Topological properties of sets definable in weakly o-minimal structures[1]

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ABSTRACT

The paper is aimed at studying the topological dimension for sets definable in weakly o-minimal structures in order to prepare background for further investigation of groups, group actions and fields definable in the weakly o-minimal context. We prove that the topological dimension of a set definable in a weakly o-minimal structure is invariant under definable injective maps, strengthening an analogous result from [MMS] for sets and functions definable in models of weakly o-minimal theories. We pay special attention to large subsets of Cartesian products of definable sets, showing that if X, Y and S are non-empty definable sets and S is a large subset of $X \times Y$, then for a large set of tuples $\langle \overline{a}_1, \ldots, \overline{a}_{2^k} \rangle \in X^{2^k}$, where $k = \dim(Y)$, the union of fibers $S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^k}}$ is large in Y. Finally, given a weakly o-minimal structure $\mathcal{M}$, we find various conditions equivalent to the fact that the topological dimension in $\mathcal{M}$ enjoys the addition property.

0 Introduction

In the theory of o-minimal structures, cell decompositions have usually been an essential tool for introducing and effective investigation of various topological invariants of definable sets, the dimension and the Euler characteristic being classical examples (see for instance [vdD2, Chapter 4]).

In this paper we concentrate on studying the topological dimension for sets definable in weakly o-minimal structures. The *dimension* of an infinite set $X \subseteq M^m$ definable in a weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$ is defined as the biggest positive integer r for which there is a projection $\pi : M^m \longrightarrow M^r$ such that $\pi[X]$ has a non-empty interior in M^r. A finite set has dimension 0 if it is non-empty and $-\infty$ otherwise (see [MMS, Definition 4.1]). If $\mathcal{M}$ is o-minimal, then this notion of dimension coincides with the usual one defined by cell decomposition. Some basic properties of the topological dimension in weakly o-minimal structures are collected in §1.

As illustrated by various examples from [MMS] and [V], the weak o-minimality of linearly ordered structures is not preserved under elementary equivalence. Therefore one cannot hope for a reasonable counterpart of the cell decomposition for arbitrary weakly o-minimal structures. A sensible way to avoid this kind of difficulty is to restrict one's attention to the class of models of weakly o-minimal theories. In such a situation, D. Macpherson, D. Marker and C. Steinhorn established a version of cell decomposition (see [MMS, Theorem 4.6]). Naturally, one could ask how much of the cell decomposition survives if the hypothesis of weak o-minimality of the theory is relaxed to that of the structure. Our attempts towards answering this question are expressed by Lemma 2.4, where we find a decomposition of a definable set into finitely many subsets of 'simple nature'. However, these simple sets are rather remote from what has traditionally been understood under the name of a 'cell'.

Macpherson, Marker and Steinhorn show that the dimension of a set definable in a model of a weakly o-minimal theory is invariant under injective definable maps (see [MMS, Theorem 4.7]). Their proof uses cell decomposition and ω_1-saturatedness, and therefore cannot be easily generalized to sets and functions definable in general weakly o-minimal structures. Nevertheless,

[1]2000 *Mathematics Subject Classification.* Primary 03C64.

[2]This research was supported by a Marie Curie Intra-European Fellowships within the 6th European Community Framework Programme. Contract number: MEIF-CT-2003-501326.

applying a completely different approach, in §2 we prove that the dimension of a set definable in a weakly o-minimal structure does not change under injective definable maps. The proof of this result easily reduces to showing that if $X \subseteq M^m$ is a non-empty definable set and $f : X \longrightarrow M$ is a definable function, then $\Gamma(f) := \{\langle \overline{a}, f(\overline{a})\rangle : \overline{a} \in X\} \subseteq M^{m+1}$, the graph of f, has dimension equal to the dimension of X. The difficulty with establishing the latter lies in showing that there is a projection witnessing the dimension of $\Gamma(f)$ which drops the last coordinate.

Imagine for example that there are some nasty one-dimensional definable set $S \subseteq M^2$ and a definable function $f : S \longrightarrow M$ such that $\dim(\Gamma(f)) = 2$, i.e. some projection of $\Gamma(f)$ contains an open box. Clearly, such a projection cannot drop the last coordinate. Suppose for instance that there are open intervals $I, J \subseteq M$ for which

$$I \times J \subseteq \{\langle y, z\rangle \in M^2 : (\exists x \in M)(z = f(x, y))\}.$$

For every $a \in I$, the set $\{x \in M : \langle x, a\rangle \in S\}$ is infinite. For the sake of simplicity, assume that $\{x \in M : \langle x, a\rangle \in S\}$ is convex and open whenever $a \in I$. Fix $b \in J$ and let $X = \{\langle x, y\rangle \in S : y \in I, f(x, y) = b\}$. Note that $\{x \in M : \langle x, a\rangle \in X\}$ is a non-empty proper subset of $\{x \in M : \langle x, a\rangle \in S\}$ whenever $a \in I$. By the monotonicity theorem (see Theorem 1.2) we can find an open interval $I' \subseteq I$ such that each of the functions

$$x \longmapsto \inf\{x \in M : \langle x, a\rangle \in S\}, x \longmapsto \sup\{x \in M : \langle x, a\rangle \in S\}$$

is constant or strictly increasing on I'. As $\dim(S) = 1$, the above functions must be both strictly increasing or both strictly decreasing. Moreover, for distinct $a_1, a_2 \in I'$

$$\{x \in M : \langle x, a_1\rangle \in S\} \cap \{x \in M : \langle x, a_1\rangle \in S\} = \emptyset.$$

Consequently, the set $\{x \in M : (\exists y \in I')(\langle x, y\rangle \in X)\}$ is not a union of finitely many convex sets, which contradicts the weak o-minimality of $\mathcal{M}$.

The general situation is much more complicated, however. Nevertheless, having in mind the above example and various special cases that may arise, we were able to state the list of inductive conditions of Theorem 2.11, from which the required result easily follows.

In §3 we study large subsets of Cartesian products of definable sets. The main result of §3 (i.e. Theorem 3.6) will constitute one of the crucial ingredients of our further study of groups, group actions and fields definable in weakly o-minimal structures. It will be used for example to show that a large subset of a group definable in a weakly o-minimal structure is generic. Theorem 3.6 says that if $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $X \subseteq M^m$, $Y \subseteq M^n$, $S \subseteq X \times Y$, are non-empty definable sets, $k = \dim(Y)$ and S is large in $X \times Y$, then the set of tuples $\langle \overline{a}_1, \ldots, \overline{a}_{2^k}\rangle \in X^{2^k}$, for which the union of fibers $S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^k}}$ is large in Y, is large in X^{2^k}.

We say that the topological dimension in a weakly o-minimal structure $\mathcal{M}$ has the addition property iff for every definable set $S \subseteq M^{m+n}$ and a projection $\pi : M^{m+n} \longrightarrow M^m$ dropping some n coordinates, if all fibers $\pi^{-1}(\overline{a}) \cap S$, $\overline{a} \in \pi[S]$ are of dimension k, then $\dim(S) = \dim(\pi[S]) + k$. The addition property holds in the o-minimal case but fails in general weakly o-minimal structures. We prove in §4 that it is closely related to the exchange property of the definable closure (Theorem 4.3) and equivalent to each of the following statements (Theorem 4.2).

- If $I \subseteq M$ is an open interval and $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ is a definable function (i.e. the set $\{\langle x, y\rangle \in I \times M : y < f(x)\}$ is definable), then there is an open interval $I' \subseteq I$ such that $f \restriction I'$ is continuous.

- If $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}^{\mathcal{M}}$ is a definable function (i.e. the set $\{\langle \overline{x}, y\rangle \in B \times M : y < f(\overline{x})\}$ is definable), then there is an open box $B' \subseteq B$ such that $f \restriction B'$ is continuous.

- If $m \in \mathbb{N}_+$, $S \subseteq M^{m+1}$ is a non-empty definable set and $\pi : M^{m+1} \longrightarrow M^m$ denotes a projection dropping one coordinate, then $\dim(S) = \dim(\pi[S])$ iff the set of tuples $\overline{a} \in \pi[S]$ for which the fiber $\pi^{-1}(\overline{a}) \cap S$ is finite is large in $\pi[S]$.

1 Notation and preliminaries

Let $(M, \leq)$ be a dense linear ordering without endpoints. A set $X \subseteq M$ is called *convex* in $(M, \leq)$ iff for any $a, b \in X$ and $c \in M$, if $a \leq c \leq b$, then $c \in X$. A non-empty convex set is called an *interval* in $(M, \leq)$ iff it has both infimum and supremum in $M \cup \{-\infty, +\infty\}$. The ordering of M determines a topology on M^m, $m \in \mathbb{N}_+$, whose basis consists of open boxes in M^m, i.e. sets of the form $(a_1, b_1) \times \ldots \times (a_m, b_m)$, where $a_i, b_i \in M$ and $a_i < b_i$, $1 \leq i \leq m$.

A first order structure $\mathcal{M} = (M, \leq, \ldots)$ equipped with a dense linear ordering $\leq$ without endpoints is called *weakly o-minimal* iff every subset of M, definable in $\mathcal{M}$, is a finite union of convex sets. Weak o-minimality, unlike o-minimality, is not preserved under elementary equivalence [MMS]. A first order complete theory is called weakly o-minimal iff all its models are weakly o-minimal. If $X \subseteq M$ is a non-empty set definable in a weakly o-minimal structure $\mathcal{M}$, then any maximal convex subset of X is said to be a *convex component* of X. If Y is a convex component of X with $\inf Y = \inf X$ [respectively: $\sup Y = \sup X$], then Y is called the first [the last] convex component of X.

Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal L-structure. A *definable cut* in $\mathcal{M}$ is an ordered pair $\langle C, D \rangle$ of non-empty definable subsets of M such that $C < D$ and $C \cup D = M$. In an obvious way the linear ordering $(M, \leq)$ extends to the linear ordering of

$$\overline{M}^{\mathcal{M}} := M \cup \{\langle C, D \rangle : \langle C, D \rangle \text{ is a definable cut in } \mathcal{M} \text{ and } C, D \text{ are not intervals in } (M, \leq)\}.$$

Note that M is dense in $\overline{M}^{\mathcal{M}}$ and every subset of M, definable in $\mathcal{M}$, has infimum and supremum in $\overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$. If $X \subseteq M^m$ is a non-empty definable [over A, $A \subseteq M$] set, then a function $f : X \longrightarrow \overline{M}^{\mathcal{M}}$ is called definable [over A] iff there is a formula $\varphi(\overline{x}, y) \in L(M)$ [respectively: $\varphi(\overline{x}, y) \in L(A)$] such that $|\overline{x}| = m$, $\varphi(M) \subseteq X \times M$ and $f(\overline{a}) = \sup \varphi(\overline{a}, M)$ whenever $\overline{a} \in X$. If $f, g : X \longrightarrow \overline{M}^{\mathcal{M}}$ are definable functions such that $f(\overline{a}) < g(\overline{a})$ for $\overline{a} \in X$, then by $(f, g)_X$ we will denote the set of tuples $\langle \overline{a}, b \rangle \in X \times M$ for which $f(\overline{a}) < b < g(\overline{a})$. Throughout the paper we will also use the following convention. We will call a function $f : X \longrightarrow M \cup \{-\infty, +\infty\}$ [$f : X \longrightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$] definable iff either f is a definable function from X to M [to $\overline{M}^{\mathcal{M}}$], or $(\forall \overline{x} \in X)(f(\overline{x}) = -\infty)$, or $(\forall \overline{x} \in X)(f(\overline{x}) = +\infty)$.

If $\mathcal{M} = (M, \ldots)$ is a first order structure, $m \in \mathbb{N}_+$ and J is a proper subset of $\{1, \ldots, m\}$, then by π^m_J we will denote the projection from M^m onto $M^{m-|J|}$ dropping all the coordinates from J. In particular, $\pi^m_\emptyset$ is the identity map on M^m. If J is a non-empty subset of $\{1, \ldots, m\}$, then by ϱ^m_J we will denote the projection from M^m onto $M^{|J|}$ dropping all the coordinates from $\{1, \ldots, m\} \setminus J$. Usually, if $J = \{j_1, \ldots, j_k\}$, we write $\pi^m_{j_1,\ldots,j_k}$ instead of $\pi^m_{\{j_1,\ldots,j_k\}}$ and $\varrho^m_{j_1,\ldots,j_k}$ instead of $\varrho^m_{\{j_1,\ldots,j_k\}}$. For example we have $\pi^5_{1,4}(x_1, x_2, x_3, x_4, x_5) = \varrho^5_{2,3,5}(x_1, x_2, x_3, x_4, x_5) = \langle x_2, x_3, x_5 \rangle$. A projection from M^m onto M^r, $r \in \{1, \ldots, m\}$, is an arbitrary map of the form π^m_J, where $J \subsetneq \{1, \ldots, m\}$ and $|J| = m - r$. Note that if $m \geq 3$ and $1 \leq i < j \leq m$, then $\pi^{m-1}_{j-1} \circ \pi^m_i = \pi^{m-1}_i \circ \pi^m_j = \pi^m_{i,j}$.

Throughout the rest of the paper, unless otherwise stated, we will work in an arbitrary weakly o-minimal structure $\mathcal{M} = (M, \leq, \ldots)$. By a definable set (function) we will always mean a set (function) definable in the structure $\mathcal{M}$. When talking about the interior or closure of a definable set $X \subseteq M^m$ (notation: $\mathrm{int}(X)$, $\mathrm{cl}(X)$ respectively) we will always refer to the topology induced on M^m by the ordering $(M, \leq)$.

Assume that $m, n \in \mathbb{N}_+$, $S \subseteq M^{m+n}$ is a definable set, $\overline{a} \in M^m$ and $\overline{b} \in M^n$. The fibers determined by $\overline{a}$ and $\overline{b}$ are defined as follows:

$$S_{\overline{a}} = \{\overline{c} \in M^n : \langle \overline{a}, \overline{c} \rangle \in S\}, \; S^{\overline{b}} = \{\overline{c} \in M^m : \langle \overline{c}, \overline{b} \rangle \in S\}.$$

Definition 1.1 *Let I be a non-empty convex open subset of M. A function $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ is called*

(a) locally constant on I iff

$$(\forall a \in I)(\exists b, c \in I)(b < a < c \text{ and } f \restriction (b, c) \text{ is constant});$$

(b) locally strictly increasing on I iff

$$(\forall a \in I)(\exists b, c \in I)(b < a < c \textit{ and } f \restriction (b,c) \textit{ is strictly increasing});$$

(c) locally strictly decreasing on I iff

$$(\forall a \in I)(\exists b, c \in I)(b < a < c \textit{ and } f \restriction (b,c) \textit{ is strictly decreasing});$$

(d) locally strictly monotone on I iff f is locally strictly increasing on I or locally strictly decreasing on I.

The following theorem, to be referred to as the monotonicity theorem, is a consequence of Theorem 3.3 from [MMS] and [Ar].

Theorem 1.2 *Assume that $A \subseteq M$. If $U \subseteq M$ is an infinite A-definable set and $f : U \longrightarrow \overline{M}^{\mathcal{M}}$ [respectively: $f : U \longrightarrow M$] is an A-definable function, then there is a partition of U into A-definable sets $X, I_0, \ldots, I_m$ such that X is finite, $I_0, \ldots, I_m$ are non-empty convex open sets, and for every $i \leq m$, $f \restriction I_i$ if locally constant or locally strictly monotone [and continuous].*

Lemma 1.3 *Assume that $I \subseteq M$ is an open interval and $f, g : I \longrightarrow \overline{M}^{\mathcal{M}}$ are definable functions such that $f(a) < g(a)$ for $a \in I$ and $\mathrm{int}((f,g)_I) = \emptyset$. Then there is an open interval $I' \subseteq I$ such that*

(a) the functions $f \restriction I'$ and $g \restriction I'$ are either both strictly increasing or both strictly decreasing;

(b) for any distinct $a, b \in I'$ we have that $(f(a), g(a)) \cap (f(b), g(b)) = \emptyset$;

(c) for any $a \in I'$ and any open interval I_1 with $\inf I_1 < f(a) < g(a) < \sup I_1$, there is an open interval $I'' \subseteq I'$, containing a, such that $\inf I_1 < f(x) < g(x) < \sup I_1$ whenever $x \in I''$.

Proof. By the monotonicity theorem, there is an open interval $I_1 \subseteq I$ such that each of the functions f, g restricted to I_1 is either strictly monotone or constant. As $\mathrm{int}((f,g)_I) = \emptyset$, the functions f, g restricted to I_1 are either both strictly decreasing or both strictly increasing. Below we only consider the first possibility.

Firstly, observe that $f(a) > g(b)$ whenever $\inf I_1 < a < b < \sup I_1$. For, if $\inf I_1 < a < c < b < \sup I_1$ and $f(a) \leq g(b)$, then $(c, b) \times (f(c), f(a)) \subseteq (f, g)_{I_1}$, which contradicts our assumption. For $a \in I_1$ define $h_1(a) = \lim\limits_{x \to a^-} f(x) = \lim\limits_{x \to a^-} g(x)$ and $h_2(a) = \lim\limits_{x \to a^+} f(x) = \lim\limits_{x \to a^+} g(x)$. By the weak o-minimality of $\mathcal{M}$, the set $X := \{a \in I_1 : h_1(a) \neq g(a) \text{ or } h_2(a) \neq f(a)\}$ is finite, so there is an open interval $I' \subseteq I_1 \setminus X$. Clearly, I' satisfies all our demands. ■

The proof of the following lemma can be easily derived from [Ar] and the proof of Theorem 4.8 from [MMS]. A similar technique will be used in the proof of Theorem 4.2.

Lemma 1.4 *Assume that $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow M$ is a definable function. Then there is an open box $B' \subseteq B$ such that $f \restriction B'$ is continuous.*

The following lemma can be deduced from [Ar] and Theorem 4.3 from [MMS].

Lemma 1.5 *Assume that $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box, $f : B \longrightarrow M \cup \{-\infty, +\infty\}$ and $g : B \longrightarrow \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$ are definable functions, f is continuous, and $b \in M$.*

(a) If $(\forall \overline{a} \in B)(f(\overline{a}) < g(\overline{a}))$, then the set $(f, g)_B$ contains an open box C. If additionally f is identically equal to b, then C may be chosen so that $\inf \varrho^{m+1}_{m+1}[C] = b$.

(b) If $(\forall \overline{a} \in B)(f(\overline{a}) > g(\overline{a}))$, then the set $(g, f)_B$ contains an open box C. If additionally f is identically equal to b, then C may be chosen so that $\sup \varrho^{m+1}_{m+1}[C] = b$.

Assume that $\mathcal{M}$ is a weakly o-minimal structure and $X \subseteq M^m$ is an infinite definable set. The *dimension* of X, denoted by $\dim(X)$, is the largest r for which there exists a projection $\pi : M^m \longrightarrow M^r$ such that $\pi[X]$ contains an open box. Non-empty finite sets are said to have dimension 0, while to an empty set we assign the dimension $-\infty$. We shall use the convention that if $d \in \mathbb{N} \cup \{-\infty\}$, then $d \geq -\infty$ and $d + (-\infty) = -\infty + d = -\infty$.

Fact 1.6 *Assume that $m, n \in \mathbb{N}_+$ and $X, Y \subseteq M^m$, $Z \subseteq M^n$ are definable sets.*
(a) If $X \subseteq Y$, then $\dim(X) \leq \dim(Y)$.
(b) If $k \in \{1, \ldots, m\}$ and $\pi : M^m \longrightarrow M^k$ is a projection, then $\dim(X) - (m-k) \leq \dim(\pi[X]) \leq \dim(X)$.
(c) If $f : M^m \longrightarrow M^m$ is a permutation of variables, then $\dim(f[X]) = \dim(X)$.
(d) $\dim(X \times Z) = \dim(X) + \dim(Z)$.
(e) $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.

Proof. (a), (b), (c) and (d) are immediate. (e) follows from [Ar] and [MMS, Theorem 4.2]. ■

Lemma 1.7 *Assume that $m \geq 3$, $S \subseteq M^m$ is a definable set, $i, j \in \{1, \ldots, m\}$ and $i \neq j$.*
(a) If $\dim(S) = \dim(\pi_j^m[S]) = \dim(\pi_i^m[S]) + 1$, then $\dim(\pi_i^m[S]) = \dim(\pi_{i,j}^m[S]) = \dim(\pi_j^m[S]) - 1$.
(b) If $\dim(S) = \dim(\pi_i^m[S]) = \dim(\pi_{i,j}^m[S])$, then $\dim(S) = \dim(\pi_j^m[S])$.

Proof. (a) $\dim(\pi_{i,j}^m[S]) \geq \dim(\pi_j^m[S]) - 1 = \dim(S) - 1 = \dim(\pi_i^m[S]) \geq \dim(\pi_{i,j}^m[S])$.
(b) $\dim(S) \geq \dim(\pi_j^m[S]) \geq \dim(\pi_{i,j}^m[S]) = \dim(S)$. ■

Definition 1.8 *Assume that $m \in \mathbb{N}_+$ and $X, Y \subseteq M^m$ are non-empty definable sets. We say that X is large in Y iff $\dim(Y \setminus X) < \dim(Y)$.*

Fact 1.9 *Assume that $m \in \mathbb{N}_+$ and $X, Y, Z \subseteq M^m$ are non-empty definable sets.*
(a) If X, Y are finite, then X is large in Y iff $Y \subseteq X$.
(b) If X is large in Y, Y is large in Z and $Y \subseteq Z$, then X is large in Z.
(c) If X and Y are both large in Z, then $X \cap Y$ and $X \cup Y$ are large in Z.

Proof. (a) is obvious; (b) and (c) follow from Fact 1.6. ■

Lemma 1.10 *Assume that $m \geq 2$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}^{\mathcal{M}}$ is a definable function. For $i \in \{1, \ldots, m\}$, $\overline{a} \in \pi_i^m[B]$ and $b \in \varrho_i^m[B]$ define $f_i^{\overline{a}}(b) = f(\overline{c})$, where $\overline{c} \in B$ is the unique tuple such that $\varrho_i^m(\overline{c}) = b$ and $\pi_i^m(\overline{c}) = \overline{a}$. Then there is an open box $C \subseteq B$ such that for every $i \in \{1, \ldots, m\}$, one of the following conditions holds.*
(a) $(\forall \overline{a} \in \pi_i^m[C])(f_i^{\overline{a}} \restriction \varrho_i^m[C]$ is strictly increasing);
(b) $(\forall \overline{a} \in \pi_i^m[C])(f_i^{\overline{a}} \restriction \varrho_i^m[C]$ is strictly decreasing);
(c) $(\forall \overline{a} \in \pi_i^m[C])(f_i^{\overline{a}} \restriction \varrho_i^m[C]$ is constant).

Proof. Let X_1 [respectively: X_2, X_3] be the set of all tuples $\overline{a} \in \pi_1^m[B]$ for which there exists an open interval $I(\overline{a}) \subseteq \varrho_1^m[B]$ such that $\inf I(\overline{a}) = \inf \varrho_1^m[B]$ and the function $f_1^{\overline{a}} \restriction I(\overline{a})$ is locally strictly increasing [respectively: locally strictly decreasing, locally constant]. By the monotonicity theorem, $\pi_1^m[B] = X_1 \cup X_2 \cup X_3$, so at least one of the sets X_1, X_2, X_3 has dimension $m-1$. Suppose for example that $\dim(X_1) = m - 1$, and fix an open box $B_0 \subseteq X_1$. For $\overline{a} \in B_0$ define

$$h_1(\overline{a}) = \sup\{y \in \varrho_1^m[B] : f_1^{\overline{a}} \restriction (\inf \varrho_1^m[B], y) \text{ is locally strictly increasing}\}.$$

By Lemma 1.5, there are an open box $B_1 \subseteq B_0$ and an open interval $I_1 \subseteq \varrho_1^m[B]$ such that $B_1 \times I_1 \subseteq \{\langle \overline{x}, y \rangle \in B_0 \times M : \inf \varrho_1^m[B] < y < h_1(\overline{x})\}$. Now, fix $b \in I_1$ and for $\overline{a} \in B_1$ define

$$h_2(\overline{a}) = \sup\{y \in (b, \sup I_1) : f_1^{\overline{a}} \restriction (b, y) \text{ is strictly increasing}\}.$$

Again, by Lemma 1.5, there are an open box $B_2 \subseteq B_1$ and an open interval $I_2 \subseteq I_1$ such that $B_2 \times I_2 \subseteq \{\langle \overline{x}, y\rangle \in B_1 \times I_1 : b < y < h_2(\overline{x})\}$. Let $B' = I_2 \times B_2$. The function $f_1^{\overline{a}} \restriction \varrho_1^m[B']$ is strictly increasing whenever $\overline{a} \in \pi_1^m[B']$.

Repeating the above procedure for the remaining coordinates, one obtains an open box $C \subseteq B'$ as required by the assertion of the lemma. ■

2 Essential dimension theory and injective maps

In this section we start to develop the dimension theory for sets definable in weakly o-minimal structures. Among several other things we prove that the topological dimension of a set definable in a weakly o-minimal structure is invariant under injective definable maps. Before formulating the main theorem we prove a series of technical and preparatory lemmas.

Lemma 2.1 *Assume that $m \geq 2$, J is a non-empty proper subset of $\{1, \dots, m\}$, $i \in \{1, \dots, m\} \setminus J$, $S \subseteq M^m$ is a definable set and $\overline{a} \in \pi_i^m[S]$. Assume also that there are infinitely many tuples $\overline{c} \in (\pi_i^m)^{-1}(\overline{a}) \cap S$ such that $\varrho_J^m(\overline{c}) \in \mathrm{int}[\varrho_J^m[(\pi_J^m)^{-1}(\pi_J^m(\overline{c})) \cap S]]$. Then there is a definable set $V \subseteq S$ such that $\varrho_{J\cup\{i\}}^m[V] \subseteq M^{|J|+1}$ is an open box and (in case $|J| < m-1$) $\pi_{J\cup\{i\}}^m$ is a singleton.*

Proof. Assume that m, J, i, S and $\overline{a} := \langle a_1, \dots, a_{i-1}, a_{i+1}, \dots, a_m\rangle$ satisfy assumptions of the lemma. Permuting variables, without loss of generality we can assume that $J = \{1, \dots, |J|\}$ and $i = |J| + 1$. In such a situation, let $\overline{a}' = \langle a_1, \dots, a_{i-1}\rangle$ and

$$S_1 = \begin{cases} S & \text{if} \quad i = m \\ \{\langle x_1, \dots, x_i\rangle \in M^i : \langle x_1, \dots, x_i, a_{i+1}, \dots, a_m\rangle \in S\} & \text{if} \quad i < m. \end{cases}$$

There is an open interval $I \subseteq M$ such that $\overline{a}' \in \mathrm{int}\{\overline{c} \in M^{i-1} : \langle \overline{c}, d\rangle \in S_1\}$ whenever $d \in I$. Let

$$S' = \bigcup_{d\in I} \mathrm{int}\{\overline{c} \in M^{i-1} : \langle \overline{c}, d\rangle \in S_1\} \times \{d\}.$$

We will be done if we demonstrate that $\mathrm{int}(S') \neq \emptyset$. For this reason we will inductively find definable sets $V_1, \dots, V_i \subseteq S'$ such that

(a)$_s$ (for $s \in \{1, \dots, i\}$) $\varrho_{i+1-s,\dots,i}^i[V_s] \subseteq M^s$ is an open box;

(b)$_s$ (for $s \in \{1, \dots, i-1\}$) $\pi_{i+1-s,\dots,i}^i[V_s] = \{\langle a_1, \dots, a_{i-s}\rangle\}$.

Note that for $V_1 := \{\overline{a}'\} \times I$, coditions (a)$_1$ and (b)$_1$ hold. Suppose that we have already found $V_s \subseteq S'$, $1 \leq s < i$, for which conditions (a)$_s$ and (b)$_s$ are satisfied. For $\overline{x} \in \varrho_{i+1-s,\dots,i}^i[V_s]$, define $f(\overline{x}) \in \overline{M}^{\mathcal{M}} \cup \{+\infty\}$, $f(\overline{x}) > a_{i-s}$, as the supremum of the convex component of $\{z \in M : \langle a_1, \dots, a_{i-s-1}, z, \overline{x}\rangle \in S'\}$ containing a_{i-s}. By Lemma 1.5, the set $\{\langle z, \overline{x}\rangle \in M^{s+1} : \overline{x} \in \varrho_{i+1-s,\dots,i}^i[V_s], a_{i-s} < z < f(\overline{x})\} \subseteq M^{s+1}$ contains an open box B. Clearly, the set $V_{s+1} := \{\langle a_1, \dots, a_{i-s-1}\rangle\} \times B$ satisfies our demands. ■

Definition 2.2 *Assume that $m \geq 2$, $J \subseteq \{1, \dots m\}$ and $\emptyset \neq S \subseteq M^m$. We say that the set S is J-open iff the following conditions are satisfied.*

(a) $(\forall i \in \{1, \dots, m\} \setminus J)(\forall \overline{a} \in \pi_i^m[S])((\pi_i^m)^{-1}(\overline{a}) \cap S$ is finite).

(b) If $0 < |J| < m$ and $\overline{c} \in S$, then there exists a definable set $U \subseteq S$ containing $\overline{c}$ such that $\varrho_J^m[U] \subseteq M^{|J|}$ is an open box and $\pi_J^m[U] = \{\pi_J^m(\overline{c})\}$.

(c) If $J = \{1, \dots, m\}$, then S is open.

Fact 2.3 *Assume that $m \geq 2$, $J, J_1, J_2 \subseteq \{1, \dots, m\}$ and $X, Y \subseteq M^m$.*

(a) If X, Y are J-open, then $X \cup Y$ is J-open.

(b) If X is J_1-open and Y is J_2-open, then $X \cap Y$ is $J_1 \cap J_2$-open or empty.

Lemma 2.4 *Assume that $m \geq 2$ and $S \subseteq M^m$ is a non-empty definable set. Then there are pairwise disjoint definable sets X_J, $J \subseteq \{1,\ldots,m\}$, such that $S = \bigcup_{J\subseteq\{1,\ldots,m\}} X_J$ and for every $J \subseteq \{1,\ldots,m\}$, the set X_J is either J-open or empty.*

Proof. Let $J_1,\ldots,J_{2^m}$ be an enumeration of all subsets of $\{1,\ldots,m\}$ such that $|J_i| \geq |J_j|$ whenever $1 \leq i \leq j \leq 2^m$. Clearly, $J_1 = \{1,\ldots,m\}$ and $J_{2^m} = \emptyset$. We will define pairwise disjoint sets $X_1,\ldots,X_{2^m}$ such that $\bigcup_{i=1}^{2^m} X_i = S$ and for every $i \in \{1,\ldots,2^m\}$, the set X_i is either J_i-open or empty.

Let $X_1 = \operatorname{int}(S)$. If $X_1 \neq \emptyset$, then X_1 is J_1-open. For the inductive step, fix $k \in \{1,\ldots,2^m-2\}$ and suppose that the (pairwise disjoint) sets $X_1,\ldots,X_k$ have already been defined. Let $Y = X \setminus (X_1\cup\ldots\cup X_k)$ and let X_{k+1} be the union of all sets of the form $(\varrho^m_{J_{k+1}})^{-1}[B]\cap(\pi^m_{J_{k+1}})^{-1}(\pi^m_{J_{k+1}}(\overline{c}))$, where $\overline{c} \in Y$ and $B \subseteq M^{|J_{k+1}|}$ is an open box such that $\overline{c} \in (\varrho^m_{J_{k+1}})^{-1}[B]\cap(\pi^m_{J_{k+1}})^{-1}(\pi^m_{J_{k+1}}(\overline{c})) \subseteq Y$. In case $X_{k+1} \neq \emptyset$, Lemma 2.1 and our enumeration of $\mathcal{P}(\{1,\ldots,m\})$ guarantee that condition (a) of Definition 2.2 is satisfied. Condition (b) is obvious. Define also X_{2^m} as $S \setminus (X_1 \cup \ldots \cup X_{2^m-1})$. Certainly, X_{2^m} is $\emptyset$-open. ■

Lemma 2.5 *Assume that $m \in \mathbb{N}_+$, $i \in \{1,\ldots,m+1\}$, $B \subseteq M^m$ is an open box, and $X \subseteq S \subseteq M^{m+1}$ are definable sets such that*

- *$\pi^{m+1}_i[S] = \pi^{m+1}_i[X] = B$;*
- *$\varrho^{m+1}_i[(\pi^{m+1}_i)^{-1}(\overline{c}) \cap S]$ is an infinite convex set whenever $\overline{c} \in B$;*
- *$\emptyset \neq (\pi^{m+1}_i)^{-1}(\overline{c}) \cap X \subsetneq (\pi^{m+1}_i)^{-1}(\overline{c}) \cap S$ whenever $\overline{c} \in B$.*

Then $\dim(S) = m+1$.

Proof. We use induction on m. For $m = 1$, the result is an easy consequence of Lemma 1.3. Suppose that it is true for dimensions smaller than m, where $m \geq 2$, and fix B, S, X and i as in the statement of the lemma. Without loss of generality we can assume that $i = m+1$. In such a situation, for $\overline{c} \in B$ define $f(\overline{c}) = \inf\{d \in M : \langle \overline{c}, d\rangle \in S\}$ and $g(\overline{c}) = \sup\{d \in M : \langle \overline{c}, d\rangle \in S\}$. Note that if one of the sets: $\{\overline{c} \in B : f(\overline{c}) = -\infty\}$, $\{\overline{c} \in B : g(\overline{c}) = +\infty\}$ has non-empty interior, then by Lemma 1.5, S contains an open box. Hence we can assume that $f(\overline{c}), g(\overline{c}) \in \overline{M}^{\mathcal{M}}$ whenever $\overline{c} \in B$. Clearly, f and g are definable functions and $f(\overline{c}) < g(\overline{c})$ for $\overline{c} \in B$. Let $B = B' \times I$, where B' is an open box and I is an open interval. By Lemma 1.10, without loss of generality we can assume that

- $(\forall \overline{a} \in B')(f(\overline{a}, y)$ is strictly increasing) or
- $(\forall \overline{a} \in B')(f(\overline{a}, y)$ is strictly decreasing), or
- $(\forall \overline{a} \in B')(f(\overline{a}, y)$ is constant),

and similarly for g.

Suppose first that for $\overline{a} \in B'$, $f(\overline{a}, y)$ is constant or strictly decreasing while $g(\overline{a}, y)$ is constant or strictly increasing. Let $b \in I$. By the inductive hypothesis, $\dim(\{\langle \overline{x}, z\rangle : \langle \overline{x}, b, z\rangle \in S\}) = m$. Consequently, the set $\{\langle \overline{x}, y, z\rangle : \langle \overline{x}, b, z\rangle \in S, b < y < \sup I\} \subseteq S$ has dimension $m+1$. Similar argument works if for $\overline{a} \in B'$, we have that $f(\overline{a}, y)$ is constant or strictly increasing while $g(\overline{a}, y)$ is constant or strictly decreasing.

To finish the proof, we have to consider the case when for every $\overline{a} \in B'$, the functions $f(\overline{a}, y)$, $g(\overline{a}, y)$ are both strictly increasing or both strictly decreasing. Below we only deal with the first possibility.

For $\langle \overline{a}, b\rangle \in B' \times I$, let $h(\overline{a}, b) = \inf\{f(\overline{a}, c) : c \in (b, \sup I)\}$. Below we consider two cases.

Case 1. There is $\overline{a} \in B'$ such that the set $\{b \in I : h(\overline{a}, b) \geq g(\overline{a}, b)\}$ is infinite, i.e. contains an open interval I'. Then the set $\{z \in M : (\exists y \in I')(\langle \overline{a}, y, z\rangle \in X)\}$ is not a union of finitely many convex sets.

Case 2. For every $\overline{a} \in B'$, the set $\{b \in I : h(\overline{a}, b) \geq g(\overline{a}, b)\}$ is finite. For $\overline{a} \in B'$ define $u(\overline{a}) = \min(\{b \in I : h(\overline{a}, b) \geq g(\overline{a}, b)\} \cup \{\sup I\})$. By Lemma 1.5, there are an open box $B_1 \subseteq B'$ and an open interval $I_1 \subseteq I$ such that for any $\overline{a} \in B_1$ and $b \in I_1$, we have that $h(\overline{a}, b) < g(\overline{a}, b)$. Again, using Lemma 1.5, without loss of generality we can assume that $g(\overline{a}, b_1) > f(\overline{a}, b_2)$ for any $\overline{a} \in B_1$ and $b_1 < b_2$ from I_1.

Now, fix $b_1 < b_2 < b_3$ from I_1 and define

$$X_1 = \{\langle \overline{x}, z\rangle \in B_1 \times M : f(\overline{x}, b_2) < z < f(\overline{x}, b_3)\};$$
$$S_1 = \{\langle \overline{x}, z\rangle \in B_1 \times M : f(\overline{x}, b_2) < z < g(\overline{x}, b_1)\}.$$

Note that $\pi^m_m[X_1] = \pi^m_m[S_1] = B_1$, and for every $\overline{a} \in B_1$, we have that $(S_1)_{\overline{a}}$ is an infinite convex set such that $\emptyset \neq (X_1)_{\overline{a}} \subsetneq (S_1)_{\overline{a}}$. By the inductive hypothesis, $\dim(S_1) = m$. So the set $\{\langle \overline{x}, y, z\rangle \in B_1 \times (b_1, b_2) \times M : f(\overline{x}, b_2) < z < g(\overline{x}, b_1)\} \subseteq S$ has dimension $m + 1$, which finishes the proof. ■

Definition 2.6 *Assume that $m \in \mathbb{N}_+$, $X \subseteq S \subseteq M^m$ are definable sets of dimension $k \geq 0$ and $\overline{a} \in X$. We say that X is smooth at $\overline{a}$ with respect to S iff $k = 0$, or $k \geq 1$ and there are an open box $B \subseteq M^m$ containing $\overline{a}$ and a projection $\pi : M^m \longrightarrow M^k$ such that $B \cap X = B \cap S$, $\pi[B \cap X]$ is an open box in M^k, and $\pi \restriction B \cap X$ is a homeomorphism from $B \cap X$ onto $\pi[B \cap X]$. We say that X is locally smooth in S iff for every $\overline{a} \in X$, X is smooth at $\overline{a}$ with respect to S.*

Lemma 2.7 *Assume that $m \in \mathbb{N}_+$, $X \subseteq M^m$ is a definable set of dimension $k \geq 1$, $B' \subseteq B \subseteq M^m$ are open boxes, $\overline{a} \in B' \cap X$, $\pi : M^m \longrightarrow M^k$ is a projection, $\pi[B \cap X]$ is an open box in M^k and $\pi \restriction B \cap X$ is a homeomorphism from $B \cap X$ onto $\pi[B \cap X]$. Then there is an open box $B'' \subseteq B'$ containing $\overline{a}$ such that $\pi[B'' \cap X]$ is an open box in M^k and $\pi \restriction B'' \cap X$ is a homeomorphism from $B'' \cap X$ onto $\pi[B'' \cap X]$.*

Proof. Let $g : \pi[B \cap X] \longrightarrow B \cap X$ be the map given by $g(\pi(\overline{c})) = \overline{c}$ for $\overline{c} \in B \cap X$. Since g is a homeomorphism from $\pi[B \cap X]$ onto $B \cap X$, the preimage $g^{-1}[B' \cap X] = \pi[B' \cap X]$ is open in M^k. Let $B_1 \subseteq g^{-1}[B' \cap X]$ be an open box containing $\pi(\overline{a})$. Then $B'' := B' \cap \pi^{-1}[B_1]$ is an open box in M^m satisfying our demands. ■

Lemma 2.8 *Assume that $m \in \mathbb{N}_+$, U, V, Y are definable subsets of M^m, $U, V \subseteq Y$, U is locally smooth in Y, V is open in Y, $U \cap V \neq \emptyset$, and $\dim(U) = \dim(Y) = k \geq 1$. Then $\dim(U \cap V) = k$ and $U \cap V$ is locally smooth in Y.*

Proof. Fix $\overline{a} \in U \cap V$. We have to show that $U \cap V$ is smooth at $\overline{a}$ with respect to Y. Since U is smooth at $\overline{a}$ with respect to Y, there are an open box $B_1 \subseteq M^m$ containing $\overline{a}$ and a projection $\pi : M^m \longrightarrow M^k$ such that $B_1 \cap Y \subseteq U$, $\pi[B_1 \cap Y]$ is an open box in M^k, and $\pi \restriction B_1 \cap Y$ is a homeomorphism from $B_1 \cap Y$ onto $\pi[B_1 \cap Y]$. Since V is open in Y, there is an open box $B_2 \subseteq M^m$ containing $\overline{a}$ such that $B_2 \cap Y \subseteq V$. Hence $B' := B_1 \cap B_2$ is an open box in M^m containing $\overline{a}$ such that $B' \cap Y \subseteq U \cap V$. By Lemma 2.7, there is an open box $B'' \subseteq B'$, containing $\overline{a}$ such that $\pi[B'' \cap Y]$ is an open box in M^k and $\pi \restriction B'' \cap Y$ is a homeomorphism from $B'' \cap Y$ onto $\pi[B'' \cap Y]$. This finishes the proof. ■

Lemma 2.9 *Assume that $m \in \mathbb{N}_+$, $i \in \{1, \ldots, m+1\}$ and $S \subseteq M^{m+1}$ is a definable set with $\dim(\pi^{m+1}_i[S]) = m$. Then $\dim(S) = m$ iff for every open interval $I \subseteq M$, there is a definable set $X \subseteq \pi^{m+1}_i[S]$ such that $\dim(X) < m$, and for any $k \in \mathbb{N}_+$ and $\overline{a}_1, \ldots, \overline{a}_k \in \pi^{m+1}_i[S] \setminus X$, we have that $I \not\subseteq \bigcup\limits_{l=1}^{k} \varrho^{m+1}_i[(\pi^{m+1}_i)^{-1}(\overline{a}_l) \cap S]$.*

Proof. Without loss of generality we can assume that $i = m+1$. For the left-to-right direction, suppose that there is an open interval $I \subseteq M$ such that for every definable set $X \subseteq \pi^{m+1}_{m+1}[S]$ with $\dim(X) < m$, there are $k \in \mathbb{N}_+$ and $\overline{a}_1, \ldots, \overline{a}_k \in \pi^{m+1}_{m+1}[S] \setminus X$ such that $I \subseteq S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_k}$. Denote by S_1 the set of all tuples $\overline{a} \in \pi^{m+1}_{m+1}[S]$ for which the set $I \cap S_{\overline{a}}$ is non-empty and contains some open interval whose infimum is equal to $\inf I$. Our assumptions guarantee that S_1 contains an open box B. For $\overline{a} \in B$, let $f(\overline{a}) \in \overline{M}^{\mathcal{M}}$ be the the supremum of the first convex component of $I \cap S_{\overline{a}}$. The set $\{\langle \overline{a}, b\rangle : \inf(I) < b < f(\overline{a})\}$ is contained in S and (by Lemma 1.5) has dimension $m+1$. Consequently, $\dim(S) = m+1$.

The right-to-left direction is trivial. ■

Lemma 2.10 *Assume that $m \in \mathbb{N}_+$, $S \subseteq M^m$ is a non-empty definable set, $I \subseteq M$ is an open interval and $X \subseteq S \times I$ is a definable set such that $\dim((S \times \{b\}) \cap X) < \dim(S)$ whenever $b \in I$. Then $\dim(X) \leq \dim(S)$.*

Proof. We use induction on m. Let $S \subseteq M$ be a non-empty definable set, $I \subseteq M$ an open interval and $X \subseteq S \times I$ a definable set such that $\dim((S \times \{b\}) \cap X) < \dim(S)$ whenever $b \in I$. If S is finite, then $(S \times \{b\}) \cap X = \emptyset$ for $b \in I$, which means that $X = \emptyset$. If S is infinite, then $(S \times \{b\}) \cap X$ is finite whenever $b \in I$. Hence X does not contain an open box, which means that $\dim(X) \leq 1$.

Now assume that $S \subseteq M^{m+1}$ is a non-empty definable set, $I \subseteq M$ is an open interval and $X \subseteq S \times I$ is a definable set such that $\dim((S \times \{b\}) \cap X) < \dim(S)$ whenever $b \in I$, and suppose that the Lemma holds for lower dimensions. If $\dim(S) = m+1$, then it is clear that X does not contain an open box. Assume that $\dim(S) \leq m$ and suppose for a contradiction that $\dim(X) = \dim(S)+1$. Let $\pi : M^{m+2} \longrightarrow M^{m+1}$ be a projection such that $\dim(\pi[X]) = \dim(X) = \dim(S)+1$. The projection π does not drop the last coordinate, so there is a unique projection $\pi' : M^{m+1} \longrightarrow M^m$ such that $\pi(\overline{a}, b) = \langle \pi'(\overline{a}), b\rangle$ for $\overline{a} \in M^{m+1}$ and $b \in M$. Then

$$\dim((\pi'[S] \times \{b\}) \cap \pi[X]) = \dim(\pi[(S \times \{b\}) \cap X]) \leq \dim((S \times \{b\}) \cap X) < \dim(S) = \dim(\pi'[S]).$$

The first equality above holds because $(\pi'[S] \times \{b\}) \cap \pi[X] = \pi[(S \times \{b\}) \cap X]$. By the inductive assumption, $\dim(\pi[X]) \leq \dim(\pi'[S])$. Hence $\dim(X) \leq \dim(S)$, a contradiction. ■

Theorem 2.11 *Let $m \in \mathbb{N}_+$.*

(a)$_m$ If $S \subseteq M^m$ is a non-empty definable set, then $\dim(\mathrm{cl}(S) \setminus S) < \dim(S)$.

(b)$_m$ If $X \subseteq S \subseteq M^m$ are non-empty definable sets and $\dim(X) = \dim(S)$, then the set $\{\overline{a} \in X : X$ is smooth at $\overline{a}$ with respect to $S\}$ is large in X.

(c)$_m$ Assume that $S \subseteq M^m$ is a non-empty definable set and $f : S \longrightarrow M$ is a definable function. Then the set of continuity points of f is large in S.

(d)$_m$ Assume that $S \subseteq M^m$ is a non-empty definable set, $f : S \longrightarrow M$ and $g : S \longrightarrow \overline{M}^{\mathcal{M}}$ are definable functions and f is continuous. If $(\forall \overline{a} \in S)(f(\overline{a}) < g(\overline{a}))$ [respectively: $(\forall \overline{a} \in S)(f(\overline{a}) > g(\overline{a}))$], then there are an open interval $I \subseteq M$ and a definable set $X \subseteq S$ such that $\dim(X) = \dim(S)$ and $X \times I \subseteq (f, g)_S$ [respectively: $X \times I \subseteq (g, f)_S$].

(e)$_m$ If $S \subseteq M^{m+1}$ is a non-empty definable set and $i \in \{1, \ldots, m+1\}$, then $\dim(S) = \dim(\pi^{m+1}_i[S])$ iff for every open interval $I \subseteq M$, there is a definable set $X \subseteq \pi^{m+1}_i[S]$ such that $\dim(X) < \dim(\pi^{m+1}_i[S])$, and for any $k \in \mathbb{N}_+$ and $\overline{a}_1, \ldots, \overline{a}_k \in \pi^{m+1}_i[S] \setminus X$,

$$I \not\subseteq \bigcup_{l=1}^{k} \varrho^{m+1}_i[(\pi^{m+1}_i)^{-1}(\overline{a}_l) \cap S].$$

(f)$_m$ Assume that $S \subseteq M^{m+1}$ is a non-empty definable set, $i \in \{1, \ldots, m+1\}$ and for every $\overline{a} \in \pi^{m+1}_i[S]$, $(\pi^{m+1}_i)^{-1}(\overline{a}) \cap S$ is finite. Then $\dim(S) = \dim(\pi^{m+1}_i[S])$.

(g)$_m$ Assume that $i \in \{1, \ldots, m+1\}$ and $X \subseteq S \subseteq M^{m+1}$ are non-empty definable sets. Assume also that $\pi^{m+1}_i[X] = \pi^{m+1}_i[S]$, $\dim(\pi^{m+1}_i[S]) \geq 1$, and for every $\overline{a} \in \pi^{m+1}_i[S]$, $\varrho^{m+1}_i[(\pi^{m+1}_i)^{-1}(\overline{a}) \cap S]$ is an infinite convex set and $\emptyset \neq (\pi^{m+1}_i)^{-1}(\overline{a}) \cap X \subsetneq (\pi^{m+1}_i)^{-1}(\overline{a}) \cap S$. Then $\dim(S) = \dim(\pi^{m+1}_i[S]) + 1$.

Proof. We use induction on m. Conditions (a)$_1$ and (b)$_1$ are obvious by the weak o-minimality of $\mathcal{M}$. (c)$_1$, (d)$_1$ and (e)$_1$ are consequences of Theorem 1.2 and Lemmas 1.5 and 2.9 respectively. (e)$_1$ implies (f)$_1$. Finally, (g)$_1$ is consequence of Lemma 1.3. For the rest of the proof suppose that $m \in \mathbb{N}_+$ and statements (a)$_m$–(g)$_m$ are true.

Proof of (a)$_{m+1}$. Let $S \subseteq M^{m+1}$ be a non-empty definable set. By Fact 1.6(e), it is enough to show that $\dim(\mathrm{cl}(S) \setminus S) < \dim(\mathrm{cl}(S))$. Suppose for a contradiction that $\dim(\mathrm{cl}(S) \setminus S) = \dim(\mathrm{cl}(S)) = k$. Clearly, this is not possible for $k \in \{0, m+1\}$, so let $1 \le k \le m$. There is $i \in \{1, \ldots, m+1\}$ such that $\dim(\pi_i^{m+1}[\mathrm{cl}(S) \setminus S]) = \dim(\pi_i^{m+1}[\mathrm{cl}(S)]) = k$. To simplify notation, assume that $i = m+1$ and let $\pi = \pi_{m+1}^{m+1}$, $\pi' = \varrho_{m+1}^{m+1}$. Note that the set

$$X := \{\overline{a} \in \pi[\mathrm{cl}(S) \setminus S] : \pi^{-1}(\overline{a}) \cap S \neq \emptyset\}$$

is large in $\pi[\mathrm{cl}(S) \setminus S]$. Otherwise, by (b)$_m$, there is an open box $B_0 \subseteq M^m$ such that

$$B_0 \cap \pi[\mathrm{cl}(S)] = B_0 \cap (\pi[\mathrm{cl}(S) \setminus S] \setminus X)$$

and $\dim(B_0 \cap \pi[\mathrm{cl}(S)]) = k$. Consequently, for every open box $B \subseteq B_0 \times M$ with $B \cap \mathrm{cl}(S) \neq \emptyset$, we have that $B \cap S = \emptyset$, which is impossible. So in particular $\dim(X) = k$.

Let X_1 be the set of all tuples $\overline{a} \in X$ such that at least one of the convex components of the set $\pi'[\pi^{-1}(\overline{a}) \cap (\mathrm{cl}(S) \setminus S)]$ precedes $\pi'[\pi^{-1}(\overline{a}) \cap S]$ and let $X_2 = X \setminus X_1$. As $X_1 \cup X_2 = X$, at least one of the sets X_1, X_2 has dimension k. The proof is similar in both situations, therefore we only consider the case when $\dim(X_1) = k$. For $\overline{a} \in X_1$ denote by $A(\overline{a})$ the last convex component of $\pi'[\pi^{-1}(\overline{a}) \cap (\mathrm{cl}(S) \setminus S)]$ preceding $\pi'[\pi^{-1}(\overline{a}) \cap S]$ and by $B(\overline{a})$ the first convex component of $\pi'[\pi^{-1}(\overline{a}) \cap S]$. Define the following sets.

$$\begin{aligned} Y_1 &= \{\overline{a} \in X_1 : \sup A(\overline{a}) = \inf B(\overline{a})\} \\ Y_2 &= \{\overline{a} \in X_1 : \sup A(\overline{a}) < \inf B(\overline{a})\} \end{aligned}$$

Again, at least one of the sets Y_1, Y_2 has dimension k. In case $\dim(Y_1) = k$, consider

$$X' := \bigcup_{\overline{a} \in Y_1} \{\overline{a}\} \times A(\overline{a}) \text{ and } S' := \bigcup_{\overline{a} \in Y_1} \{\overline{a}\} \times (A(\overline{a}) \cup B(\overline{a})).$$

By (g)$_m$, $\dim(S') = k+1$, a contradiction. If $\dim(Y_2) = k$, then define

$$X'' := \bigcup_{\overline{a} \in Y_2} \{\overline{a}\} \times A(\overline{a}) \text{ and } S'' := \bigcup_{\overline{a} \in Y_2} \{\overline{a}\} \times \{b \in M : A(\overline{a}) < b < B(\overline{a})\}.$$

(g)$_m$ implies that $\dim(X'' \cup S'') = k+1$. But $\dim(X'') = k$, so $\dim(S'') = k+1$. By lemma 2.10, there is $b \in M$ such that $\dim(S'' \cap (M^m \times \{b\})) = k$. For such b, let $Y_3 = \pi[S'' \cap (M^m \times \{b\})]$. By (b)$_m$, there is an open box $B_1 \subseteq M^m$ such that $B_1 \cap Y_3 = B_1 \cap \pi[\mathrm{cl}(S)]$ and $\dim(B_1 \cap Y_3) = k$. There is $b_1 \in (-\infty, b)$ such that $(B_1 \times (b_1, b)) \cap \mathrm{cl}(S) \neq \emptyset$ and $(B_1 \times (b_1, b)) \cap S = \emptyset$, which is impossible.

Proof of (b)$_{m+1}$. Assume that $X \subseteq S \subseteq M^{m+1}$ are definable sets of dimension $k \ge 0$. The assertion of (b)$_{m+1}$ is obvious for $k \in \{0, m+1\}$. So let $1 \le k \le m$ and suppose for a contradiction that the set $X' := \{\overline{a} \in X : X \text{ is not smooth at } \overline{a} \text{ with respect to } S\}$ has dimension k. By Lemma 2.4 and Fact 1.6(e), there are $J \subsetneq \{1, \ldots, m+1\}$ and a k-dimensional J-open set $X_0 \subseteq X'$. By (a)$_{m+1}$, the set $X_0' := X_0 \setminus \mathrm{cl}(S \setminus X_0)$ is large in X_0. Fix $i \in \{1, \ldots, m\} \setminus J$. Clearly, for every $\overline{a} \in \pi_i^{m+1}[X_0']$, the fiber $(\pi_i^{m+1})^{-1}(\overline{a}) \cap X_0'$ is finite. Hence, by (f)$_m$, $\dim(\pi_i^{m+1}[X_0']) = k$.

For $\overline{a} \in \pi_i^{m+1}[X_0']$ define

$$\begin{aligned} f(\overline{a}) &= \min \varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap X_0']; \\ g(\overline{a}) &= \min((\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap X_0'] \setminus \{f(\overline{a})\}) \cup \{+\infty\}). \end{aligned}$$

Let $Y_1 = \{\overline{a} \in \pi_i^{m+1}[X_0'] : g(\overline{a}) \in M\}$ and $Y_2 := \{\overline{a} \in \pi_i^{m+1}[X_0'] : g(\overline{a}) = +\infty\}$. Of course, $Y_1 \cup Y_2 = \pi_i^{m+1}[X_0']$, so at least one of the sets Y_1, Y_2 has dimension k. Below we only consider the case when $\dim(Y_1) = k$.

By (b)$_m$ and (c)$_m$, there are an open box $B_1 \subseteq M^m$ and a projection $\pi : M^m \longrightarrow M^k$ such that $B_1 \cap Y_1 = B_1 \cap \pi_i^{m+1}[S]$, $\pi[B_1 \cap Y_1]$ is an open box in M^k, $\pi \restriction B_1 \cap Y_1$ is a homeomorphism from $B_1 \cap Y_1$ onto $\pi[B_1 \cap Y_1]$, and the functions f, g are continuous on $B_1 \cap Y_1$. Fix $\overline{a} \in B_1 \cap Y_1$ and $b, c \in M$ such that $b < f(\overline{a}) < c < g(\overline{a})$. There is an open box $B_2 \subseteq B_1$ containing $\overline{a}$ such that $b < f(\overline{x}) < c < g(\overline{x})$ whenever $\overline{x} \in B_2 \cap Y_1$. Let $\overline{c} \in X_0'$ be the unique tuple such that $\varrho_i^{m+1}(\overline{c}) = f(\overline{a})$ and $\pi_i^{m+1}(\overline{c}) = \overline{a}$. There is an open box $B \subseteq M^{m+1}$ containing $\overline{c}$ such that $B \cap \mathrm{cl}(S \setminus X_0) = \emptyset$, $\pi_i^{m+1}[B] \subseteq B_2$ and $b < \inf \varrho_i^{m+1}[B] < f(\overline{x}) < \sup \varrho_i^{m+1}[B] < g(\overline{x})$ for $\overline{x} \in \pi_i^{m+1}[B \cap X_0']$. By Lemma 2.7, there is an open box $B_3 \subseteq \pi_i^{m+1}[B]$ containing $\overline{a}$ such that $\pi \restriction B_3 \cap Y_1$ is a homeomorphism from $B_3 \cap Y_1$ onto $\pi[B_3 \cap Y_1]$, an open box in M^k. Let $B' = B \cap (\pi_i^{m+1})^{-1}[B_3]$ and $\pi' = \pi \circ \pi_i^{m+1}$. Then $B' \cap X = B' \cap S$, $\pi' \restriction B' \cap X$ is a homeomorphism from $B' \cap X$ onto $\pi'[B' \cap X]$, an open box in M^k. This finishes the proof of (b)$_{m+1}$.

Proof of (c)$_{m+1}$. Assume that $S \subseteq M^{m+1}$ is a non-empty definable set, $f : S \longrightarrow M$ is a definable function and denote by X the set of discontinuity points of f. Suppose for a contradiction that $\dim(X) = \dim(S) = k$. By Lemma 1.4, without loss of generality we can assume that $1 \leq k \leq m$. By (b)$_{m+1}$, there are an open box $B \subseteq M^{m+1}$ and $i \in \{1, \ldots, m+1\}$ such that $B \cap S = B \cap X$, $\dim(B \cap X) = k$ and the projection π_i^{m+1} restricted to $B \cap X$ is a homeomorphism from $B \cap X$ onto $\pi_i^{m+1}[B \cap X]$. Let g denote the inverse of this homeomorphism. By (c)$_m$, the function $f \circ g$ has a continuity point in $\pi_i^{m+1}[B \cap X]$. Consequently, f has a continuity point in $B \cap X$, which contradicts our choice of X.

Proof of (d)$_{m+1}$. Assume that $S \subseteq M^{m+1}$ is a non-empty definable set, $f : S \longrightarrow M$ and $g : S \longrightarrow \overline{M}^{\mathcal{M}}$ are definable functions, f is continuous, and $f(\overline{a}) < g(\overline{a})$ whenever $\overline{a} \in S$. In case $\dim(S) = m+1$, the assertion of (d)$_{m+1}$ is a consequence of Lemma 1.5. The case $\dim(S) = 0$ is trivial. So assume that $1 \leq \dim(S) = k \leq m$. By (b)$_{m+1}$, there are an open box $B \subseteq M^{m+1}$ and $i \in \{1, \ldots, m+1\}$ such that the projection π_i^{m+1} restricted to $B \cap S$ is a homeomorphism from $B \cap S$ onto $\pi_i^{m+1}[B \cap S]$ and $\dim(\pi_i^{m+1}[B \cap S]) = k$. Let h denote the inverse of this homeomorphism. By (d)$_m$, there are an open interval I and a definable set $Z \subseteq \pi_i^{m+1}[B \cap S]$ such that $\dim(Z) = k$ and $Z \times I \subseteq \{\langle \overline{x}, y\rangle : \overline{x} \in Z, f(h(\overline{x})) < y < g(h(\overline{x}))\}$. Clearly, $\dim(h[Z]) = k$. Moreover, $h[Z] \times I \subseteq \{\langle \overline{x}, y\rangle : \overline{x} \in B \cap S \text{ and } f(\overline{x}) < y < g(\overline{x})\}$.

Proof of (e)$_{m+1}$. Assume that $S \subseteq M^{m+2}$ is a non-empty definable set and $i \in \{1, \ldots, m+2\}$. If $\dim(S) = 0$, then both sides of the equivalence in (e)$_{m+1}$ are true. In case $\dim(\pi_i^{m+2}[S]) = \dim(S) - 1 \in \{0, m+1\}$, they are both false. Also, by Lemma 2.9, if $\dim(S) = \dim(\pi_i^{m+2}[S]) = m+1$, then the right side of the equivalence in (e)$_{m+1}$ is valid. Having considered all the trivial cases, assume that $1 \leq \dim(\pi_i^{m+2}[S]) \leq m$.

For the left-to-right direction, assume that $\dim(S) = \dim(\pi_i^{m+2}[S])$ and $I \subseteq M$ is an open interval. Denote by J the (necessarily non-empty) set of all $j \in \{1, \ldots, m+2\} \setminus \{i\}$ for which $\dim(\pi_{i,j}^{m+2}[S]) = \dim(\pi_i^{m+2}[S]) = \dim(S)$. By Lemma 1.7(b), we have that $\dim(\pi_{i,j}^{m+2}[S]) = \dim(\pi_j^{m+2}[S])$ whenever $j \in J$. Without loss of generality we can assume that $i < j$ for all $j \in J$. By (e)$_m$, there are definable sets $X_j \subseteq \pi_{i,j}^{m+2}[S]$, $j \in J$, such that $\dim(X_j) < \dim(\pi_{i,j}^{m+2}[S])$, and for any $k \in \mathbb{N}_+$ and any $\overline{a}_1, \ldots, \overline{a}_k \in \pi_{i,j}^{m+2}[S] \setminus X_j$, $I \not\subseteq \bigcup_{l=1}^{k} \varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}_l) \cap \pi_j^{m+2}[S]]$. This implies that for any $j \in J$, $k \in \mathbb{N}_+$ and $\overline{b}_1, \ldots, \overline{b}_k \in \pi_i^{m+2}[S] \setminus (\pi_{j-1}^{m+1})^{-1}[X_j]$, we have that $I \not\subseteq \bigcup_{l=1}^{k} \varrho_i^{m+2}[(\pi_i^{m+2})^{-1}(\overline{b}_l) \cap S]$. Let $X = \bigcap_{j \in J} (\pi_{j-1}^{m+1})^{-1}[X_j] \cap \pi_i^{m+2}[S]$. Certainly, for any $k \in \mathbb{N}_+$

and $\overline{b}_1, \ldots, \overline{b}_k \in \pi_i^{m+2}[S] \setminus X$ we have that

$$I \not\subseteq \bigcup_{l=1}^{k} \varrho_i^{m+2}[(\pi_i^{m+2})^{-1}(\overline{b}_l) \cap S]$$

and $\dim(\pi_j^{m+1}[X]) < \dim(\pi_{i,j}^{m+2}[S]) = \dim(S)$ whenever $j \in J$. We claim that $\dim(X) < \dim(\pi_i^{m+2}[S])$. Suppose otherwise. Then there is $j_0 \in J$ such that

$$\dim(S) = \dim(\pi_{i,j_0}^{m+2}[S]) \geq \dim(\pi_{j_0-1}^{m+1}[X]) = \dim(X) = \dim(\pi_i^{m+2}[S]) = \dim(S),$$

a contradiction.

For the right-to-left direction, assume that $\dim(S) = \dim(\pi_i^{m+2}[S]) + 1$ and fix $j \in \{1, \ldots, m+2\} \setminus \{i\}$ such that $\dim(\pi_j^{m+2}[S]) = \dim(S)$. Again, without loss of generality, we can assume that $i < j$. By Lemma 1.7, $\dim(\pi_i^{m+2}[S]) = \dim(\pi_{i,j}^{m+2}[S]) = \dim(\pi_j^{m+2}[S]) - 1$. By (e)$_m$, there is an open interval $I \subseteq M$ such that for any definable set $X \subseteq \pi_{i,j}^{m+2}[S]$ with $\dim(X) < \dim(\pi_{i,j}^{m+2}[S])$, there are $k \in \mathbb{N}_+$ and $\overline{a}_1, \ldots, \overline{a}_k \in \pi_{i,j}^{m+2}[S] \setminus X$ such that

$$I \subseteq \bigcup_{l=1}^{k} \varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}_l) \cap \pi_j^{m+2}[S]].$$

Denote by S_1 the set of all $\overline{a}$'s from $\pi_{i,j}^{m+2}[S]$ for which the set $\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap \pi_j^{m+2}[S]]$ contains an interval of the form $(\inf I, y)$ with $y \in I$. Our assumptions guarantee that $\dim(S_1) = \dim(\pi_{i,j}^{m+2}[S]) = \dim(\pi_i^{m+2}[S])$. For $\overline{a} \in S_1$ denote by $g(\overline{a})$ the supremum of the first convex component of the set $\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap \pi_j^{m+2}[S]] \cap I$. By (d)$_m$, there are an open interval $I_1 \subseteq I$ and a definable set $S_2 \subseteq S_1$ such that $\dim(S_2) = \dim(S_1) = \dim(\pi_i^{m+2}[S])$ and $S_2 \times I_1 \subseteq \{\langle \overline{x}, y \rangle : \overline{x} \in S_1, \inf I < y < g(\overline{x})\}$. Hence, for every $\overline{a} \in S_2$, $I_1 \subseteq \varrho_{i,j}^{m+2}[(\pi_{i,j}^{m+2})^{-1}(\overline{a}) \cap S]$.

Now, suppose that $Z \subseteq \pi_i^{m+2}[S]$ is a definable set such that $\dim(Z) < \dim(\pi_i^{m+2}[S])$. For $\overline{a} \in S_2$ denote by $R(\overline{a})$ the set of all elements $b \in I_1$ for which there is a (necessarily unique) tuple $\overline{c} \in \pi_j^{m+2}[S]$ such that $\pi_i^{m+1}(\overline{c}) = \overline{a}$, $\varrho_i^{m+1}(\overline{c}) = b$ and $\varrho_j^{m+2}[(\pi_j^{m+2})^{-1}(\overline{c}) \cap S]$ (necessarily a non-empty set) is a union of some convex components of $\varrho_{j-1}^{m+1}[(\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap \pi_i^{m+2}[S]]$. Let

$$S_3 = \{\overline{a} \in S_2 : R(\overline{a}) \text{ contains an open interval of the form } (\inf I_1, y) \text{ with } y \in I_1\}.$$

Below we consider two cases.

Case 1. $\dim(S_3) = \dim(S_2) = \dim(\pi_i^{m+2}[S])$. For $\overline{a} \in S_3$ let $h(\overline{a}) \in \overline{M}^{\mathcal{M}}$ denote the supremum of the first convex component of $R(\overline{a})$. As previously, there are an open interval $I_2 \subseteq I_1$ and a definable set $S_4 \subseteq S_3$ such that $\dim(S_4) = \dim(S_3) = \dim(\pi_i^{m+2}[S])$ and $S_4 \times I_2 \subseteq \{\langle \overline{x}, y \rangle : \overline{x} \in S_3, \inf I_1 < y < h(\overline{x})\}$. Fix $\overline{a} \in S_4 \setminus \pi_{j-1}^{m+1}[Z]$ and choose $\overline{b}_1, \ldots, \overline{b}_k \in (\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap \pi_i^{m+2}[S]$ so that $\varrho_{j-1}^{m+1}(\overline{b}_1), \ldots, \varrho_{j-1}^{m+1}(\overline{b}_k)$ are representatives of all the convex components of $\varrho_{j-1}^{m+1}[(\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap \pi_i^{m+2}[S]]$. Clearly, $I_2 \subseteq \bigcup_{l=1}^{k} \varrho_i^{m+2}[(\pi_i^{m+2})^{-1}(\overline{b}_l) \cap S]$.

Case 2. $\dim(S_3) < \dim(\pi_i^{m+2}[S])$. Let $S_4 = S_2 \setminus S_3$. Of course, $\dim(S_4) = \dim(\pi_i^{m+2}[S])$. For $\overline{a} \in S_4$ denote by $T(\overline{a})$ the set of all elements $b \in I_1$ for which there is a (necessarily unique) tuple $\overline{c} \in \pi_j^{m+2}[S]$ such that $\pi_i^{m+1}(\overline{c}) = \overline{a}$, $\varrho_i^{m+1}(\overline{c}) = b$ and at least one of the convex components of $\varrho_j^{m+2}[(\pi_j^{m+2})^{-1}(\overline{c}) \cap S]$ is a proper subset of some of the convex components of $\varrho_{j-1}^{m+1}[(\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap \pi_i^{m+2}[S]]$. Note that for every $\overline{a} \in S_4$, $T(\overline{a})$ contains an open interval of the form $(\inf I, y)$, $y \in I$. As in Case 1, by (d)$_m$, there are an open interval $I_2 \subseteq I_1$ and a definable set $S_5 \subseteq S_4$ such that $\dim(S_5) = \dim(S_4) = \dim(\pi_i^{m+2}[S])$ and for every $\overline{c} \in \pi_j^{m+2}[S]$ with $\varrho_i^{m+1}(\overline{c}) \in I_2$ and $\pi_i^{m+1}(\overline{c}) \in S_5$, at least one of the convex components of $\varrho_j^{m+2}[(\pi_j^{m+2})^{-1}(\overline{c}) \cap S]$ is a proper subset of some of the convex components of $\varrho_{j-1}^{m+1}[(\pi_{j-1}^{m+1})^{-1}(\pi_i^{m+1}(\overline{c})) \cap \pi_i^{m+2}[S]]$.

Fix $b \in I_2$ and for $\overline{a} \in S_5$ denote by $\overline{c}(\overline{a})$ the unique tuple from $\pi_j^{m+2}[S]$ such that $\varrho_i^{m+1}(\overline{c}(\overline{a})) = b$ and $\pi_i^{m+1}(\overline{c}(\overline{a})) = \overline{a}$. Let $U(\overline{a})$ be the first convex component of $\varrho_j^{m+1}[(\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap \pi_i^{m+2}[S]]$ such that $\varrho_j^{m+2}[(\pi_j^{m+2})^{-1}(\overline{c}(\overline{a})) \cap S] \cap U(\overline{a})$ is a non-empty proper subset of $U(\overline{a})$. Denote by $V(\overline{a})$ the first convex component of $\varrho_j^{m+2}[(\pi_j^{m+2})^{-1}(\overline{c}(\overline{a})) \cap S] \cap U(\overline{a})$. For every $\overline{a} \in S_5$ denote by $U'(\overline{a})$ the unique definable subset of $(\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap \pi_i^{m+2}[S]$ such that $\varrho_{j-1}^{m+1}[U'(\overline{a})] = U(\overline{a})$ and by $V'(\overline{a})$ the unique definable subset of $\pi_i^{m+2}[(\pi_j^{m+2})^{-1}(\overline{c}(\overline{a})) \cap S]$ such that $\varrho_{j-1}^{m+1}[V'(\overline{a})] = V(\overline{a})$. Let $S' = \bigcup_{\overline{a} \in S_5} U'(\overline{a})$ and $X' = \bigcup_{\overline{a} \in S_5} V'(\overline{a})$. Note that $\dim(S') = \dim(\pi_{j-1}^{m+1}[S'])$, $\pi_{j-1}^{m+1}[S'] = \pi_{j-1}^{m+1}[X'] = S_5$, and for every $\overline{a} \in \pi_{j-1}^{m+1}[S']$ we have that $\varrho_{j-1}^{m+1}[(\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap S']$ is an infinite convex set and $\emptyset \neq (\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap X' \subsetneq (\pi_{j-1}^{m+1})^{-1}(\overline{a}) \cap S'$. This contradicts (g)$_m$.

Proof of (f)$_{m+1}$. Assume that $S \subseteq M^{m+2}$ is a non-empty definable set, $i \in \{1, \dots, m+2\}$, and for every $\overline{a} \in \pi_i^{m+2}[S]$, the set $(\pi_i^{m+2})^{-1}(\overline{a}) \cap S$ is finite. Then for any open interval $I \subseteq M$, $k \in \mathbb{N}_+$ and $\overline{a}_1, \dots, \overline{a}_k \in \pi_i^{m+2}[S]$, we have that $I \not\subseteq \bigcup_{l=1}^{k} \varrho_i^{m+2}[(\pi_i^{m+2})^{-1}(\overline{a}_l) \cap S]$. By (e)$_{m+1}$, $\dim(S) = \dim(\pi_i^{m+2}[S])$.

Proof of (g)$_{m+1}$. Of course, it is sufficient to prove (g)$_{m+1}$ for $i = m+1$. To make the notation simpler, let $\pi = \pi_{m+2}^{m+2}$. Assume that $X \subseteq S \subseteq M^{m+2}$ are non-empty definable sets such that $\dim(\pi[S]) = \dim(S) = k \geq 1$ and $\pi[S] = \pi[X]$. Assume also that for every $\overline{a} \in \pi[S]$, the set $B(\overline{a}) := \{b \in M : \langle \overline{a}, b\rangle \in S\}$ is an infinite convex set of which $\{b \in M : \langle \overline{a}, b\rangle \in X\}$ is a non-empty proper subset. Denote by X' the unique definable subset of X such that for every $\overline{a} \in \pi[S]$, the set $A(\overline{a}) := \{b \in M : \langle \overline{a}, b\rangle \in X'\}$ is the first convex component of $\{b \in M : \langle \overline{a}, b\rangle \in X\}$. Define the following sets.

$$\begin{aligned}
Y_1 &= \{\overline{a} \in \pi[S] : \inf B(\overline{a}) < \inf A(\overline{a}) < \sup B(\overline{a})\};\\
Y_2 &= \{\overline{a} \in \pi[S] : \inf B(\overline{a}) < \sup A(\overline{a}) < \sup B(\overline{a})\};\\
Y_3 &= \{\overline{a} \in \pi[S] : \inf A(\overline{a}) = \inf B(\overline{a}), \sup A(\overline{a}) = \sup B(\overline{a}) \text{ and } \inf B(\overline{a}) \in B(\overline{a}) \setminus A(\overline{a})\};\\
Y_4 &= \{\overline{a} \in \pi[S] : \inf A(\overline{a}) = \inf B(\overline{a}), \sup A(\overline{a}) = \sup B(\overline{a}) \text{ and } \sup B(\overline{a}) \in B(\overline{a}) \setminus A(\overline{a})\};\\
Y_5 &= \{\overline{a} \in \pi[S] : \inf A(\overline{a}) = \sup A(\overline{a}) = \inf B(\overline{a})\};\\
Y_6 &= \{\overline{a} \in \pi[S] : \inf A(\overline{a}) = \sup A(\overline{a}) = \sup B(\overline{a})\}.
\end{aligned}$$

Clearly, $\pi[S] = Y_1 \cup Y_2 \cup Y_3 \cup Y_4 \cup Y_5 \cup Y_6$, so at least one of the sets $Y_1, Y_2, Y_3, Y_4, Y_5, Y_6$ has dimension k. Below we consider 4 cases.

Case 1. $\dim(Y_1) = k$. Lemma 2.5 implies that $k < m+1$. By (b)$_{m+1}$, there are $j \in \{1, \dots, m+1\}$ and an open box $B \subseteq M^{m+1}$ such that $B \cap \pi[S] = B \cap Y_1$, $\dim(B \cap Y_1) = \dim(\pi_j^{m+1}[B \cap Y_1]) = k$ and the projection π_j^{m+1} restricted to $B \cap Y_1$ is a homeomorphism. By (g)$_m$, the set $\pi_j^{m+2}[\pi^{-1}[B \cap Y_1] \cap S] \subseteq M^{m+1}$ has dimension $k+1$, which implies that $\dim(S) = k+1$, a contradiction.

Case 2. $\dim(Y_2) = k$. This case is similar to Case 1.

Case 3. $\dim(Y_3) = k$. Define functions $f : Y_3 \longrightarrow M$ and $g : Y_3 \longrightarrow \overline{M}^{\mathcal{M}}$ as follows:

$$f(\overline{a}) = \inf A(\overline{a}),\ g(\overline{a}) = \sup A(\overline{a}).$$

Our assumptions guarantee that $f(\overline{a}) < g(\overline{a})$ whenever $\overline{a} \in Y_1$. By (c)$_{m+1}$, there is a definable set $Z \subseteq Y_3$, large in Y_3, such that the function f restricted to Z is continuous. By (d)$_{m+1}$, there are an open interval I and a definable set $Z' \subseteq Z$ such that $\dim(Z') = k$ and $Z' \times I \subseteq \{\langle \overline{x}, y\rangle : \overline{x} \in Z \text{ and } f(\overline{x}) < y < g(\overline{x})\}$. Hence $\dim(S) = k+1$.

Case 4. One of the sets Y_4, Y_5, Y_6 has dimension k. This case is similar to Case 3. ■

The following corollary is a direct consequence of condition (f)$_m$ from Theorem 2.11.

Corollary 2.12 *Assume that $m \in \mathbb{N}_+$, $S \subseteq M^m$ is a non-empty definable set and $f : S \longrightarrow M$ is a definable function. Then* $\dim(\Gamma(f)) = \dim(S)$.

Theorem 2.13 *Assume that $\mathcal{M} = (M, \leq, \ldots)$ is a weakly o-minimal structure, $m \in \mathbb{N}_+$, $S \subseteq M^m$ is a non-empty definable set and $f : S \longrightarrow M^n$ is an injective definable map. Then* $\dim(S) = \dim(\Gamma(f)) = \dim(f[S])$.

Proof. Let $f = \langle f_1, \ldots, f_n \rangle$, where $f_1, \ldots, f_n$ are definable maps from S into M. Let $g_1 = f_1$ and for $1 \leq k < n$ define a map $g_{k+1} : \Gamma(g_k) \longrightarrow M$ as follows: $g_{k+1}(\overline{a}, g_k(\overline{a})) = f_{k+1}(\overline{a})$. It is clear that $\Gamma(f) = \Gamma(g_n)$. By Corollary 2.12, $\dim(\Gamma(f)) = \dim(S)$. The same argument with f replaced by f^{-1} shows that $\dim(\Gamma(f)) = \dim(f[S])$. ■

Corollary 2.14 *Assume that $m \in \mathbb{N}_+$, J_1, J_2 are distinct subsets of $\{1, \ldots, m+1\}$ and $X_1, X_2 \subseteq M^{m+1}$ are definable sets such that X_1 is J_1-open and X_2 is J_2-open. Then*

$$\dim(X_1 \cap X_2) < \max\{\dim(X_1), \dim(X_2)\}.$$

Proof. Assume that m, J_1, J_2, X_1, X_2 satisfy assumptions of the lemma. In case $\dim(X_1) \neq \dim(X_2)$ or $X_1 \cap X_2 = \emptyset$, the assertion of the lemma is trivial. So let $\dim(X_1) = \dim(X_2) = k$ and let $X_1 \cap X_2 \neq \emptyset$. Without loss of generality we can assume that $J_2 \setminus J_1 \neq \emptyset$. Below, for a fixed $i \in J_2 \setminus J_1$ we consider two cases.

Case 1. $\dim(\pi_i^{m+1}[X_2]) < k$. Then for every $\overline{a} \in \pi_i^{m+1}[X_1 \cap X_2]$, the fiber $(\pi_i^{m+1})^{-1}(\overline{a}) \cap X_1 \cap X_2$ is finite, and using condition $(\mathrm{f})_m$ from Theorem 2.11 we conclude that

$$\dim(X_1 \cap X_2) = \dim(\pi_i^{m+1}[X_1 \cap X_2]) \leq \dim(\pi_i^{m+1}[X_2]) < k.$$

Case 2. $\dim(\pi_i^{m+1}[X_2]) = k$. Suppose for a contradiction that $\dim(X_1 \cap X_2) = k$ and let $Z = \pi_i^{m+1}[X_1 \cap X_2]$. Again, by condition $(\mathrm{f})_m$ from Theorem 2.11, $\dim(Z) = k$. There is a unique definable set $X \subseteq X_1 \cap X_2$ such that $\pi_i^{m+1}[X] = Z$ and for every $\overline{a} \in Z$, we have that

$$\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap X] = \{\min \varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap X_1 \cap X_2]\}.$$

Clearly, $\dim(X) = k$. Moreover, there is a unique definable set $S \subseteq X_2$ such that $\pi_i^{m+1}[S] = Z$ and for every $\overline{a} \in Z$, $\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap S]$ is the convex component of $\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap X_2]$ containing $\varrho_i^{m+1}[(\pi_i^{m+1})^{-1}(\overline{a}) \cap X]$. Now, condition $(\mathrm{g})_m$ from Theorem 2.11 implies that $\dim(X_2) = k + 1$, a contradiction. ■

3 Large subsets of Cartesian products

This section has been motivated in two ways. Firstly, if $\mathcal{M}$ is an o-minimal structure, X, Y are non-empty definable subsets of M^m, M^n respectively, and S is a large subset of $X \times Y$, then the set of all $\overline{a}$'s from X for which the fiber $S_{\overline{a}}$ is large in Y, is large in X. Of course, such a statement fails in general weakly o-minimal structures. Nevertheless, I was eager to know if there exists a number k such that for "almost all" tuples $\langle \overline{a}_1, \ldots, \overline{a}_k \rangle \in X^k$, the union $S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_k}$ is large in Y. Another motivation comes from the area of groups and fields definable in the o-minimal context. One could easily rewrite all proofs from [Pi] so that the use of generic types is replaced by the addition property of the dimension. Such proofs works for weakly o-minimal structures with the addition property (for the definition see §4). A natural question to ask was how important is the assumption of the addition property. It turned out that a weaker version of topologization of groups and fields in weakly o-minimal context is possible modulo some technical fact concerning large definable subsets of cartesian products of definable sets. The aim of this section is to provide a proof of that technical fact. Below in a series of lemmas we will show that if $X \subseteq M^m$, $Y \subseteq M^n$ and $S \subseteq X \times Y$ are definable sets and S is large in $X \times Y$, then we can find $\overline{a}_1, \ldots, \overline{a}_k \in X$, where $k = 2^{\dim(Y)}$ such that the set $S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_k}$ is large in Y.

Lemma 3.1 *Let $m, n \in \mathbb{N}_+$ and assume that $X \subseteq M^m$, $Y \subseteq M^n$ and $S \subseteq X \times Y$ are non-empty definable sets. The following conditions are equivalent.*

(a) S is large in $X \times Y$.

(b) For any definable sets $U \subseteq X$ and $V \subseteq Y$ with $\dim(U) = \dim(X)$ and $\dim(V) = \dim(Y)$, we have that $(U \times V) \cap S \neq \emptyset$.

Proof. (a)$\Longrightarrow$(b). If there are definable sets $U \subseteq X$ and $V \subseteq Y$ such that $\dim(U) = \dim(X)$, $\dim(V) = \dim(Y)$ and $(U \times V) \cap S = \emptyset$, then $\dim((X \times Y) \setminus S) = \dim(U \times V) = \dim(X \times Y)$, which means that S is not large in $X \times Y$.

(b)$\Longrightarrow$(a). Assume that S is not large in $X \times Y$ and let $W = (X \times Y) \setminus S$. By condition (b)$_{m+n}$ from Theorem 2.11, there are open boxes $B \subseteq M^m$ and $C \subseteq M^n$ such that $(B \times C) \cap W = (B \times C) \cap (X \times Y) = (B \cap X) \times (C \cap Y)$ and $\dim((B \times C) \cap W) = \dim(X \times Y)$. Clearly, $\dim(B \cap X) = \dim(X)$, $\dim(C \cap Y) = \dim(Y)$ and $((B \cap X) \times (C \cap Y)) \cap S = \emptyset$. ■

Lemma 3.2 *Let $m, n \in \mathbb{N}_+$ and assume that $X \subseteq M^m$, $Y \subseteq M^n$ and $S \subseteq X \times Y$ are non-empty definable sets.*

(a) If the set $\{\overline{a} \in X : S_{\overline{a}}$ is large in $Y\}$ is large in X, then S is large in $X \times Y$.

(b) If the set $\{\overline{b} \in Y : S^{\overline{b}}$ is large in $X\}$ is large in Y, then S is large in $X \times Y$.

Proof. As both cases are similar, we only prove (a). Suppose that S is not large in $X \times Y$. Then by Lemma 3.1, there are definable sets $U \subseteq X$ and $V \subseteq Y$ such that $\dim(U) = \dim(X)$, $\dim(V) = \dim(Y)$ and $(U \times V) \cap S = \emptyset$. Hence $\dim(\{\overline{a} \in X : S_{\overline{a}}$ is not large in $Y\} = \dim(X)$. ■

Lemma 3.3 *Let $m \in \mathbb{N}_+$ and assume that $X \subseteq M^m, U \subseteq M, S \subseteq X \times U$ are non-empty definable sets, U is open and S is large in $X \times U$. Then the set $X' := \{\langle \overline{a}, \overline{b} \rangle \in X \times X : S_{\overline{a}} \cup S_{\overline{b}}$ is large in $U\}$ is large in $X \times X$.*

Proof. The assertion of the lemma is obvious for X finite. So assume that $\dim(X) = k \geq 1$ and suppose for a contradiction that X' is not large in $X \times X$. Then by Lemma 3.1, there are definable sets $X_1, X_2 \subseteq X$ such that $\dim(X_1) = \dim(X_2) = k$ and $(\forall \overline{a} \in X_1)(\forall \overline{b} \in X_2)(S_{\overline{a}} \cup S_{\overline{b}}$ is not large in $U)$. Let $Y_1 = \bigcup\limits_{\overline{a} \in X_1} \{\overline{a}\} \times \mathrm{int}(U \setminus S_{\overline{a}})$ and $Y_2 = \bigcup\limits_{\overline{b} \in X_2} \{\overline{b}\} \times \mathrm{int}(U \setminus S_{\overline{b}})$. Clearly, $(Y_1 \cup Y_2) \cap S = \emptyset$ and

$$(\forall \overline{a} \in X_1)(\forall \overline{b} \in X_2)((Y_1)_{\overline{a}} \cap (Y_2)_{\overline{b}} \text{ is a non-empty open subset of } U).$$

Below we consider two cases.

Case 1. There is $\overline{a} \in X_1$ such that

$$(*)\ \dim(\{\overline{b} \in X_2 : (Y_1)_{\overline{a}} \cap (Y_2)_{\overline{b}} \text{ is a union of some convex components of } (Y_1)_{\overline{a}}\}) = k.$$

Fix $\overline{a} \in X_1$ for which $(*)$ holds. There are a definable set $X_2' \subseteq X_2$ and an open interval $I \subseteq U$ such that $\dim(X_2') = k$ and $(\forall \overline{b} \in X_2')(I \subseteq (Y_1)_{\overline{a}} \cap (Y_2)_{\overline{b}})$. Consequently, $(X_2' \times I) \cap S = \emptyset$, which means that S is not large in $X \times U$.

Case 2. For every $\overline{a} \in X_1$, the set

$$\{\overline{b} \in X_2 : (Y_1)_{\overline{a}} \cap (Y_2)_{\overline{b}} \text{ is not a union of some convex components of } (Y_1)_{\overline{a}}\}$$

is large in X_2. By Lemma 3.2, the set

$$\{\langle \overline{a}, \overline{b} \rangle \in X_1 \times X_2 : (Y_1)_{\overline{a}} \cap (Y_2)_{\overline{b}} \text{ is not a union of some convex components of } (Y_1)_{\overline{a}}\}$$

is large in $X_1 \times X_2$. Hence, by Lemma 3.1, there are definable sets $X_1' \subseteq X_1$ and $X_2' \subseteq X_2$ such that $\dim(X_1') = \dim(X_2') = k$ and for any $\overline{a} \in X_1'$ and $\overline{b} \in X_2'$, $(Y_1)_{\overline{a}} \cap (Y_2)_{\overline{b}}$ is not a union of some convex components of $(Y_1)_{\overline{a}}$. Fix $\overline{b}_0 \in X_2'$ and for $\overline{a} \in X_1'$ denote by $A(\overline{a})$ the first convex component of $(Y_1)_{\overline{a}}$ which is not contained in $(Y_2)_{\overline{b}_0}$ and has a non-empty intersection with $(Y_2)_{\overline{b}_0}$. Denote also by $B(\overline{a})$ the first convex component of $A(\overline{a}) \cap (Y_2)_{\overline{b}_0}$. Let $P = \bigcup_{\overline{a} \in X_1'} \{\overline{a}\} \times A(\overline{a})$ and $R = \bigcup_{\overline{a} \in X_1'} \{\overline{a}\} \times B(\overline{a})$. Our construction guarantees that $\pi^{m+1}_{m+1}[P] = \pi^{m+1}_{m+1}[R] = X_1'$ and $\dim(P) = \dim(R) = k$. Moreover, for every $\overline{a} \in X_1'$, $P_{\overline{a}}$ is an infinite convex set and $R_{\overline{a}}$ is a non-empty proper subset of $P_{\overline{a}}$. By condition $(\mathrm{g})_m$ from Theorem 2.11, this is impossible. ■

Lemma 3.4 *Let $m, n \in \mathbb{N}_+$ and assume that $X \subseteq M^m, U \subseteq M^n, S \subseteq X \times U$ are non-empty definable sets, U is open, and S is large in $X \times U$. Then the set*

$$\{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle \in X^{2^n} : S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^n}} \textit{ is large in } U\}$$

is large in X^{2^n}.

Proof. We use induction on n. The case $n = 1$ is a consequence of Lemma 3.3, so suppose that the result holds for dimension n. Assume that $X \subseteq M^m$ is a nonempty definable set of dimension k, $U \subseteq M^{n+1}$ is a non-empty open definable set and $S \subseteq X \times U$ is a definable set, large in $X \times U$. We will show that the set

$$\{\langle \overline{a}_1, \ldots, \overline{a}_{2^{n+1}} \rangle \in X^{2^{n+1}} : S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^{n+1}}} \text{ is large in } U\}$$

is large in $X^{2^{n+1}}$. Let $Y = \bigcup_{\overline{a} \in X} \{\overline{a}\} \times \mathrm{int}(U \setminus S_{\overline{a}})$. We will be done if we prove that the set

$$\{\langle \overline{a}_1, \ldots, \overline{a}_{2^{n+1}} \rangle \in X^{2^{n+1}} : Y_{\overline{a}_1} \cap \ldots \cap Y_{\overline{a}_{2^{n+1}}} = \emptyset\}$$

is large in $X^{2^{n+1}}$.

Clearly, $\dim(Y) \leq k + n$. For every $\overline{a} \in X$, the fiber $Y_{\overline{a}}$ is an open (possibly empty) subset of U. This implies that for every $\overline{d} \in Y$, $\varrho^{m+n+1}_{m+1}[(\pi^{m+n+1}_{m+1})^{-1}(\pi^{m+n+1}_{m+1}(\overline{d})) \cap Y] \subseteq M$ is an open set containing $\varrho^{m+n+1}_{m+1}(\overline{d})$.

Claim 1. For every $b \in M$, $\dim((\varrho^{m+n+1}_{m+1})^{-1}(b) \cap Y) < k + n$.

Proof of Claim 1. Suppose for a contradiction that for some $b \in M$, $\dim((\varrho^{m+n+1}_{m+1})^{-1}(b) \cap Y) = k+n$. For $\langle \overline{a}, \overline{c} \rangle \in \pi^{m+n+1}_{m+1}[(\varrho^{m+n+1}_{m+1})^{-1}(b) \cap Y]$, where $|\overline{a}| = m$ and $|\overline{c}| = n$, let $f(\overline{a}, \overline{c}) \in \overline{M}^{\mathcal{M}} \cup \{+\infty\}$ be the supremum of the set

$$\{b_1 > b : (b, b_1) \subseteq \varrho^{m+n+1}_{m+1}[(\pi^{m+n+1}_{m+1})^{-1}(\overline{a}\overline{c}) \cap Y]\}.$$

By condition $(\mathrm{d})_{m+n}$ from Theorem 2.11, there are a definable set $X_1 \subseteq \pi^{m+n+1}_{m+1}[(\varrho^{m+n+1}_{m+1})^{-1}(b) \cap Y]$ and an open interval $I \subseteq M$ such that $\dim(X_1) = k + n$ and

$$X_1 \times I \subseteq \{\langle \overline{a}, \overline{c}, b_1 \rangle : \langle \overline{a}, \overline{c} \rangle \in \pi^{m+n+1}_{m+1}[(\varrho^{m+n+1}_{m+1})^{-1}(b) \cap Y] \text{ and } b < b_1 < f(\overline{a}, \overline{c})\}.$$

This means that $\dim(Y) = k + n + 1$, a contradiction.

Claim 1 implies that for every $b \in M$, the set $Y(b) := \pi^{m+n+1}_{m+1}[(\varrho^{m+n+1}_{m+1})^{-1}(b) \cap Y] \subseteq X \times M^n$ has dimension lower than $k + n$. Moreover, for any $\overline{a} \in X$ and $b \in M$, $Y(b)_{\overline{a}}$ is an open subset of M^n. By the inductive hypothesis, the set

$$V(b) := \{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle \in X^{2^n} : Y(b)_{\overline{a}_1} \cap \ldots \cap Y(b)_{\overline{a}_{2^n}} = \emptyset\}$$

is large in X^{2^n}. Consequently, by Lemma 3.2, the set $Z := \bigcup_{b \in M} V(b) \times \{b\} \subseteq X^{2^n} \times M$ is large in $X^{2^n} \times M$.

Claim 2. For any $\overline{a}_1, \ldots, \overline{a}_{2^n} \in X$, the fiber $((X^{2^n} \times M) \setminus Z)_{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle} \subseteq M$ is open.

Proof of Claim 2. Let $\overline{a}_1, \ldots, \overline{a}_{2^n} \in X$. By our choice of Y, the set $Y_{\overline{a}_1} \cap \ldots \cap Y_{\overline{a}_{2^n}} \subseteq M^{m+1}$ is open. Moreover, for every $b \in M^{m+1}$ we have that

$$\begin{array}{rl} b \in ((X^{2^n} \times M) \setminus Z)_{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle} & \iff \\ \langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle \notin V(b) & \iff \\ Y(b)_{\overline{a}_1} \cap \ldots \cap Y(b)_{\overline{a}_{2^n}} \neq \emptyset & \iff \\ (\exists \overline{c} \in M^n)(\langle b, \overline{c} \rangle \in Y_{\overline{a}_1} \cap \ldots \cap Y_{\overline{a}_{2^n}}) & . \end{array}$$

Thus, if $b \in ((X^{2^n} \times M) \setminus Z)_{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle}$, then there are an open interval I containing b and an open box $B \subseteq M^n$ such that $I \times B \subseteq Y_{\overline{a}_1} \cap \ldots \cap Y_{\overline{a}_{2^n}}$, which implies that $I \subseteq ((X^{2^n} \times M) \setminus Z)_{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle}$.

By Lemma 3.3, the set

$$Z' := \{\langle \overline{a}_1, \ldots, \overline{a}_{2^{n+1}} \rangle \in X^{2^{n+1}} : Z_{\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle} \cup Z_{\langle \overline{a}_{2^n+1}, \ldots, \overline{a}_{2^{n+1}} \rangle} = M\}$$

is large in $X^{2^{n+1}}$. Clearly, if $\langle \overline{a}_1, \ldots, \overline{a}_{2^{n+1}} \rangle \in Z'$, then for every $b \in M$, $\langle \overline{a}_1, \ldots, \overline{a}_{2^n} \rangle \in V(b)$ or $\langle \overline{a}_{2^n+1}, \ldots, \overline{a}_{2^{n+1}} \rangle \in V(b)$. Consequently, $Y_{\overline{a}_1} \cap \ldots \cap Y_{\overline{a}_{2^{n+1}}} = \emptyset$. This finishes the proof. ■

The following lemma is obvious.

Lemma 3.5 *Assume that $m, n \in \mathbb{N}_+$, $X \subseteq M^m$ is a non-empty definable set, $Y \subseteq M^n$ is a non-empty finite set and $S \subseteq X \times Y$ is a definable set, large in $X \times Y$. Then the set $\{\overline{a} \in X : S_{\overline{a}} = Y\}$ is large in X.*

Theorem 3.6 *Let $m, n \in \mathbb{N}_+$ and assume that $X \subseteq M^m, Y \subseteq M^n, S \subseteq X \times Y$ are non-empty definable sets, S is large in $X \times Y$, and $\dim(Y) = k$. Then the set*

$$\{\langle \overline{a}_1, \ldots, \overline{a}_{2^k} \rangle \in X^{2^k} : S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^k}} \textit{ is large in } Y\}$$

is large in X^{2^k}.

Proof. We proceed inductively on n. For $n = 1$ the result easily follows from Lemmas 3.3 and 3.5. Suppose that it holds for dimension n and assume that $X \subseteq M^m$ and $Y \subseteq M^{n+1}$ are non-empty definable sets, $k = \dim(Y)$ and $S \subseteq X \times Y$ is a definable set, large in $X \times Y$. For $k = 0$ the assertion of the theorem holds by Lemma 3.5. In case $k = n + 1$, it is true by Lemma 3.4 and the fact that $\dim(Y \setminus \mathrm{int}(Y)) \leq n$. To complete the proof, assume that $k \in \{1, \ldots, n\}$. By Lemma 2.4, there are $J_0, \ldots, J_l$, distinct proper subsets of $\{1, \ldots, n+1\}$ and pairwise disjoint definable sets $Y_0, \ldots, Y_l$ such that $Y_0 \cup \ldots \cup Y_l = Y$ and for every $i \leq l$, the set Y_i is J_i-open. Clearly, without loss of generality we can assume that $\dim(Y_i) = k$ whenever $i \leq l$. Then for any $i \leq l$, S is large in $X \times Y_i$. Since the intersection of finitely many subsets of X^{2^k} all of which are large in X^{2^k} is large in X^{2^k}, without loss of generality we can assume that $Y = Y_i$ for some $i \leq l$. As Y is J_i-open, there is a projection $\pi : M^{n+1} \longrightarrow M^n$ such that for every $\overline{a} \in \pi[Y]$, the fiber $\pi^{-1}(\overline{a}) \cap Y$ is finite. By condition $(\mathrm{f})_m$ of Theorem 2.11, $\dim(Y) = \dim(\pi[Y])$. Consequently, $\dim(X \times Y) = \dim(X \times \pi[Y])$. For $\overline{a} \in M^m$ and $\overline{b} \in M^{n+1}$ define $\pi'(\overline{a}, \overline{b}) = \langle \overline{a}, \pi(\overline{b}) \rangle$. The set $S' := (X \times \pi[Y]) \setminus \pi'[(X \times Y) \setminus S]$ is large in $X \times \pi[Y]$ because

$$\dim(\pi'[(X \times Y) \setminus S]) \leq \dim((X \times Y) \setminus S) < \dim(X \times Y) = \dim(X \times \pi[Y]).$$

By the inductive assumption, the set

$$X' := \{\langle \overline{a}_1, \ldots, \overline{a}_{2^k} \rangle \in X^{2^k} : S'_{\overline{a}_1} \cup \ldots \cup S'_{\overline{a}_{2^k}} \text{ is large in } \pi[Y]\}$$

is large in X^{2^k}. Note that for any $\overline{a}_1, \ldots, \overline{a}_{2^k} \in X$, $\pi^{-1}[S'_{\overline{a}_1} \cup \ldots \cup S'_{\overline{a}_{2^k}}] \cap Y \subseteq S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^k}}$. Consequently, if $\langle \overline{a}_1, \ldots, \overline{a}_{2^k} \rangle \in X'$, then $S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_{2^k}}$ is large in Y. This finishes the proof. ■

A natural question that appears in mind after having completed the proof of Theorem 3.6 is whether one could replace the number 2^k by a smaller one. To be more precise, given a weakly o-minimal structure $\mathcal{M}$, define a function $f_{\mathcal{M}} : \mathbb{N} \longrightarrow \mathbb{N}$ as follows: $f_{\mathcal{M}}(k)$ is the smallest number l such that if $m, n \in \mathbb{N}_+$, $X \subseteq M^m, Y \subseteq M^n$ and $S \subseteq X \times Y$ are non-empty definable sets, S is large in $X \times Y$, and $\dim(Y) = k$, then the set $\{\langle \overline{a}_1, \ldots, \overline{a}_l \rangle \in X^l : S_{\overline{a}_1} \cup \ldots \cup S_{\overline{a}_l} \text{ is large in } Y\}$ is large in X^l. It is well known that if $\mathcal{M}$ is o-minimal, then $f_{\mathcal{M}}(k) = 1$. According to Theorem 3.6, $f_{\mathcal{M}}(k) \leq 2^k$ for any weakly o-minimal structure $\mathcal{M}$. Below we give an example of a weakly o-minimal structure $\mathcal{M}$ with $f_{\mathcal{M}}(k) \geq k + 1$ for $k \in \mathbb{N}$.

Example. Let $\mathcal{M} = (M, \leq, \ldots)$ be a weakly o-minimal structure in which there are: a convex open definable set $U \subseteq M$ and a definable function $f : U \longrightarrow M$ which is locally constant but not piecewise constant (see [MMS, Example 2.6.1]). For $k \geq 2$ and $1 \leq j < k$ define

$$X_{j,k} = \{\langle x_1, \ldots, x_k \rangle \in M^k : x_1 = f(x_{1+j})\}; S_k = M^k \setminus (X_{1,k} \cup \ldots \cup X_{k-1,k}). \tag{1}$$

It is easy to see that S_{k+1} is large in $f[M] \times M^k$ for $k \in \mathbb{N}_+$. Moreover, if $a_1, \ldots, a_k \in f[U]$, then the set $(S_{k+1})_{a_1} \cup \ldots \cup (S_{k+1})_{a_k}$ is not large in M^k.

4 The addition property and the exchange property

For a definable set $S \subseteq M^{m+n}$ and numbers $d \in \{-\infty, 0, 1, \ldots, n\}$ and $d' \in \{-\infty, 0, 1, \ldots, m\}$ we define

$$\begin{aligned} X(S, m, d) &= \{\overline{a} \in M^m : \dim(S_{\overline{a}}) = d\}, \\ Y(S, n, d') &= \{\overline{b} \in M^n : \dim(S^{\overline{b}}) = d'\}. \end{aligned}$$

Clearly, the sets $X(S, m, d)$ and $Y(S, n, d')$ are definable.

Definition 4.1 *We say that* dim *has the addition property in* $\mathcal{M}$ *iff one [equivalently: both] of the following conditions is [are] true.*

(a) If $m, n \in \mathbb{N}_+$, $S \subseteq M^{m+n}$ *is a definable set, and* $d \in \{-\infty, 0, 1, \ldots, n\}$, *then*

$$\dim \left(\bigcup_{\overline{a} \in X(S,m,d)} \{\overline{a}\} \times S_{\overline{a}} \right) = \dim(X(S, m, d)) + d.$$

(b) If $m, n \in \mathbb{N}_+$, $S \subseteq M^{m+n}$ *is a definable set, and* $d' \in \{-\infty, 0, 1, \ldots, m\}$, *then*

$$\dim \left(\bigcup_{\overline{a} \in Y(S,n,d')} S^{\overline{b}} \times \{\overline{b}\} \right) = \dim(Y(S, n, d')) + d'.$$

The following theorem relates the addition property of $\mathcal{M}$ to two statements concerning definable functions with values in $\overline{M}^{\mathcal{M}}$. The addition property (condition (d)) is also shown to be equivalent to a seemingly weaker statement (c). In fact the proof of equivalence of (c) and (d) does not depend on the assumption of weak o-minimality of $\mathcal{M}$. It goes through in every first order structure with a sufficiently good dimension function (for details we refer the reader to [vdD1], section 1).

Theorem 4.2 *The following conditions are equivalent.*

(a) For any open interval $I \subseteq M$ and any definable function $f : I \longrightarrow \overline{M}^{\mathcal{M}}$, there is an open interval $I' \subseteq I$ such that $f \restriction I'$ is continuous.

(b) If $m \in \mathbb{N}_+$, $B \subseteq M^m$ is an open box and $f : B \longrightarrow \overline{M}^{\mathcal{M}}$ is a definable function, then there is an open box $B' \subseteq B$ such that $f \restriction B'$ is continuous.

(c) If $m \in \mathbb{N}_+$, $i \in \{1, \ldots, m\}$ and $S \subseteq M^{m+1}$ is a non-empty definable set, then $\dim(S) = \dim(\pi_i^{m+1}[S])$ *iff the set of tuples $\overline{a} \in \pi_i^{m+1}[S]$ for which the fiber $(\pi_i^{m+1})^{-1}(\overline{a}) \cap S$ is finite, is large in $\pi_i^{m+1}[S]$.*

(d) dim *has the addition property in $\mathcal{M}$.*

Proof. (a)$\Longrightarrow$(b). Assume that (a) holds. In order to prove (b) we use induction on m. There is nothing to do in dimension 1, so fix a positive integer m, and suppose that for any open box $B \subseteq M^m$ and any definable function $f : B \longrightarrow \overline{M}^{\mathcal{M}}$, there is an open box $B' \subseteq B$ such that $f \restriction B'$ is continuous.

Let $C \subseteq M^m$ be an open box, $I \subseteq M$ an open interval and $f : C \times I \longrightarrow \overline{M}^{\mathcal{M}}$ a definable function. By Lemma 1.10, without loss of generality we can assume that for every $\overline{a} \in C$, the function $f(\overline{a}, y)$ is either constant or strictly monotone. For $b \in I$, denote by $S(b)$ the set of continuity points of $f(\overline{x}, b)$. By the inductive hypothesis, $S(b)$ is large in C whenever $b \in I$. By Lemma 3.2, the set $\bigcup\limits_{b \in I} S(b) \times \{b\}$ is large in $C \times I$, so it contains an open box $C_1 \times I_1$, where I_1 is an open interval. Clearly, for every $b \in I_1$, the function $f(\overline{x}, b)$ restricted to C_1 is continuous. By (a), for every $\overline{a} \in C_1$, the set of discontinuity points of $f(\overline{a}, y)$ restricted to I_1 is finite. For $\overline{a} \in C_1$, let

$$u(\overline{a}) = \min(\{b \in I_1 : b \text{ is a discontinuity point of } f(\overline{a}, y)\} \cup \{\sup I_1\}).$$

By Lemma 1.5, there are an open box $C_2 \subseteq C_1$ and an open interval $I_2 \subseteq I_1$ such that for every $\overline{a} \in C_2$, the function $f(\overline{a}, y)$ is continuous on I_2. Now, it is easy to check that $f \restriction C_2 \times I_2$ is continuous.

(b)$\Longrightarrow$(c) Assume that (b) holds. Clearly, we will be done if we show that for any $m \in \mathbb{N}_+$, the following condition $(*)_m$ is true.

$(*)_m$ If $S \subseteq M^{m+1}$ is a non-empty definable set and $\pi : M^{m+1} \longrightarrow M^m$ denotes the projection dropping the last coordinate, then $\dim(S) = \dim(\pi[S])$ iff the set $\{\overline{a} \in \pi[S] : \pi^{-1}(\overline{a}) \cap S \text{ is finite}\}$ is large in $\pi[S]$.

In order to prove $(*)_1$, consider a non-empty definable set $S \subseteq M^2$ and let $\pi : M^2 \longrightarrow M$ be the projection dropping the second coordinate. If $\dim(S) = 2$, then $\dim(\pi[S]) = 1$ and the set $\{a \in \pi[S] : \pi^{-1}(a) \cap S \text{ is finite}\}$ is not large in $\pi[S]$. If $\dim(S) = 0$, then $\dim(\pi[S]) = 0$ and $\pi^{-1}(a) \cap S$ is finite for every $a \in \pi[S]$. Thus in both cases the equivalence from $(*)_1$ holds. To complete the proof of $(*)_1$, assume that $\dim(S) = 1$. Then $\dim(\pi[S]) \in \{0, 1\}$.

Case 1. $\dim(\pi[S]) = 1$.

Suppose for a contradiction that the set $\{a \in \pi[S] : \pi^{-1}(a) \cap S \text{ is finite}\}$ is not large in $\pi[S]$. Then the set $\{a \in \pi[S] : \pi^{-1}(a) \cap S \text{ is infinite}\}$ contains an open interval I. For $a \in I$ define $f(a), g(a) \in \overline{M}^{\mathcal{M}} \cup \{-\infty, +\infty\}$ as infimum and supremum (respectively) of the first convex component of $\mathrm{int}(\{b \in M : \langle a, b\rangle \in S\})$. By (b), there is an open interval $J \subseteq I$ such that the functions f, g are both continuous on J. By Lemma 1.5, the set $(f, g)_J$ contains an open box, which means that $\dim(S) = 2$, a contradiction.

Case 2. $\dim(\pi[S]) = 0$, i.e. $\pi[S]$ is finite.

Since $\dim(S)=1$, there is $a\in\pi[S]$ such that $\pi^{-1}(a)\cap S$ is infinite, which means that the set $\{a\in\pi[S]:\pi^{-1}(a)\cap S \text{ is finite}\}$ is not large in $\pi[S]$. This finishes the proof of $(*)_1$.

Assume that $m>1$, $\pi:M^{m+1}\longrightarrow M^m$ is the projection dropping the last coordinate, $S\subseteq M^{m+1}$ is a non-empty definable set and suppose that $(*)_k$ holds if $1\le k<m$. For the left-to-right direction, assume that $\dim(S)=\dim(\pi_i^{m+1}[S])$.

Case 1'. $\dim(\pi_i^{m+1}[S])=m$.

Suppose for a contradiction that the set $\{\overline{a}\in\pi[S]:\pi^{-1}(\overline{a})\cap S \text{ is finite}\}$ is not large in $\pi[S]$. Then the set $\{\overline{a}\in\pi[S]:\pi^{-1}(\overline{a})\cap S \text{ is infinite}\}$ contains an open box B. For $\overline{a}\in B$ define $f(\overline{a}),g(\overline{a})\in\overline{M}^{\mathcal{M}}\cup\{-\infty,+\infty\}$ as infimum and supremum (respectively) of the first convex component of $\mathrm{int}(\{b\in M:\langle\overline{a},b\rangle\in S\})$. By (b), there is an open box $B'\subseteq B$ such that the functions f and g restricted to B' are both continuous. Hence (by Lemma 1.5), the set $(f,g)_{B'}$ contains an open box, which implies that $\dim(S)=m+1>\dim(\pi[S])$, a contradiction.

Case 2'. $\dim(\pi[S])<m$.

Denote by J the set of all j's from $\{1,\ldots,m\}$ for which $\dim(\pi_{j,m+1}^{m+1}[S])=\dim(\pi[S])=\dim(S)$. The assumption $\dim(\pi[S])<m$ guarantees that $J\neq\emptyset$. Let $j\in J$. By $(*)_{m-1}$, the set

$$X_j:=\{\overline{b}\in\pi_{j,m+1}^{m+1}[S]:(\pi_j^m)^{-1}(\overline{b})\cap\pi[S] \text{ is finite}\}$$

is large in $\pi_{j,m+1}^{m+1}[S]$. By Lemma 1.7(b), $\dim(\pi_j^{m+1}[S])=\dim(\pi_{j,m+1}^{m+1}[S])$. Again, by $(*)_{m-1}$, the set

$$X_j':=\{\overline{b}\in\pi_{j,m+1}^{m+1}[S]:(\pi_m^m)^{-1}(\overline{b})\cap\pi_j^{m+1}[S] \text{ is finite}\}$$

is large in $\pi_{j,m+1}^{m+1}[S]$. So $Y_j:=X_j\cap X_j'$ is also large in $\pi_{j,m+1}^{m+1}[S]$. Let

$$Z=\pi[S]\cap\bigcup_{j\in J}(\pi_j^m)^{-1}[Y_j].$$

The definition of Z guarantees that

$$(*)\ \dim(\pi_j^m[\pi[S]\setminus Z])<\dim(\pi_{j,m+1}^{m+1}[S]) \text{ for } j\in J,$$

and $\pi^{-1}(\overline{a})\cap S$ is finite whenever $\overline{a}\in Z$. We claim that Z is large in $\pi[S]$. Suppose not. Then

$$(**)\ \dim(\pi[S]\setminus Z)=\dim(\pi[S])=\dim(S).$$

Since $\dim(\pi[S])<m$, there is $j_0\in\{1,\ldots,m\}$ such that

$$(***)\ \dim(\pi_{j_0}^m[\pi[S]\setminus Z])=\dim(\pi[S]\setminus Z).$$

$(*)$, $(**)$ and $(***)$ imply that $j_0\in J$ and $\dim(\pi_{j_0,m+1}^{m+1}[S])>\dim(S)$, which is impossible.

For the right-to-left direction, assume that the set $\{\overline{a}\in\pi[S]:\pi^{-1}(\overline{a})\cap S \text{ is finite}\}$ is large in $\pi[S]$ and $\dim(S)=\dim(\pi[S])+1$. We consider three cases.

Case 1". $\dim(\pi[S])=m$.

In this situation, $\dim(S)=m+1$, i.e. S contains an open box B. This implies that the set $\{\overline{a}\in\pi[S]:\pi^{-1}(\overline{a})\cap S \text{ is finite}\}$ is not large in $\pi[S]$, a contradiction.

*Case 2". * $0 < \dim(\pi[S]) < m$.

Let $T = \{\overline{a} \in \pi[S] : \pi^{-1}(\overline{a}) \cap S$ is infinite$\}$. Since $\dim(S) \le m$, there are distinct $j_1, \ldots, j_k \in \{1, \ldots m\}$ such that $\dim(\pi^{m+1}_{j_1,\ldots,j_k}[S]) = \dim(S) = m+1-k$. Then

$$\dim(\pi^{m+1}_{j_1,\ldots,j_k,m+1}[S]) \ge \dim(\pi^{m+1}_{j_1,\ldots,j_k}[S]) - 1 = \dim(S) - 1 = \dim(\pi[S]) \ge \dim(\pi^{m+1}_{j_1,\ldots,j_k,m+1}[S]).$$

Consequently, $\dim(\pi^{m+1}_{j_1,\ldots,j_k,m+1}[S]) = \dim(\pi[S])$. Let

$$X = \{\overline{a} \in \pi^{m+1}_{j_1,\ldots,j_k,m+1}[S] : (\pi^{m}_{j_1,\ldots,j_k})^{-1}(\overline{a}) \cap \pi[S] \text{ is finite}\} \cap (\pi^{m+1}_{j_1,\ldots,j_k,m+1}[S] \setminus \pi^{m}_{j_1,\ldots,j_k}[T]).$$

Since

$$\dim(\pi^{m}_{j_1,\ldots,j_k}[T]) \le \dim(T) < \dim(\pi[S]),$$

by the inductive hypothesis and Fact 1.9, X is large in $\pi^{m+1}_{j_1,\ldots,j_k,m+1}[S]$. Note that $(\pi^{m+1-k}_{m+1-k})^{-1}(\overline{a}) \cap \pi^{m+1}_{j_1,\ldots,j_k}[S]$ is a finite subset of M^{m+1-k} whenever $\overline{a} \in X$. Again, by the inductive hypothesis, $\dim(\pi^{m+1}_{j_1,\ldots,j_k,m+1}[S]) = \dim(\pi^{m+1}_{j_1,\ldots,j_k}[S])$. Summing up, $\dim(\pi[S]) = \dim(S)$, a contradiction.

*Case 3". * $\dim(\pi[S]) = 0$.

The assumptions guarantee that $\pi[S]$ is finite and $\pi^{-1}(\overline{a}) \cap S$ is finite for all $\overline{a} \in \pi[S]$. Hence S is finite and $\dim(S) = 0$.

(c)$\Longrightarrow$(d) Suppose that (c) holds. Using induction on n, we will prove condition (a) from Definition 4.1. Let $S \subseteq M^{m+1}$ be a definable set. Applying (c) to the sets $\bigcup_{\overline{a} \in X(S,m,0)} \{\overline{a}\} \times S_{\overline{a}}$ and $\bigcup_{\overline{a} \in X(S,m,1)} \{\overline{a}\} \times S_{\overline{a}}$, we easily get

$$\begin{aligned} \dim\left(\bigcup_{\overline{a} \in X(S,m,0)} \{\overline{a}\} \times S_{\overline{a}}\right) &= \dim(X(S,m,0)) \text{ and} \\ \dim\left(\bigcup_{\overline{a} \in X(S,m,1)} \{\overline{a}\} \times S_{\overline{a}}\right) &= \dim(X(S,m,1)) + 1. \end{aligned}$$

Now, assume that S is a definable subset of $M^m \times M^{n+1}$, $d \in \{-\infty, 0, \ldots, n+1\}$, and suppose that condition (d) holds for definable subsets of $M^m \times M^n$. By (c), without loss of generality we can assume that $d \ge 1$. For

$$\langle i, j\rangle \in Q_d := (\{d\} \times \{-\infty, 0, \ldots, d-1\}) \cup (\{-\infty, 0, \ldots, d-1\} \times \{d-1\})$$

define

$$X_{i,j}(S,m,d) = \{\overline{a} \in M^m : \dim(S_{\overline{a}}) = d, \dim(X(S_{\overline{a}}, n, 0)) = i \text{ and } \dim(X(S_{\overline{a}}, n, 1)) = j\}.$$

It is clear that $X(S,m,d)$ is a disjoint union of the sets $X_{i,j}(S,m,d)$ as $\langle i, j\rangle$ ranges over Q_d. Let

$$\begin{aligned} Y^0_{i,j} &= \{\overline{a}\overline{b} \in (X_{i,j}(S,m,d) \times M^n) \cap \pi^{m+n+1}_{m+n+1}[S] : \dim(S_{\overline{a}\overline{b}}) = 0\} \text{ and} \\ Y^1_{i,j} &= \{\overline{a}\overline{b} \in (X_{i,j}(S,m,d) \times M^n) \cap \pi^{m+n+1}_{m+n+1}[S] : \dim(S_{\overline{a}\overline{b}}) = 1\}. \end{aligned}$$

One easily sees that $Y^0_{i,j} \subseteq X(S, m+n, 0)$ and $Y^1_{i,j} \subseteq X(S, m+n, 1)$ whenever $\langle i,j\rangle \in Q_d$. For every $\overline{a} \in X_{i,j}(S,m,d)$, $\dim((Y^0_{i,j})_{\overline{a}}) = i$ and $\dim((Y^1_{i,j})_{\overline{a}}) = j$. Using Fact 1.6(e), condition (c) and the inductive hypothesis, we obtain

$$\dim\left(\bigcup_{\overline{a}\in X_{i,j}(S,m,d)} \{\overline{a}\}\times S_{\overline{a}}\right) = \dim\left(\bigcup_{\overline{a}\overline{b}\in (X_{i,j}(S,m,d)\times M^n)\cap \pi^{m+n+1}_{m+n+1}[S]} \{\overline{a}\overline{b}\}\times S_{\overline{a}\overline{b}}\right) =$$
$$\max\left\{\dim\left(\bigcup_{\overline{a}\overline{b}\in Y^0_{i,j}} \{\overline{a}\overline{b}\}\times S_{\overline{a}\overline{b}}\right), \dim\left(\bigcup_{\overline{a}\overline{b}\in Y^1_{i,j}} \{\overline{a}\overline{b}\}\times S_{\overline{a}\overline{b}}\right)\right\} =$$
$$\max\{\dim(Y^0_{i,j}), \dim(Y^1_{i,j})+1\} = \max\{\dim(X_{i,j}(S,m,d))+i, \dim(X_{i,j}(S,m,d))+j+1\} =$$
$$\dim(X_{i,j}(S,m,d)) + \max(i, j+1) = \dim(X_{i,j}(S,m,d)) + d.$$

Now, by Fact 1.6(e),

$$\dim\left(\bigcup_{\overline{a}\in X(S,m,d)} \{\overline{a}\}\times S_{\overline{a}}\right) = \dim\left(\bigcup_{\langle i,j\rangle\in Q_d}\ \bigcup_{\overline{a}\in X_{i,j}(S,m,d)} \{\overline{a}\}\times S_{\overline{a}}\right) =$$
$$\max\left\{\dim\left(\bigcup_{\overline{a}\in X_{i,j}(S,m,d)} \{\overline{a}\}\times S_{\overline{a}}\right) : \langle i,j\rangle \in Q_d\right\} =$$
$$\max\{\dim(X_{i,j}(S,m,d)) : \langle i,j\rangle\in Q_d\} + d = \dim(X(S,m,d)) + d.$$

(d)$\Longrightarrow$(a) Assume that $I \subseteq M$ is an open interval and $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ is a definable function such that for every $a \in I$, f is not continuous at a. By the monotonicity theorem, there is an open interval $I' \subseteq I$ such that $f \restriction I'$ is strictly monotone. Suppose for instance that $f \restriction I'$ is strictly increasing. For $a \in I'$, define

$$f_1(a) = \sup_{x\in I'\cap(-\infty,a)} f(x) \text{ and } f_2(a) = \inf_{x\in I'\cap(a,+\infty)} f(x).$$

As I' is contained in the set of discontinuity points of f, we have that $f_1(a) < f_2(a)$ for $a \in I'$. Moreover, $(f_1(a), f_2(a)) \cap (f_1(b), f_2(b)) = \emptyset$ whenever a, b are distinct elements from I'. Let $S = \bigcup_{a\in I} \{a\} \times (f_1(a), f_2(a))$. Clearly, S witnesses the fact that the addition property of dim fails in $\mathcal{M}$. ■

Lemma 4.3 *(a) If for every $\mathcal{N} \succ \mathcal{M}$,* dcl *has the exchange property in $\mathcal{N}$, then* dim *has the addition property in $\mathcal{M}$.*

(b) If dim *has the addition property in $\mathcal{M}$, then* dcl *has the exchange property in $\mathcal{M}$.*

Proof. (a) Suppose that dim does not have the addition property in $\mathcal{M}$. By Theorem 4.2, there are an open interval $I \subseteq M$ and a definable function $f : I \longrightarrow \overline{M}^{\mathcal{M}}$ such that each element $a \in I$ is a discontinuity point of f. By Theorem 1.2, there is an open interval $I_1 \subseteq I$ such that $f \restriction I_1$ is strictly monotone. Suppose for example that $f \restriction I_1$ is strictly increasing. For $a \in I_1$ define $g(a) = \lim_{x\longrightarrow a^-} f(x)$ and $h(a) = \lim_{x\longrightarrow a^+} f(x)$. Our assumptions guarantee that $g(a) \le f(a) \le h(a)$ and $g(a) < h(a)$ for $a \in I_1$. Hence at least one of the sets

$$X_1 := \{a \in I_1 : g(a) < f(a)\},\ X_2 := \{a\in I_1 : f(a) < h(a)\}$$

contains an open interval I_2. Suppose for instance that $I_2 \subseteq X_1$ and define an $L(M)$-formula $\varphi(x,y)$ as follows:

$$\varphi(x,y) \equiv (x \in I_2) \wedge (g(x) < y < f(x)).$$

Fix $a \in I_2$ and $\mathcal{N} \succ \mathcal{M}$ such that $\varphi(a, N)$ is not contained in $\mathrm{dcl}(a)$. Clearly, dcl does not have the exchange property in $\mathcal{N}$.

(b) Suppose that dcl does not have the exchange property in $\mathcal{M}$. There are $A \subseteq M$ and $a, b \in M$ such that $a \in \mathrm{dcl}(Ab) \setminus \mathrm{dcl}(A)$ and $b \notin \mathrm{dcl}(Aa)$. There is a formula $\varphi(x, y) \in L(A)$ such that $\varphi(M, b) = \{a\}$ and $|\varphi(M, d)| \leq 1$ whenever $d \in M$. Let $X = \{c \in M : \varphi(c, M) \neq \emptyset\}$. Note that $a \in \mathrm{int}(X)$, because $a \notin \mathrm{dcl}(A)$. Denote by I convex component of $\mathrm{int}(X)$ containing a. Again, $b \in \mathrm{int}(\varphi(a, M))$, because $b \notin \mathrm{acl}(Aa)$. Assume that b belongs to the k-th convex component of $\mathrm{int}(\varphi(a, M))$. As $a \notin \mathrm{dcl}(A)$, there is an A-definable convex open set $I' \subseteq I$ containing a such that for every $c \in I'$, the set $\mathrm{int}(\varphi(c, M))$ has at least k convex components. There is a formula $\psi(x, y) \in L(A)$ such that $\psi(M) \subseteq I' \times M$ and for every $c \in I'$, $\psi(c, M)$ is the k-th convex component of $\mathrm{int}(\varphi(c, M))$. Now, for $c \in I'$ let $f(c) = \sup(\psi(c, M))$. Clearly, there is an open interval $I'' \subseteq I'$ on which f is strictly monotone. But then f is not continuous on any subinterval of I''. This finishes the proof. ■

References

[Ar] R. Arefiev, *On monotonicity for weakly o-minimal structures*, preprint.

[MMS] D. Macpherson, D. Marker, and C. Steinhorn, *Weakly o-minimal structures and real closed fields*, Trans. Amer. Math. Soc. 352 (2000), 5435-5483.

[Pi] A. Pillay, *On groups and fields definable in o-minimal structures*, J. Pure Appl. Algebra 53 (1988), 239-255.

[V] V. Verbovskiy, *Non-uniformly weakly o-minimal group.* Algebra and model theory, 3 (Erlogol, 2001), 136–145, 161, Novosibirsk State Tech. Univ., Novosibirsk, 2001.

[vdD1] L. van den Dries, *Dimension of definable sets, algebraic boundedness and henselian fields*, Ann. Pure Appl. Logic 45 (1989), 189-209.

[vdD2] L. van den Dries, **Tame Topology and o-minimal Structures**, London Mathematical Society Lecture Notes Series, vol. 248, Cambridge: Cambridge University Press 1998.

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