+$ and any refined strong cell $C \subseteq M^m$ definable in $\mathcal{M}$, denote by R_C an m-ary relational symbol. If we interpret R_C in M^m as C, then clearly the structures

$$\mathcal{M} \text{ and } \mathcal{M}' := (M, \leq, R_C^{\mathcal{M}'} : C \text{ is a refined strong cell})$$

have the same definable sets. Moreover, $Th(\mathcal{M}')$ admits elimination of quantifiers. Now, interpret R_C in $\overline{M}^m$ as $\overline{C}$, the completion of C. In what follows we will show that the structure $\overline{\mathcal{M}} := (\overline{M}, \leq, R_C^{\overline{\mathcal{M}}})$ is o-minimal.

We will start the proof by showing that for every $m \in \mathbb{N}_+$, the following conditions are satisfied.

(a) $(D_m(\overline{M}), \cap, \cup, ^c, \emptyset, M^m)$ is a Boolean algebra.

(b) If $X \in D_m(\overline{M})$, then $X \times \overline{M}, \overline{M} \times X \in D_{m+1}(\overline{M})$.

(c) If $1 \leq i \leq j \leq m$, then $X_m^{i,j} := \{\langle x_1, \ldots, x_m\rangle \in \overline{M}^m : x_i = x_j\} \in D_m(\overline{M})$.

(d) If $X \in D_{m+1}(\overline{M})$ and $\pi : \overline{M}^{m+1} \longrightarrow \overline{M}^m$ is the projection dropping the last coordinate, then $\pi[X] \in D_m(\overline{M})$.

(e) $\{\langle x, y\rangle \in \overline{M}^2 : x < y\} \in D_2(\overline{M})$.

(f) $D_1(\overline{M}) = \{X \subseteq \overline{M} : X \text{ is a finite union of intervals in } (\overline{M}, \leq)\}$.

That $D_m(\overline{M})$ is a Boolean algebra follows easily from Lemma 3.3 and the fact that $\emptyset, \overline{M}^m \in D_m(\overline{M})$. Note that if C is a cell in $\overline{M}^m$, then $C \times \overline{M} = (-\infty, +\infty)_C$ is also a cell in $\overline{M}^{m+1}$. To complete the proof of (b) we show inductively on m the following.

$(*)_m$ If C is a cell in $\overline{M}^m$, then $\overline{M} \times C$ is a cell in $\overline{M}^{m+1}$.

It is clear that $(*)_1$ holds. So let C be a cell in $\overline{M}^{m+1}$ and suppose that $(*)_m$ holds. Let $D = \pi[C]$, where $\pi : \overline{M}^{m+1} \longrightarrow \overline{M}^m$ is the projection dropping the last coordinate. Clearly D is a cell in $\overline{M}^m$. If $C = \Gamma(f)$, where $f : D \longrightarrow \overline{M}$ is an elementary function, then $\overline{M} \times C = \Gamma(f_1)$, where $f_1 : \overline{M} \times D \longrightarrow \overline{M}$ is an elementary function defined as $f_1(a, \bar{b}) = f(\bar{b})$ for $a \in \overline{M}$ and $\bar{b} \in D$. By the inductive hypothesis, $\overline{M} \times D$ is a cell in $\overline{M}^m$, so $\Gamma(f_1)$ is a cell in $\overline{M}^{m+1}$. If $C = (f, g)_D$, where D is a cell in $\overline{M}^m$ and $f, g : D \longrightarrow \overline{M} \cup \{-\infty. +\infty\}$ are elementary functions such that $f(\bar{b}) < g(\bar{b})$ for $\bar{b} \in D$, then $\overline{M} \times C = (f_1, g_1)_{\overline{M} \times D}$, where $f_1, g_1 : \overline{M} \times D \longrightarrow \overline{M}$ are given by $f_1(a, \bar{b}) = f(\bar{b})$ and $g_1(a, \bar{b}) = g(\bar{b})$ for $a \in \overline{M}$ and $\bar{b} \in D$.

For the proof of (c), notice that
- if $i = j$, then $X_m^{i,j} = \overline{M}^m$;
- if $i < j < m$, then $X_m^{i,j} = X_{m-1}^{i,j} \times \overline{M}$;
- if $i < j = m$, then $X_m^{i,j} = \Gamma(f)$, where $f : \overline{M}^{m-1} \longrightarrow \overline{M}$ is given by $f(x_1, \ldots, x_{m-1}) = x_i$.

In each case $X_m^{i,j}$ is a cell in $\overline{M}^m$.

If C is an $\langle i_1, \ldots, i_m, i_{m+1}\rangle$-cell in $\overline{M}^{m+1}$, then $\pi[C]$ is an $\langle i_1, \ldots, i_m\rangle$-cell in $\overline{M}^m$. From this (d) follows.

(e) holds because $\{\langle x, y\rangle \in \overline{M}^2 : x < y\}$ is a cell in $\overline{M}^2$. Note that $C \subseteq \overline{M}$ is a cell in $\overline{M}$ iff C is a singleton or C is an open interval in $(\overline{M}, \leq)$. This implies (f).

Now, by [vdD, Chapter I], for every positive integer m, the family of subsets of $\overline{M}^m$ definable in $\overline{\mathcal{M}}$ coincides with $\mathcal{D}_m(\overline{M})$ and the structure $\overline{\mathcal{M}}$ is o-minimal. The structure $\mathcal{M}'$ is a substructure of $\mathcal{M}$, but in general not an elementary substructure.

Note that if C is a cell in $\overline{M}^m$, then $C \cap M^m$ is either empty or is a refined strong cell in M^m. Consequently, if $X \subseteq \overline{M}^m$ is a set definable in $\overline{\mathcal{M}}$, then $X \cap M^m$ is definable in $\mathcal{M}$.

Fact 3.4 *If $\mathcal{M} = (M, \leq, +, \ldots)$ is a weakly o-minimal non-valuational expansion of an ordered group $(M, \leq, +)$, then the canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ expands the ordered group $(\overline{M}, \leq, +)$. If $\mathcal{M} = (R, \leq, +, \cdot, \ldots)$ is a weakly o-minimal non-valuational expansion of a real closed field $(R, \leq, +, \cdot)$, then the canonical o-minimal extension $\overline{\mathcal{M}}$ of $\mathcal{M}$ expands the real closed field $(\overline{R}, \leq, +, \cdot)$.*

Proof. The addition operation in M is a strongly continuous function from $M \times M$ onto M, so its graph is a refined strong cell in M^3. The completion of this cell is the graph of addition in $\overline{M}$, as well as a cell in $\overline{M}^3$. The second part is proved in a similar manner. ■

4 An Euler characteristic

The usual Euler characteristic for an o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$ assigns to each definable set $X \subseteq M^m$ an integer $E(X)$. If $\mathcal{D}$ is a cell decomposition of X, then $E(X) = \sum_{D \in \mathcal{D}} (-1)^{\dim(D)}$. An analogous definition for sets definable in models of weakly o-minimal theories does not make sense since it depends on the cell decomposition. In this section we introduce a reasonably well behaving generalization of the Euler characteristic to sets definable in weakly o-minimal structures with the strong cell decomposition property.

For a weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$ with the strong cell decomposition property and $m \in \mathbb{N}_+$, denote by $\mathrm{Def}_m(\mathcal{M})$ the family of all subsets of M^m definable in $\mathcal{M}$. Below we inductively define a map $\chi_m : \mathrm{Def}_m(\mathcal{M}) \longrightarrow \mathbb{Z}\left[\frac{1}{2}\right]$ and call it Euler characteristic. To simplify notation we omit the subscript m.

(0) $\chi(\emptyset) := 0$.

(1) If $a \in M$, then $\chi(\{a\}) := 1$.

(2) If $I \subseteq M$ is a non-empty convex open definable set, then

$$\chi(I) := \begin{cases} -1 & \text{if } \inf I \in M \cup \{-\infty\} \text{ and } \sup I \in M \cup \{+\infty\} \\ -\frac{1}{2} & \text{if } \inf I \in M \cup \{-\infty\} \text{ and } \sup I \in \overline{M} \setminus M \\ -\frac{1}{2} & \text{if } \inf I \in \overline{M} \setminus M \text{ and } \sup I \in M \cup \{+\infty\} \\ 0 & \text{if } \inf I, \sup I \in \overline{M} \setminus M. \end{cases}$$

Assume that $m \in \mathbb{N}_+$ and suppose that we have already defined the Euler characteristic for refined strong cells in M^m.

(3) If $C \subseteq M^m$ is a refined strong cell and $f : C \longrightarrow M$ is a strongly continuous definable function, then $\chi(\Gamma(f)) := \chi(C)$.

(4) If $C \subseteq M^m$ is a refined strong cell and $f : C \longrightarrow \overline{M} \cup \{-\infty\}$, $g : C \longrightarrow \overline{M} \cup \{+\infty\}$ are strongly continuous definable functions such that $(f, g)_C$ is a refined strong cell in M^{m+1} (so in particular $\chi((f(\overline{a}), g(\overline{a})))$ is constant as $\overline{a}$ varies over C), then $\chi((f, g)_C) := \chi(C) \cdot \chi((f(\overline{a}), g(\overline{a})))$.

(5) If $X \subseteq M^m$ is a non-empty definable set and $\mathcal{D}$ is a decomposition of X into refined strong cells, then $\chi_{\mathcal{D}}(X) := \sum_{D \in \mathcal{D}} \chi(D)$ (this definition a priori depends on $\mathcal{D}$, so for the time being we will use the notation $\chi_{\mathcal{D}}$ instead of χ).

Lemma 4.1 *If $C \subseteq M^m$ is a refined strong cell and $\mathcal{D}$ is a decomposition of C into refined strong cells in M^m, then $\chi(C) = \chi_{\mathcal{D}}(C)$.*

Proof. The assertion of the Lemma is clear for $m = 1$. Suppose it is true for decompositions of refined strong cells in M^m into refined strong cells. Let $C \subseteq M^{m+1}$ be a refined strong cell, $\mathcal{D}$ its decomposition into refined strong cells in M^{m+1}, and $\pi : M^{m+1} \longrightarrow M^m$ the projection dropping the last coordinate. If C is a refined strong $\langle i_1, \ldots, i_m, 0\rangle$-cell, i.e. $C = \Gamma(f)$, where $f : \pi[C] \longrightarrow M$ is a strongly continuous definable function, then by the inductive assumption we obtain

$$\chi(C) = \chi(\pi[C]) = \chi_{\pi[\mathcal{D}]}(\pi[C]) = \sum_{B \in \pi[\mathcal{D}]} \chi(B) =$$
$$\sum_{B \in \pi[\mathcal{D}]} \chi(\Gamma(f \restriction B)) = \sum_{D \in \mathcal{D}} \chi(D) = \chi_{\mathcal{D}}(C)$$

Assume that C is a refined strong $\langle i_1, \ldots, i_m, 1\rangle$-cell in M^{m+1}. There are strongly continuous definable functions $f : \pi[C] \longrightarrow \overline{M} \cup \{-\infty\}$ and $g : \pi[C] \longrightarrow \overline{M} \cup \{+\infty\}$ such that $C = (f, g)_{\pi[C]}$.

For every $B \in \pi[\mathcal{D}]$, there are definable functions $f_0^B, \ldots, f_{t(B)}^B : B \longrightarrow \overline{M} \cup \{-\infty, +\infty\}$ such that $(f_0^B, f_{t(B)}^B)_B = C \cap (B \times M)$ and for every $i \in \{0, \ldots, t(B)-1\}$, $(f_i^B, f_{i+1}^B)_B$ is a refined strong cell in M^{m+1}. For $B \in \pi[\mathcal{D}]$ fix $\overline{a}_B \in B$ and define

$$J_B = \{i \in \{1, \ldots, t(B)-1\} : f_i^B \text{ is a function from } B \text{ to } M\}.$$

Fix also $\overline{a} \in \pi[C]$. The cells from $\mathcal{D}$ that map onto B under π are:

$$(f_i^B, f_{i+1}^B)_B,\ i < t(B), \text{ and } \Gamma(f_i^B), i \in J_B.$$

Using this notation and the inductive assumption we obtain

$$\begin{aligned}\chi(C) &= \chi(\pi[C]) \cdot \chi((f(\overline{a}), g(\overline{a}))) = \chi_{\pi[\mathcal{D}]}(\pi[C]) \cdot \chi((f(\overline{a}), g(\overline{a}))) = \\ &\sum_{B \in \pi[\mathcal{D}]} \chi(B) \cdot \chi((f(\overline{a}), g(\overline{a}))) = \sum_{B \in \pi[\mathcal{D}]} \chi(B) \cdot \chi((f_0^B(\overline{a}_B), f_{t(B)}^B(\overline{a}_B))) = \\ &\sum_{B \in \pi[\mathcal{D}]} \chi(B) \cdot \left(\sum_{i<t(B)} \chi(f_i^B(\overline{a}_B), f_{i+1}^B(\overline{a}_B)) + \sum_{i \in J_B} \chi(\{f_i^B(\overline{a}_B)\}) \right) = \\ &\sum_{B \in \pi[\mathcal{D}]} \chi(B) \cdot \left(\sum_{i<t(B)} \chi((f_i^B(\overline{a}_B), f_{i+1}^B(\overline{a}_B)) + |J_B| \right) = \\ &\sum_{B \in \pi[\mathcal{D}]} \left(\sum_{i<t(B)} \chi((f_i^B, f_{i+1}^B)_B) + \sum_{i \in J_B} \chi(\Gamma(f_i^B)) \right) = \chi_{\mathcal{D}}(C).\end{aligned}$$

■

Now, repeating the argument from the proof of Proposition 2.2 from [vdD], we can easily show that the Euler characteristic $\chi_{\mathcal{D}}(S)$ does not depend on $\mathcal{D}$. Note that if $\mathcal{M}$ is an o-minimal structure, then χ is $\mathbb{Z}$-valued and coincides with the o-minimal Euler characteristic.

To prove the next theorem one can essentially rewrite the proofs of analogous results from [vdD].

Theorem 4.2 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure with the strong cell decomposition property and $m, n \in \mathbb{N}_+$.*

(a) If $X, Y \subseteq M^m$ are definable sets, then $\chi(X \cup Y) = \chi(X) + \chi(Y) - \chi(X \cap Y)$.

(b) If $S \subseteq M^{m+n}$ is a definable set and $k \in \mathbb{Z}\left[\frac{1}{2}\right]$, then the set $X := \{\overline{a} \in M^m : \chi(S_{\overline{a}}) = k\}$ is definable and $\chi(\bigcup_{\overline{a} \in X} \{\overline{a}\} \times S_{\overline{a}}) = \chi(X) \cdot k$.

(c) If $X \subseteq M^m$ and $Y \subseteq M^n$ are definable sets, then $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$.

Assume that $\mathcal{M}$ is an o-minimal expansion of a real closed field. If $m, n \in \mathbb{N}_+$ and $S_1 \subseteq M^m$, $S_2 \subseteq M^n$ are definable sets of equal dimensions and Euler characteristics, then there is a definable bijection $f : S_1 \longrightarrow S_2$ (see [vdD, Chapter 8]). Moreover, the topological dimension and the Euler characteristic for sets definable in $\mathcal{M}$ are invariant under definable bijections. As observed in [KS], this means that the Grothendieck ring of $\mathcal{M}$ is isomorphic to $\mathbb{Z}$.

By Theorem 2.13 from [We] we know that the topological dimension of a set definable in a weakly o-minimal structure is invariant under injective definable maps. This does not apply to

χ, even if the structure has the strong cell decomposition property. Consequently, our function χ is not even a weak Euler characteristic in the sense of [KS, Definition 3.1]. Neverheless, as Theorem 4.2 shows, it enjoys some properties of the o-minimal Euler characteristic. On the other hand, equality of dimensions and Euler characteristics (in our sense) of sets definable in a weakly o-minimal non-valuational expansion of a real closed field does not guarantee the existence of definable bijection between them (see the following example). Consequently, there is no reason for the Grothendieck ring of a weakly o-minimal non-valuational expansion of a real closed field to be isomorphic to $\mathbb{Z}\left[\frac{1}{2}\right]$.

Example. Let $\mathcal{R} = (R_{\text{alg}}, \leq, +, \cdot)$ be the ordered field of all real algebraic numbers. Let $\mathcal{M}_1 = (M_1, \leq, \ldots)$ and $\mathcal{M}_2 = (M_2, \leq, \ldots)$ be two isomorphic copies of $\mathcal{R}$ such that $M_1 \cap M_2 = \emptyset$. Extend the linear orderings $(M_1, \leq)$ and $(M_2, \leq)$ to the linear ordering of $M := M_1 \cup M_2$ by setting $x < y$ whenever $x \in M_1$ and $y \in M_2$. Expand $(M, \leq)$ to a first order structure $\mathcal{M}$ so that the family of sets definable in $\mathcal{M}$ is the smallest family containing all sets definable in $\mathcal{M}_1$ and in $\mathcal{M}_2$. Clearly, $\mathcal{M}$ is weakly o-minimal and has strong the cell decomposition property. If $a, b \in M_1$, $a < b$ and $f : (a, b) \longrightarrow M_1$ is a strictly increasing continuous function mapping (a, b) onto $(0, \sup M_1)$, then $\chi((a, b)) \neq \chi(f[(a, b)])$.

For a real transcendental number α let $P_\alpha = \{x \in R_{\text{alg}} : x > \alpha\}$. Fix transcendental numbers α, β such that P_β is not definable in $(\mathcal{R}, P_\alpha)$, and let $\mathcal{M}' = (\mathcal{R}, P_\alpha, P_\beta)$. Then $\chi(P_\alpha) = \chi(P_\beta) = -\frac{1}{2}$, $Th(\mathcal{M}')$ is weakly o-minimal (by [BP] or [Bz]), but there is no definable (in $\mathcal{M}'$) bijection between P_α and P_β.

Problem 4.3 *Investigate the Grothendieck ring for weakly o-minimal non-valuational expansions of real closed fields.*

Acknowledgements. The author expresses his gratitude towards the referee whose detailed comments helped to improve the presentation of results.

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