BAD GROUPS IN THE SENSE OF CHERLIN

OLIVIER FRÉCON

Abstract. There exists no bad group (in the sense of Gregory Cherlin), namely any simple group of Morley rank 3 is isomorphic to $\mathrm{PSL}_2(K)$ for an algebraically closed field K.

1. Introduction

Model theory is a branch of mathematical logic concerned with the study of classes of mathematical structures. There are numerous interactions between model theory and the other areas of mathematics, sometimes spectacular as Wilkie's article on the field of real numbers with the exponential function, Hrushovski's proofs of the geometric Mordell-Lang conjecture and of the Manin-Mumford conjecture, Sela's work connecting logic with geometric group theory, Pila's paper on the André-Oort conjecture, the manuscript on Berkovich spaces by Hrushovski and Loeser, or the theory of motivic integration of Cluckers, Loeser and others.

Model theoreticians study mathematical structures by considering first-order sentences and formulas. The *Morley rank* is a model-theoretical notion of abstract dimension. It generalizes the dimension of an algebraic variety (when the ground field is algebraically closed). There are other notions of abstract dimension, the importance of the Morley rank lies on *Morley's Categoricity Theorem* below, which "can be thought of as the beginning of modern model theory" (David Marker [11, p. 2]) and the following Baldwin and Zilber Theorems.

We remember that a *theory* is a set of first-order $\mathscr{L}$-sentences for a language $\mathscr{L}$, it is *complete* if for any sentence ϕ, either ϕ or $\neg\phi$ belongs to T, and a theory is *κ-categorical* for some cardinal κ if, up to isomorphism, it has exactly one model of cardinality κ (cf. [11, Chapters 1 and 2] for more details).

Fact 1.1. – *Let T be a complete theory in a countable language.*

- (Morley's Categoricity Theorem, [12]) *If T is κ-categorical for some uncountable κ, then T is κ-categorical for every uncountable κ.*
- (Baldwin, [2]) *If T is uncountably categorical, then it is* of finite Morley rank.
- (Zilber, [18]) *The theory of an* infinite simple group *of finite Morley rank is uncountably categorical.*

In this paper, we are concerned with *groups* of finite Morley rank. The main example of such a group is an algebraic group defined over an algebraically closed field in the field language (Zilber, [18]). In the late seventies, Gregory Cherlin [6, §6] and Boris Zilber [18] formulated independently the following algebraicity conjecture.

Date: July 1, 2016.

2000 *Mathematics Subject Classification.* 20F11, 03C45, 20A15.

Key words and phrases. Groups of finite Morley rank, Bad groups, Projective space.

Conjecture 1.2. – *(Cherlin-Zilber Conjecture* or *Algebraicity Conjecture)* An infinite simple group of finite Morley rank is algebraic over an algebraically closed field.

This is the main conjecture on groups of finite Morley rank, and it is still open. Most of studies on groups of finite Morley rank focus on this conjecture. Actually, the original Cherlin Conjecture concerned simple ω*-stable* groups, but the substantial litterature on the Algebraicity Conjecture treats only the finite Morley rank case.

The Algebraicity Conjecture has been proved for several important classes of groups including locally finite groups [17]. The main theorem on groups of finite Morley rank ensures that any simple group of finite Morley rank with an infinite abelian subgroup of exponent 2 satisfies the Cherlin-Zilber Conjecture [1].

However, in despite of numerous papers on the subject, the Cherlin-Zilber Conjecture is still open, even for groups of Morley rank 3. As a matter of fact, in [6], the Algebraicity Conjecture was formulated as a result from an analysis of simple groups of Morley rank 3. The main result of [6] can be summarized as follows, where a *bad group* is a nonsolvable group of Morley rank 3 containing no definable subgroup of Morley rank 2.

Fact 1.3. – (Cherlin, [6]) *Let G be an infinite simple group of Morley rank at most 3. Then G has Morley rank 3, and one of the following two assertions is satisfied:*

- *there is an algebraically closed field K such that $G \simeq \mathrm{PSL}_2(K)$,*
- *G is a bad group.*

Thus bad groups became a major obstacle to the Cherlin-Zilber Conjecture. These groups have been studied in [6, 14] and [15], whose results are summarized in Facts 2.3 and 2.4 respectively. Later, it was shown that no bad group is existentially closed [10] or linear [13]. However, these groups appeared very resistant, and very sparse other information was known on bad groups.

Furthermore, Nesin has shown in [15] that a bad group acts on a natural geometry, which is not very far from being a non-Desarguesian projective plane of Morley rank 2. However, Baldwin discovered non-Desarguesian projective planes of Morley rank 2 [3]. Thus, the question of the existence, or not, of a bad group was still fully open. In this paper, we show that bad groups do not exist.

Main Theorem 1.4. – *There is no bad group.*

Note other more general notions of bad groups have been introduced independently by Corredor [7] and by Borovik and Poizat [4], where a *bad group* is defined to be a nonsolvable connected group of finite Morley rank all of whose proper connected definable subgroups are nilpotent. Such a bad group has similar properties to original bad groups. Moreover, later Jaligot will introduce a more general notion of bad groups [9], and he will obtain similar results. However, we recall that, in this paper, a *bad group* is defined to be nonsolvable, of Morley rank 3, and containing no definable subgroup of Morley rank 2.

Our proof of Main Theorem goes as follows. First we note that it is sufficying to study *simple* bad groups since for any bad group G, the quotient group $G/Z(G)$ is a *simple* bad group by [14, §4, Introduction].

Then we fix a simple bad group G, and we introduce a notion of lines as cosets of Borel subgroups of G (Definition 3.1). In §3, we study their behavior, mainly in regards with conjugacy classes of elements of G.

In §4, we propose a definition of a plane (Definition 4.1). This section is dedicated to prove that G contains a plane (Theorem 4.14). This result is the key point of our demonstration. Roughly speaking, we show that for each nontrivial element g of G such that $g = [u, v]$ for $(u, v) \in G \times G$, the union of the preimages of g, by maps of the form $\mathrm{ad}_v : G \to G$ defined by $\mathrm{ad}_v(x) = [x, v]$, is almost a plane, and from this, we obtain a plane.

In last section §5, we try to show that our notions of lines and planes provide a structure of projective space over the group G. Indeed, such a structure would provide a division ring (see [8, p. 124, Theorem 7.15]), and probably it would be easy to conclude. However, a contradiction occurs along the way, and achieves our proof.

2. Background material

A thorough analysis of groups of finite Morley rank can be found in [5] and [1]. In this section we recall some definitions and known results.

2.1. **Axioms.** We present the rank axioms exactly as in [1, p. 23-24].

"We consider a group G equipped with additional structure. We suppose that G carries a rank function *in the sense of Borovik and Poizat, namely a function "*rk*" which assigns to each set S definable over G, its "dimension"* $\mathrm{rk}\,(S)$*, and which satisfies the following four axioms.*

Monotonicity *For any integer n,* $\mathrm{rk}\,(S) > n$ *if and only if S contains an infinite family of disjoint definable subsets S_i of rank n.*

Additivity *If $f : A \to B$ is definable and surjective, and if the fibers $f^{-1}(b)$ have constant rank r for $b \in B$, then* $\mathrm{rk}\,(A) = r + \mathrm{rk}\,(B)$.

Definability *For any uniformly definable family $\{S_b \ : \ b \in B\}$ of definable sets, and for any $n \in \mathbb{N}$, the set $\{b \in B \ : \ \mathrm{rk}\,(S_b) = n\}$ is also definable.*

Finite Bounds *For any uniformly definable family $\mathfrak{F}$ of finite subsets, the sizes of the sets in $\mathfrak{F}$ are bounded.*

The definable subsets over G are quotients by definable equivalence relations of definable subsets of G^n for some n. A family $\{S_b \ : \ b \in B\}$ is uniformly definable *if, first, B is itself definable, and second, the relation "$x \in S_y$" is definable."*

It is shown in [16] that the groups $(G, \cdot, \cdots)$ as above are precisely the groups of finite Morley rank, and that the function rk assigns to each definable set its Morley rank. In particular, in this paper, the Morley rank of a definable set S will be denoted by $\mathrm{rk}\,S$, as in [5] and [1].

2.2. **Morley degree.** A nonempty definable set A is said to have *Morley degree 1* if for any definable subset B of A, either $\mathrm{rk}\,B < \mathrm{rk}\,A$ or $\mathrm{rk}\,(A \setminus B) < \mathrm{rk}\,A$. The set A is said to have *Morley degree d* if A is the disjoint union of d definable sets of Morley degree 1 and Morley rank $\mathrm{rk}\,A$.

Fact 2.1. –

- [5, Lemmas 4.12 and 4.14] *Every nonempty definable set has a unique degree.*

- [5, Proposition 4.2] *Let X and Y be definable subsets of Morley degree d and d' respectively. Then $X \times Y$ has Morley degree dd'.*
- [6, §2.2] *A group of finite Morley rank has Morley degree 1 if and only if it is* connected*, namely it has no proper definable subgroup of finite index.*

Moreover, the following elementary result will be useful for us.

Fact 2.2. – *Let $f : E \to F$ be a definable map. If the set E has Morley degree 1 and $r = \mathrm{rk}\, f^{-1}(y)$ is constant for $y \in F$, then the Morley degree of F is 1.*

Proof – Let B be a definable subset of F of Morley rank $\mathrm{rk}\, F$. We show that $\mathrm{rk}\,(F \setminus B) < \mathrm{rk}\, F$. By the additivity axiom, we have $\mathrm{rk}\, E = r + \mathrm{rk}\, F$ and

$$\mathrm{rk}\, f^{-1}(B) = r + \mathrm{rk}\, B = r + \mathrm{rk}\, F = \mathrm{rk}\, E$$

Since E has Morley degree 1, we obtain $\mathrm{rk}\, f^{-1}(F \setminus B) = \mathrm{rk}\,(E \setminus f^{-1}(B)) < \mathrm{rk}\, E$, and by the additivity axiom again,

$$\mathrm{rk}\,(F \setminus B) = \mathrm{rk}\, f^{-1}(F \setminus B) - r < \mathrm{rk}\, E - r = \mathrm{rk}\, F$$

so F has Morley degree 1. □

2.3. **Bad groups.** Main properties of bad groups are summarized in the following facts, where a *Borel subgroup* of a bad group G is defined to be an infinite definable proper subgroup of G.

Fact 2.3. – ([6, §5.2] and [14]) *Let G be a simple bad group, and B be a Borel subgroup of G.*

(1) $B = C_G(b)$ for $b \in G \setminus \{1\}$,
(2) B is connected, abelian, self-normalizing and of Morley rank 1,
(3) $C_G(x)$ is a Borel subgroup for each nontrivial element x of G,
(4) if A is another Borel subgroup of G, then A is conjugate with B, and either $A = B$ or $A \cap B = \{1\}$,
(5) $G = \bigcup_{g \in G} B^g$,
(6) G has no involution.

Fact 2.4. – [15, Lemma 18] *Let A and B be two distinct Borel subgroups of a simple bad group G. Then $\mathrm{rk}\,(ABA) = 3$, $\mathrm{rk}\,(AB) = 2$, and AB has Morley degree 1.*

3. Lines

In this paper, G denotes a fixed simple bad group. We fix a Borel subgroup B of G and we denote by $\mathscr{B}$ the set of Borel subgroups of G.

In this section, we define a *line* of G, and we provide their basic properties. We note that, by conjugation of Borel subgroups (Fact 2.3 (4)), any Borel subgroup is a *line* in the following sense.

Definition 3.1. – *A* line *of G is a subset of the form uBv for two elements u and v of G.*

We denote by Λ the set of lines of G.

We note that, by Fact 2.3 (2), each line has Morley rank 1 and Morley degree 1.

3.1. **Basic properties.** The elementary properties below of lines are used throughout this paper. In particular, they show that any two distinct elements of G belong to a unique line (Lemma 3.4).

Lemma 3.2. – *Let uBv and rBs be two lines. Then $uBv = rBs$ if and only if $uB = rB$ and $Bv = Bs$.*

Proof – We may assume that $uBv = rBs$. Then we have

$$B = u^{-1}rBsv^{-1} = u^{-1}rsv^{-1}B^{sv^{-1}}$$

so $u^{-1}rsv^{-1} \in B^{sv^{-1}}$ and $B = B^{sv^{-1}}$. Now sv^{-1} belongs to B since B is self-normalizing by Fact 2.3. Hence we obtain $Bv = Bs$, and the equality $uB = rB$ follows from $uBv = rBs$. □

By the above lemma, the set Λ identifies with $(G/B)_l \times (G/B)_r$ where $(G/B)_l$ (resp. $(G/B)_r$) denotes the set of left cosets (resp. right cosets) of B in G. Then Λ is a definable set. Moreover, since G is connected of Morley rank 3 and B has Morley rank 1, the Morley rank of Λ is 4 and its Morley degree is 1.

Lemma 3.3. – *The set Λ is a uniformly definable family.*

Proof – We consider the set $A = \{(x, u, v) \in G \times G \times G \mid x = uv\}$ and the map $f : A \to G \times \Lambda$ defined by $f(x, u, v) = (x, (uB, Bv))$. Then f is definable, so its image $f(A)$ is definable too. But if $(x, (uB, Bv))$ belongs to $f(A)$, then there are $u' \in uB$ and $v' \in Bv$ such that $x = u'v'$, so x belongs to uBv (which identifies with (uB, Bv)). Moreover, for any $(x, (uB, Bv)) \in G \times \Lambda$ such that x belongs to uBv, there is $b \in B$ such that $x = ubv$, so (x, ub, v) belongs to A and $(x, (uB, Bv)) = f(x, ub, v) \in f(A)$. Hence $f(A)$ is the graph of the membership relation $\in$ of an element of G to a line of G, so $\in$ is definable, and Λ is a uniformly definable family. □

Lemma 3.4. – *Two distinct elements x and y of G lie in one and only one line $l(x, y)$.*

Proof – By Fact 2.3 (5), there exists $v \in G$ such that $y^{-1}x$ belongs to B^v. Then x and y lie in uBv for $u = yv^{-1}$.

Now, if rBs is a line containing x and y, then we find two elements b_1 and b_2 of B such that $x = rb_1s$ and $y = rb_2s$. Thus $y^{-1}x = s^{-1}b_2^{-1}b_1s$ is a nontrivial element of B^s. But $y^{-1}x$ belongs to B^v by the choice of v, hence we have $B^s = B^v$ (Fact 2.3 (4)). Since B is self-normalizing, sv^{-1} belongs to B and we obtain $Bs = Bv$, so there exists $b \in B$ such that $s = bv$. This implies that $u = yv^{-1} = (rb_2s)(s^{-1}b) = rb_2b$ belongs to rB, and $rBs = uBv$. □

Corollary 3.5. – *The map $l : \{(x, y) \in G \times G \mid x \neq y\} \to \Lambda$ is definable.*

Proof – We consider the set

$$\Gamma = \{((x, y), uBv) \in (G \times G) \times \Lambda \mid x \neq y,\ x \in uBv,\ y \in uBv\}$$

By Lemma 3.3, it is a definable subset of $\{(x, y) \in G \times G \mid x \neq y\} \times \Lambda$. But by Lemma 3.4, it is the graph of the map l, hence l is definable. □

Lemma 3.6. – *If $uBv = (uBv)^g$ for $uBv \in \Lambda \setminus \mathscr{B}$ and $g \in G$, then $g = 1$.*

Proof – We have $uBv = g^{-1}uBvg$, so $uB = g^{-1}uB$ and $Bv = Bvg$ by Lemma 3.2, and g belongs to the Borel subgroups $B^{u^{-1}}$ and B^v. If g is nontrivial, then $B^{u^{-1}} = B^v$ (Fact 2.3 (4)), and vu belongs to $N_G(B) = B$. Consequently u belongs to $v^{-1}B$, and we obtain $uBv = B^v$, contradicting $uBv \notin \mathscr{B}$. Thus $g = 1$. □

3.2. **Sets of lines.** By Lemma 3.3, the set Λ is uniformly definable. In this section, we are interested by its definable subsets.

Definition 3.7. – *For each $g \in G$ and each definable subset X of G, we consider the following subsets of Λ:*

$$\mathscr{L}(g, X) = \{l(g, x) \in \Lambda \mid x \in X \setminus \{g\}\}$$
$$\Lambda_X = \{\lambda \in \Lambda \mid \lambda \cap X \text{ is infinite}\}$$

Lemma 3.8. – *Let X be a definable subset of G. Then the sets Λ_X and $\mathscr{L}(g, X)$ are definable subsets of Λ for each $g \in G$.*

Proof – The set $\mathscr{L}(g, X)$ is definable since it is the image of the definable set $\{g\} \times (X \setminus \{g\})$ by the map l, which is definable by Corollary 3.5.

Now we consider the definable map l_X from $\{(x, y) \in X \times X \mid x \neq y\}$ to Λ defined by $l_X(x, y) = l(x, y)$. We show that $\Lambda_X = \{\lambda \in \Lambda \mid \mathrm{rk}\,(l_X^{-1}(\lambda)) = 2\}$. We note that $l_X^{-1}(\lambda)$ is contained in $(\lambda \cap X) \times (\lambda \cap X)$. Thus, if λ does not belong to Λ_X, then $\lambda \cap X$ is finite, and $l_X^{-1}(\lambda)$ has not Morley rank 2. If λ belongs to Λ_X, then $\lambda \cap X$ is infinite, and for each pair (x, y) of elements of $\lambda \cap X$ with $x \neq y$, we have $l_X(x, y) = \lambda$. This implies that $l_X^{-1}(\lambda)$ is a generic subset of $(\lambda \cap X) \times (\lambda \cap X)$, with Morley rank 2. Consequently $\Lambda_X = \{\lambda \in \Lambda \mid \mathrm{rk}\,(l_X^{-1}(\lambda)) = 2\}$, and Λ_X is a definable subset of Λ. □

Lemma 3.9. – *Let $\lambda_1, \ldots, \lambda_n$ be n lines. Then $\lambda_1 \cup \cdots \cup \lambda_n$ is a definable set of Morley rank 1 and Morley degree n.*

Proof – For each i, the set $A_i = \lambda_i \cap (\bigcup_{j \neq i} \lambda_j)$ has at most $n-1$ elements by Lemma 3.4, and $\lambda_1 \cup \cdots \cup \lambda_n$ is the disjoint union of $\lambda_1 \setminus A_1, \ldots, \lambda_n \setminus A_n, A_1, \ldots, A_n$. Since each line λ_i has Morley rank 1 and Morley degree 1 (Fact 2.3 (2)), the result follows. □

Lemma 3.10. – *If Λ_0 is a definable subset of Λ, then $\bigcup \Lambda_0$ is a definable subset of G. Moreover, if Λ_0 is infinite, then $\bigcup \Lambda_0$ has Morley rank at least 2.*

Proof – Since Λ_0 is a definable subset of Λ, the set $\bigcup \Lambda_0 = \{x \in G \mid \exists \lambda \in \Lambda_0,\ x \in \lambda\}$ is definable by Lemma 3.3. Moreover, if Λ_0 is infinite, then $\bigcup \Lambda_0$ has Morley rank at least 2 by Lemma 3.9. □

Corollary 3.11. – *The subset $\bigcup \Lambda_X$ of G is definable for each definable subset X of G.*

4. Planes

Our aim is to find a definable structure of projective space on our bad group G. In this section, we introduce a notion of planes, and we show that G has such a plane (Theorem 4.14).

Definition 4.1. – *A definable subset X of G is said to be a* plane *if it has Morley rank 2 and satisfies $\overline{X} = X$ where*

$$\overline{X} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g, X)) = 1\}$$

Lemma 4.2. – *Let X be a definable set of Morley rank 2. For each $g \in G$, we have* $\mathrm{rk}\,(\mathscr{L}(g,X)) \in \{1,2\}$. *In particular,* $\overline{X} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g,X)) \leq 1\}$ *and* $G \setminus \overline{X} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g,X)) = 2\}$.

Proof – Let $g \in G$. We remember that, by Lemma 3.8, the set $\mathscr{L}(g,X)$ is definable. We consider the map $l_g : X \setminus \{g\} \to \mathscr{L}(g,X)$ defined by $l_g(x) = l(g,x)$. This map is definable and surjective, and since $\mathrm{rk}\,X = 2$, we have $\mathrm{rk}\,\mathscr{L}(g,X) \leq 2$.

Moreover, the set $\cup\mathscr{L}(g,X)$ is definable (Lemma 3.10), and it contains X. Since $\mathrm{rk}\,X = 2$, we obtain $\mathrm{rk}\,(\cup\mathscr{L}(g,X)) \geq 2$, and the set $\mathscr{L}(g,X)$ is infinite (Lemma 3.9), so $\mathrm{rk}\,\mathscr{L}(g,X) \geq 1$. □

Lemma 4.3. – *Let X be a definable subset of G of Morley rank 2. Then the set $\overline{X}$ is a definable subset of $\bigcup \Lambda_X$.*

Proof – If $g \in G$ does not belong to $\bigcup \Lambda_X$, then $l(g,x) \cap X$ is finite for each $x \in X$, and since X has Morley rank 2, the set $\mathscr{L}(g,X)$ has Morley rank 2, so $g \notin \overline{X}$. Thus $\overline{X}$ is contained in $\bigcup \Lambda_X$.

We show that $\overline{X}$ is definable. We consider the set

$$A = \{(g,\lambda) \in G \times \Lambda \mid g \in \lambda,\ \exists x \in X \setminus \{g\},\ x \in \lambda\}$$

and the map $f : A \to G$ defined by $f(g,\lambda) = g$. We note that A is definable by Lemma 3.3, so f is definable too. Moreover, the preimage by f of each $g \in G$ is $f^{-1}(g) = \{g\} \times \mathscr{L}(g,X)$, and we have $\mathrm{rk}\,(f^{-1}(g)) = \mathrm{rk}\,\mathscr{L}(g,X)$. Consequently, we obtain $\overline{X} = \{g \in G \mid \mathrm{rk}\,(f^{-1}(g)) = 1\}$, and $\overline{X}$ is definable. □

Lemma 4.4. – *Let X be a definable subset of G of Morley rank 2. Then* $\mathrm{rk}\,\Lambda_X$ *and* $\mathrm{rk}\bigcup \Lambda_X$ *are at most 2. Moreover, Λ_X is infinite if and only if* $\mathrm{rk}\bigcup\Lambda_X = 2$.

Proof – We consider the surjective definable map

$$l_0 : (X \times X) \cap l^{-1}(\Lambda_X) \to \Lambda_X$$

defined by $l_0(x,y) = l(x,y)$. For each $\lambda \in \Lambda_X$, we have $l_0^{-1}(\lambda) = \{(x,y) \in (\lambda \cap X) \times (\lambda \cap X) \mid x \neq y\}$, and since $\mathrm{rk}\,\lambda = 1$, we obtain $\mathrm{rk}\,(\lambda \cap X) = 1$ and $\mathrm{rk}\,l_0^{-1}(\lambda) = 2$. But we have

$$\mathrm{rk}\,((X \times X) \cap l^{-1}(\Lambda_X)) \leq \mathrm{rk}\,(X \times X) = 2\mathrm{rk}\,X = 4$$

hence $\mathrm{rk}\,\Lambda_X$ is at most $4 - 2 = 2$.

We show that $\mathrm{rk}\bigcup\Lambda_X \leq 2$. We consider the definable set

$$A = \{(x,\lambda) \in G \times \Lambda_X \mid x \in \lambda \setminus X\}$$

and the definable map $l_1 : A \to \Lambda_X$ defined by $l_1(x,\lambda) = \lambda$. For each $\lambda \in \Lambda_X$, we have $\mathrm{rk}\,\lambda = 1 = \mathrm{rk}\,(\lambda \cap X)$, so $l_1^{-1}(\lambda)$ is finite. Consequently we obtain $\mathrm{rk}\,A \leq \mathrm{rk}\,\Lambda_X \leq 2$. But the definable map $l_2 : A \to (\bigcup\Lambda_X) \setminus X$, defined by $l_2(x,\lambda) = x$, is surjective, hence the Morley rank of $(\bigcup\Lambda_X) \setminus X$ is at most $\mathrm{rk}\,A \leq 2$. Since X has Morley rank 2, we obtain $\mathrm{rk}\bigcup\Lambda_X \leq 2$.

Now it follows from Lemmas 3.9 and 3.10 that Λ_X is infinite if and only if $\mathrm{rk}\bigcup\Lambda_X = 2$. □

For each $a \in G$, let $\mathscr{L}(a) = \mathscr{L}(a,G)$ be the set of lines containing a. It is definable by Lemma 3.8. Moreover, we note that $\mathscr{L}(1) = \mathscr{B}$.

Lemma 4.5. – *Let Λ_0 be a definable subset of Λ. If* $\mathrm{rk} \cup \Lambda_0 = 2$*, then we have* $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) \leq 1$ *for each* $g \in G$*.*

Moreover, if further $\mathrm{rk}\,\Lambda_0 = 2$*, then the set* $\{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$ *has Morley rank 2.*

Proof – We recall that $\mathscr{L}(g) = \mathscr{L}(g, G)$ is definable by Lemma 3.8. We show that $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) \leq 1$ for each $g \in G$. Let $g \in G$ and $l_g : \cup(\mathscr{L}(g) \cap \Lambda_0) \setminus \{g\} \to \mathscr{L}(g) \cap \Lambda_0$ be the map defined by $l_g(x) = l(g, x)$. Since each line has Morley rank 1, the preimage of each element of $\mathscr{L}(g) \cap \Lambda_0$ has Morley rank 1. Consequently, we have

$$\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = \mathrm{rk} \cup (\mathscr{L}(g) \cap \Lambda_0) - 1 \leq \mathrm{rk} \cup \Lambda_0 - 1 = 1$$

as desired.

We suppose further that $\mathrm{rk}\,\Lambda_0 = 2$, and we show that $\{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$ has Morley rank 2. Let $U = \cup\Lambda_0$, $A = \{(u, \lambda) \in U \times \Lambda_0 \mid u \in \lambda\}$ and $f : A \to \Lambda_0$ be the map defined by $f(u, \lambda) = \lambda$. Then A and f are definable, and the preimage $f^{-1}(\lambda)$ of each $\lambda \in \Lambda_0$ has Morley rank $\mathrm{rk}\,\lambda = 1$, so $\mathrm{rk}\,A = 1 + \mathrm{rk}\,\Lambda_0 = 3$. Now let $h : A \to U$ be the map defined by $g(u, \lambda) = u$. It is a definable map, and the preimage $h^{-1}(u)$ of each $u \in U$ has Morley rank either 0, or 1 by the previous paragraph.

But the preimage of $U_0 = \{u \in U \mid \mathrm{rk}\,h^{-1}(u) = 0\}$ has Morley rank

$$\mathrm{rk}\,h^{-1}(U_0) = \mathrm{rk}\,U_0 \leq \mathrm{rk}\,U = 2 < \mathrm{rk}\,A$$

so the preimage of $U_1 = \{u \in U \mid \mathrm{rk}\,h^{-1}(u) = 1\}$ has Morley rank 3. Hence we obtain $\mathrm{rk}\,U_1 = 3 - 1 = 2$. Moreover, we note that

$$U_1 = \{u \in U \mid \mathrm{rk}\,(\mathscr{L}(u) \cap \Lambda_0) = 1\} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$$

so $\{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_0) = 1\}$ has Morley rank 2. □

Proposition 4.6. – *Let X be a definable subset of G of Morley rank 2. For each* $g \in \overline{X}$*, we have* $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = 1$*.*

Moreover, if X has Morley degree 1, then $\overline{X} = \{g \in G \mid \mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = 1\}$ *and* $G \setminus \overline{X} = \{g \in G \mid \mathscr{L}(g) \cap \Lambda_X \text{ is finite}\}$*.*

Proof – First we note that $\mathscr{L}(g) \cap \Lambda_X = \mathscr{L}(g, X) \cap \Lambda_X$ for any $g \in G$. For each $g \in G$, we consider the definable map $l_g : X \setminus \{g\} \to \mathscr{L}(g, X)$ defined by $l_g(x) = l(g, x)$. In particular, the preimage $l_g^{-1}(\lambda)$ of each $\lambda \in \mathscr{L}(g, X)$ is $(\lambda \cap X) \setminus \{g\}$.

We show that $\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) \leq 1$ for each $g \in G$. We may assume that Λ_X is infinite. Then, by Lemma 4.4, the set $\cup\Lambda_X$ has Morley rank 2, and by Lemma 4.5, we obtain $\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) \leq 1$ for each $g \in G$.

Let $g \in \overline{X}$. We show that $\mathrm{rk}\,(\mathscr{L}(g) \cap \Lambda_X) = 1$. For each $\lambda \in \mathscr{L}(g, X) \setminus \Lambda_X$, the set $l_g^{-1}(\lambda) = (\lambda \cap X) \setminus \{g\}$ is finite, and since $g \in \overline{X}$, we have $\mathrm{rk}\,\mathscr{L}(g, X) = 1$. Consequently, $l_g^{-1}(\mathscr{L}(g, X) \setminus \Lambda_X)$ has Morley rank at most 1, and $l_g^{-1}(\mathscr{L}(g, X) \cap \Lambda_X)$ has Morley rank $\mathrm{rk}\,X = 2$. But the set $l_g^{-1}(\lambda) = (\lambda \cap X) \setminus \{g\}$ is infinite of Morley rank 1 for each $\lambda \in \mathscr{L}(g, X) \cap \Lambda_X$. Hence we obtain $\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) = 2 - 1 = 1$.

Now we assume that X has Morley degree 1. Let $g \in G$ such that $\mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) = 1$. We show that $g \in \overline{X}$. Since the set $l_g^{-1}(\lambda) = (\lambda \cap X) \setminus \{g\}$ is infinite of Morley rank 1 for each $\lambda \in \mathscr{L}(g, X) \cap \Lambda_X$, the set $l_g^{-1}(\mathscr{L}(g, X) \cap \Lambda_X)$ has Morley rank

$$1 + \mathrm{rk}\,(\mathscr{L}(g, X) \cap \Lambda_X) = 2 = \mathrm{rk}\,X$$

Then, since X has Morley degree 1, the preimage of $\mathscr{L}(g,X)\setminus\Lambda_X$ has Morley rank at most 1. Moreover, for each $\lambda\in\mathscr{L}(g,X)\setminus\Lambda_X$, the preimage $l_g^{-1}(\lambda)=(\lambda\cap X)\setminus\{g\}$ is finite and non-empty, so we obtain

$$\operatorname{rk}(\mathscr{L}(g,X)\setminus\Lambda_X)=\operatorname{rk} l_g^{-1}(\mathscr{L}(g,X)\setminus\Lambda_X)\leq 1$$

This shows that $\operatorname{rk}\mathscr{L}(g,X)=1$ and $g\in\overline{X}$.

Furthermore, since $\operatorname{rk}(\mathscr{L}(g,X)\cap\Lambda_X)\leq 1$ for each $g\in G$, we obtain $G\setminus\overline{X}=\{g\in G \mid \mathscr{L}(g)\cap\Lambda_X \text{ is finite}\}$, as desired. □

Corollary 4.7. – *If X is a definable subset of G of Morley rank 2, then* $\operatorname{rk}(\overline{X}\setminus X)\leq 1$.

Proof – We remember that $\overline{X}$ is definable by Lemma 4.3, so the sets $Y=\overline{X}\setminus X$ and $A=\{(y,\lambda)\in Y\times\Lambda_X \mid y\in\lambda\}$ are definable too. Let $l_Y : A\to Y$ and $l_D : A\to\Lambda_X$ be the definable maps defined by $l_Y(y,\lambda)=y$ and $l_D(y,\lambda)=\lambda$ respectively. On the one hand, for each $\lambda\in\Lambda_X$, the set $\lambda\cap X$ is infinite, and since λ has Morley rank 1 and Morley degree 1 (Fact 2.3 (2)), the set $\lambda\cap Y$ is finite and $l_D^{-1}(\lambda)$ has Morley rank at most 0. This implies $\operatorname{rk} A\leq\operatorname{rk}\Lambda_X\leq 2$ (Lemma 4.4). On the other hand, for each $y\in Y$, we have $\operatorname{rk}(\mathscr{L}(y)\cap\Lambda_X)=1$ by Proposition 4.6, so $l_Y^{-1}(y)$ has Morley rank 1, and we obtain $\operatorname{rk} A=1+\operatorname{rk} Y$. Consequently, the Morley rank of Y is at most 1. □

Corollary 4.8. – *Let X be a definable subset of G of Morley rank 2. If the Morley degree of X is not 1, then* $\operatorname{rk}\overline{X}<2$. *In particular, any plane has Morley degree 1.*

Proof – Let n be the Morley degree of X, and $X_1,\ldots,X_n$ be n definable subsets of X of Morley rank 2 and Morley degree 1 such that X is the disjoint union of $X_1,\ldots,X_n$. For each $g\in\overline{X}$, we have $\operatorname{rk}\mathscr{L}(g,X)\leq 1$ (Lemma 4.2), so we obtain $\operatorname{rk}\mathscr{L}(g,X_i)\leq 1$ for each i, and $g\in\overline{X_i}$ for each i by Lemma 4.2 again. Thus $\overline{X}$ is contained in $\overline{X_1}\cap\overline{X_2}$. Since $X_1\cap X_2=\emptyset$, the set $\overline{X}$ is contained in $(X_1\cap Y_2)\cup(Y_1\cap X_2)\cup(Y_1\cap Y_2)$ where $Y_1=\overline{X_1}\setminus X_1$ and $Y_2=\overline{X_2}\setminus X_2$. Since Y_1 and Y_2 have Morley rank at most 1 by Corollary 4.7, we obtain $\operatorname{rk}\overline{X}<2$. □

Proposition 4.9. – *Let X be a definable subset of G of Morley rank 2 and Morley degree 1. Then* $\operatorname{rk}\overline{X}=2$ *if and only if Λ_X has Morley rank 2.*

In this case, Λ_X and $\overline{X}$ have Morley degree 1, and $\overline{X}$ contains a generic definable subset of X.

Proof – We consider the definable set $A=\{(x,\lambda)\in\overline{X}\times\Lambda_X \mid x\in\lambda\}$ and the definable maps $l_1 : A\to\overline{X}$ and $l_2 : A\to\Lambda_X$ defined by $l_1(x,\lambda)=x$ and $l_2(x,\lambda)=\lambda$ respectively. By Proposition 4.6, the preimage $l_1^{-1}(g)$ of each element g of $\overline{X}$ has Morley rank 1, so $\operatorname{rk} A=1+\operatorname{rk}\overline{X}$. Moreover, the preimage $l_2^{-1}(\lambda)$ of each $\lambda\in\Lambda_X$ has Morley rank at most 1, so $\operatorname{rk} A\leq 1+\operatorname{rk}\Lambda_X$. Then we obtain $\operatorname{rk}\overline{X}\leq\operatorname{rk}\Lambda_X$. In particular, it follows from Lemma 4.4 that if $\operatorname{rk}\overline{X}=2$, then $\operatorname{rk}\Lambda_X=2$. Hence we may assume that $\operatorname{rk}\Lambda_X=2$.

At this stage, Lemma 4.4 gives $\operatorname{rk}\Lambda_X=2$, and by Lemma 4.5 and Proposition 4.6, we obtain $\operatorname{rk}\overline{X}=2$. Moreover, it follows from Corollary 4.7 that $\overline{X}$ has Morley degree 1 and that $X\cap\overline{X}$ is a generic definable subset of $\overline{X}$.

We show that the Morley degree of Λ_X is 1. Let $l_0 : \{(x,y)\in X\times X \mid x\neq y\}\to\Lambda$ be the definable map defined by $l_0(x,y)=l(x,y)$. Since the Morley degree of X is 1, the one of $\{(x,y)\in X\times X \mid x\neq y\}$ is 1 too. For each $\lambda\in\Lambda_X$, we have

$\operatorname{rk} l_0^{-1}(\lambda) = \operatorname{rk}((\lambda \cap X) \times (\lambda \cap X)) = 2$. Since $\operatorname{rk} \Lambda_X = 2$, we obtain

$$\operatorname{rk} l_0^{-1}(\Lambda_X) = 2 + \operatorname{rk} \Lambda_X = 4 = \operatorname{rk}\{(x,y) \in X \times X \mid x \neq y\}$$

and since the Morley degree of $\{(x,y) \in X \times X \mid x \neq y\}$ is 1, the Morley degree of $l_0^{-1}(\Lambda_X)$ is 1 too. Now the Morley degree of Λ_X is 1 by Fact 2.2. □

Lemma 4.10. – *Let X and Y be two definable subsets of G of Morley rank 2 and Morley degree 1. If $X \cap Y$ has Morley rank 2, then $\overline{X} = \overline{Y}$.*

Proof – Let $g \in G$. If g belongs to $\overline{X \cap Y}$, then we have $\operatorname{rk} \mathscr{L}(g, X \cap Y) \leq 1$ (Lemma 4.2). Since X has Morley degree 1 and $X \cap Y$ has Morley rank 2, the set $X \setminus Y$ has Morley rank at most 1, and the set $\mathscr{L}(g, X \setminus Y)$ has Morley rank at most 1. Thus $\mathscr{L}(g, X)$ has Morley rank at most 1, and g belongs to $\overline{X}$ by Lemma 4.2 again.

Conversely, if $g \in \overline{X}$, then $\mathscr{L}(g, X)$ has Morley rank 1, so $\mathscr{L}(g, X \cap Y) \subseteq \mathscr{L}(g, X)$ has Morley rank at most 1. Then Lemma 4.2 gives $g \in \overline{X \cap Y}$. This shows that $\overline{X \cap Y} = \overline{X}$. By the same way, we obtain $\overline{X \cap Y} = \overline{Y}$, so $\overline{X} = \overline{Y}$. □

Lemma 4.11. – *Let g be a nontrivial element such that $g = [u, v]$ for $(u, v) \in G \times G$. Then we have $\{x \in G \mid [x, v] = g\} = C_G(v)u$ and $\{y \in G \mid [u, y] = g\} = C_G(u)v$. In particular, they are two lines and have Morley rank 1 and Morley degree 1.*

Proof – The equalities are obvious. Moreover, by Fact 2.3, the sets $C_G(v)u$ and $C_G(u)v$ are two lines, and they have Morley rank 1 and Morley degree 1. □

Lemma 4.12. – *For each $a \in G$, the set $a^G \cap B$ has exactly one element.*

Proof – We may assume $a \neq 1$. By Fact 2.3 (5), there is $g \in G$ such that a^g belongs to B. If $a^h \in B$ for $h \in G$, then a is a nontrivial element of $B^{g^{-1}} \cap B^{h^{-1}}$. By Fact 2.3 (4), we obtain $B^{g^{-1}} = B^{h^{-1}}$, and $h^{-1}g$ belongs to $N_G(B) = B$. But B is abelian (Fact 2.3 (2)), so $h^{-1}g$ centralizes a^h, and $a^h = (a^h)^{h^{-1}g} = a^g$. Hence $a^G \cap B = \{a^g\}$. □

The following result isolates a step of the proof of Theorem 4.14. The proof of second point was initially more complicated, and Bruno Poizat proposed a simplification.

Proposition 4.13. – *Let g be a nontrivial element of G. If the set $\{(x,y) \in G \times G \mid [x,y] = g\}$ is non-empty, then it has Morley rank 3 and the definable set $\{x \in G \mid \exists y \in G,\ [x,y] = g\}$ has Morley rank 2.*

Proof – We consider the definable map $\operatorname{ad} : G \times G \to G$ defined by $\operatorname{ad}(x,y) = [x,y]$.

1. $\operatorname{rk} \operatorname{ad}^{-1}(g) \geq 3$.

Since $\operatorname{ad}^{-1}(g)$ is non-empty, there is $(x,y) \in G \times G$ such that $g = [x,y]$. Since g is nontrivial, y is nontrivial too, and $C = C_G(y)$ is a Borel subgroup. By the same way, x is nontrivial, and since g is nontrivial, x does not belong to C and $1 \notin Cx$.

We show that the set $Y = \{dy \in G \mid \exists c \in C,\ d \in C_G(cx)\}$ has Morley rank at least 2. For each $c \in C$, we have $cx \in Cx$, so $cx \neq 1$ and $C_G(cx)$ is a Borel subgroup. In particular, there exists $u \in G$ such that $C_G(cx) = B^u$ (Fact 2.3 (4)), and $C_G(cx)y = u^{-1}Buy$ is a line of G. If we have $C_G(c_1x)y = C_G(c_2x)y$ for $c_1 \in C$ and $c_2 \in C$ such that $c_1 \neq c_2$, then $C_G(c_1x) = C_G(c_2x)$ is a line containing c_1x

and c_2x, and this line contains 1. But Cx is a line containing c_1x and c_2x, and Cx does not contain 1. Hence we have $Cx \neq C_G(c_1x)$, contradicting Lemma 3.4. This proves that $Y = \bigcup_{c\in C} C_G(cx)y$ is a union of infinitely many lines, and $\mathrm{rk}\, Y$ is at least 2 by Lemma 3.10.

For each $\alpha \in Y$, there is $c \in C$ such that $\alpha \in C_G(cx)y$, and we have

$$\mathrm{ad}(cx, \alpha) = [cx, \alpha] = [cx, y] = [x, y] = g$$

Thus, the infinite set $C_G(\alpha)cx \times \{\alpha\}$ is contained in $\mathrm{ad}^{-1}(g)$. This shows that the preimage of each element $\alpha \in Y$, by the definable map $f : \mathrm{ad}^{-1}(g) \to Y$ defined by $f(r, s) = s$, is infinite. Then we obtain $\mathrm{rk}\, \mathrm{ad}^{-1}(g) \geq 1 + \mathrm{rk}\, Y \geq 3$.

2. For each $g \in G \setminus \{1\}$, the definable set $X(g) = \{x \in G \mid \exists y \in G,\ [x, y] = g\}$ has not Morley rank 3.

We assume toward a contradiction that $X(g)$ has Morley rank 3. Let $U = \{z \in G \setminus \{1\} \mid \mathrm{rk}\, X(z) = 3\}$, and let $V = \{(x, y) \in G \times G \mid [x, y] \in U\}$. For each $z \in G \setminus \{1\}$ and each $x \in G$, we have $x \in X(z)$ if and only if $\exists y \in G$, $[x, y] = z$, so the family $\{X(z) \mid z \in G \setminus \{1\}\}$ is uniformly definable, and U and V are definable sets.

We recall that, by Fact 2.3, each nontrivial conjugacy class a^G of an element a of G has Morley rank $\mathrm{rk}\, a^G = \mathrm{rk}\, G - \mathrm{rk}\, C_G(a) = 2$

We consider the definable surjective map $f : V \to U$ defined by $f(x, y) = [x, y]$. For each $z \in U$, we have

$$f^{-1}(z) = \{(x, y) \in G \times G \mid x \in X(z),\ [x, y] = z\}$$

and by Lemma 4.11, this set has Morley rank $\mathrm{rk}\, f^{-1}(z) = \mathrm{rk}\, X(z) + 1 = 4$, so $\mathrm{rk}\, U = \mathrm{rk}\, V - 4$. But U is a normal set containing g, hence $\mathrm{rk}\, U \geq \mathrm{rk}\, g^G = 2$, and we obtain $\mathrm{rk}\, V = 6$ and $\mathrm{rk}\, U = 2$.

Moreover, since G is connected, $G \times G$ is connected too, and since V is a subset of $G \times G$ of Morley rank 6, the Morley degree of V is 1. Then, by Fact 2.2, the Morley degree of U is 1 too. Since U is normal in G, and since each nontrivial conjugacy class of G has Morley rank 2, the set U is a union of finitely many conjugacy classes of G, and since the Morley degree of U is 1, there exists $a \in G$ such that $U = a^G$.

Now, V is a generic definable subset of $G \times G$, so there is $(x, y) \in V$ such that (y, x) belongs to V. Thus $[x, y] \in a^G$ and its inverse $[y, x] \in a^G$ are conjugate, and they are equal by Lemma 4.12, contradicting Fact 2.3 (6).

3. Conclusion.

Let $p_X : \mathrm{ad}^{-1}(g) \to X(g)$ be the map defined by $p_X(x, y) = x$. Then p_X is a surjective definable map. By Lemma 4.11, we have $\mathrm{rk}\, p_X^{-1}(x) = 1$ for each $x \in X(g)$. Consequently, we obtain $\mathrm{rk}\, \mathrm{ad}^{-1}(g) = 1 + \mathrm{rk}\, X(g)$, and since $\mathrm{rk}\, X(g) \leq 2$ by 2, we find $\mathrm{rk}\, \mathrm{ad}^{-1}(g) \leq 3$. Thus we have $\mathrm{rk}\, \mathrm{ad}^{-1}(g) = 3$ by 1, so $1 + \mathrm{rk}\, X(g) = \mathrm{rk}\, \mathrm{ad}^{-1}(g) = 3$ and $\mathrm{rk}\, X(g) = 2$. This finishes the proof. □

Theorem 4.14. – *There is a plane in G.*

Proof – It is sufficing to show that there is a definable subset X of G satisfying the following properties:

(1) its Morley rank is 2 and its Morley degree is 1,
(2) Λ_X has Morley rank 2.

Indeed, by Proposition 4.9, for such a subset X, the set $\overline{X}$ has Morley rank 2 and Morley degree 1, and it contains a generic definable subset Y of X. At this stage, Lemma 4.10 shows that $\overline{X}$ is a plane.

We fix a nontrivial element g such that $g = [u, v]$ for $(u, v) \in G \times G$, and we consider the definable map $\mathrm{ad} : G \times G \to G$ defined by $\mathrm{ad}(x, y) = [x, y]$. Let $X(g) = \{x \in G \mid \exists y \in G,\ [x, y] = g\}$. By Proposition 4.13, it is a definable set of Morley rank 2. Let d be its Morley degree. Then $X(g)$ is the disjoint union of definable subsets $X_1, \dots, X_d$ of Morley rank 2 and Morley degree 1.

Let $Y(g) = \{y \in G \mid \exists x \in G,\ [x, y] = g\}$. Then $Y(g) = \{y \in G \mid \exists x \in G,\ [y, x] = g^{-1}\}$ has Morley rank 2 by Proposition 4.13. By Lemma 4.11, the set $l_y = \{x \in G \mid [x, y] = g\}$ is a line for each $y \in Y(g)$. We consider the following definable subset of Λ:

$$\mathscr{L}_X = \{\lambda \in \Lambda \mid \exists y \in G,\ \forall x \in \lambda,\ [x, y] = g\}$$

that is $\mathscr{L}_X = \{l_y \in \Lambda \mid y \in Y(g)\}$. We note that $\cup \mathscr{L}_X = X(g)$. Indeed, $\cup \mathscr{L}_X$ is contained in $X(g)$, and conversely, for each $x \in X(g)$ there is $y \in Y(g)$ such that $[x, y] = g$ so $x \in l_y \subseteq \cup \mathscr{L}_X$. In particular, $\mathscr{L}_X$ is contained in $\Lambda_{X(g)}$.

We show that $l_a \neq l_b$ for any distinct elements a and b of $Y(g)$. Indeed, suppose toward a contradiction that $l_a = l_b$ for two distinct elements a and b of $Y(g)$. We note that a and b are nontrivial because g is nontrivial. There exist $r \in G$ and $s \in G$ such that $[r, a] = [s, b] = g$, and we have $l_a = C_G(a)r$ and $l_b = C_G(b)s$ by Lemma 4.11. Since r belongs to $l_a = l_b = C_G(b)s$, we obtain $C_G(b)r = C_G(b)s = l_a = C_G(a)r$ and $C_G(b) = C_G(a)$. Moreover, since $r \in l_b$, we have $[r, b] = g = [r, a]$ and $r^b = r^a$, so ba^{-1} is a nontrivial element of $C_G(r)$. But $[r, a] = g$ is nontrivial, so r and a are nontrivial too, and $C_G(r)$ and $C_G(a)$ are Borel subgroups (Fact 2.3). Since they contain ba^{-1}, we find $C_G(r) = C_G(a)$ and $g = [r, a] = 1$, contradicting that g is nontrivial. Thus, for each $\lambda \in \mathscr{L}_X$, there exists a unique $y \in Y(g)$ such that $\lambda = l_y$.

We consider the definable set

$$\mathscr{A} = \{(\lambda, y) \in \mathscr{L}_X \times Y(g) \mid \forall x \in \lambda,\ [x, y] = g\}$$

that is $\mathscr{A} = \{(\lambda, y) \in \mathscr{L}_X \times Y(g) \mid \lambda = l_y\}$. By the previous paragraph, the projection map $p : \mathscr{A} \to \mathscr{L}_X$ defined by $p(\lambda, y) = \lambda$ is bijective, and since it is definable, we obtain $\mathrm{rk}\, \mathscr{A} = \mathrm{rk}\, \mathscr{L}_X$. Moreover, by Lemma 4.11, the definable map $q : \mathscr{A} \to Y(g)$ defined by $q(\lambda, y) = y$ is bijective too, so $\mathrm{rk}\, \mathscr{A} = \mathrm{rk}\, Y(g) = 2$. Hence the Morley rank of $\mathscr{L}_X$ is 2. Then, since $\mathscr{L}_X$ is contained in $\Lambda_{X(g)}$, the Morley rank of $\Lambda_{X(g)}$ is 2 by Lemma 4.4.

Now, for each element λ of $\Lambda_{X(g)}$, since $\lambda \cap X(g)$ is infinite and since λ has Morley rank 1 and Morley degree 1, there is a unique $i \in \{1, \dots, d\}$ such that $\lambda \cap X_i$ is infinite, that is $\lambda \in \Lambda_{X_i}$. Thus, each $\lambda \in \Lambda_{X(g)}$ belongs to a unique definable set Λ_{X_i} for $i \in \{1, \dots, d\}$. Hence there exists $i \in \{1, \dots, d\}$ such that $\mathrm{rk}\, \Lambda_{X_i} = 2$. Now the set X_i satisfies the conditions (1) and (2) of the beginning of our proof, so $\overline{X_i}$ is a plane. □

5. A projective space ?

In this section, we analyze planes. We remember that, by Theorem 4.14, the group G has a plane, and that by Corollary 4.8, any plane has Morley degree 1. We show that, if X and Y are two distinct planes, then $\Lambda_X \cap \Lambda_Y$ has a unique element (Proposition 6.1). However, along the way, we will obtain our final contradiction.

Definition 5.1. – *For each line λ, we consider the following subset of Λ:*

$$\mathscr{L}(\lambda) = \{m \in \Lambda \mid \lambda \cap m \text{ is not empty}\}$$

Lemma 5.2. – *For any line λ, the set $\mathscr{L}(\lambda)$ is definable, it has Morley rank 3 and Morley degree 1.*

Proof – We consider the definable map $f : \lambda \times (G \setminus \lambda) \to \mathscr{L}(\lambda) \setminus \{\lambda\}$ defined by $f(x, g) = l(x, g)$. By Lemma 3.4, for each $m \in \mathscr{L}(\lambda) \setminus \{\lambda\}$, there is a unique element x in $\lambda \cap m$. Moreover, for any $g \in G \setminus \lambda$, we have $f(x, g) = m$ if and only if $g \in m \setminus \{x\}$. Consequently we have $\operatorname{rk} f^{-1}(m) = \operatorname{rk} m = 1$, and

$$\operatorname{rk} \mathscr{L}(\lambda) = \operatorname{rk} (\lambda \times (G \setminus \lambda)) - 1 = 3$$

Furthermore, since λ and G have Morley degree 1, the Morley degree of $\lambda \times G$ and $\lambda \times (G \setminus \lambda)$ is 1, and the Morley degree of $\mathscr{L}(\lambda) \setminus \{\lambda\}$ and $\mathscr{L}(\lambda)$ is 1 too (Fact 2.2). □

Lemma 5.3. – *Let X be a plane, and $\lambda \in \Lambda_X$. Then $\mathscr{L}(\lambda) \cap \Lambda_X$ has Morley rank 2.*

Proof – Since λ belongs to Λ_X, the set $\lambda \cap X$ is infinite, and since λ is a line, we have $\operatorname{rk} (\lambda \cap X) = 1$. We consider the definable set

$$\mathscr{A} = \{(x, m) \in (\lambda \cap X) \times \Lambda_X \mid m \neq \lambda,\ x \in m\}$$

and the definable maps $p : \mathscr{A} \to \lambda \cap X$ and $q : \mathscr{A} \to \Lambda_X$ defined by $p(x, m) = x$ and $q(x, m) = m$ respectively. By Proposition 4.6, the set $p^{-1}(x)$ has Morley rank 1 for each $x \in \lambda \cap X$, so $\operatorname{rk} \mathscr{A} = 1 + \operatorname{rk} (\lambda \cap X) = 2$.

Moreover, each $m \in \Lambda_X \setminus \{\lambda\}$ contains at most one element of λ (Lemma 3.4), so q is an injective map and its image has Morley rank $\operatorname{rk} \mathscr{A} = 2$. But the image of q is contained in $(\mathscr{L}(\lambda) \cap \Lambda_X) \setminus \{\lambda\}$, and we have $\operatorname{rk} \Lambda_X \leq 2$ (Lemma 4.4), hence $\mathscr{L}(\lambda) \cap \Lambda_X$ has Morley rank 2. □

Lemma 5.4. – *Let λ_1 and λ_2 be two distinct lines. Then $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)$ has Morley rank 2 and Morley degree 1.*

Proof – Let $A = \{(x, y) \in \lambda_1 \times \lambda_2 \mid x \neq y\}$, and let $f : \lambda_1 \times \lambda_2 \to (\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)) \setminus \{\lambda_1, \lambda_2\}$ be the map defined by $f(x, y) = l(x, y)$. This map is definable by Corollary 3.5 and it is a bijection by Lemma 3.4. Since λ_1 and λ_2 are two lines, the set $\lambda_1 \times \lambda_2$ has Morley rank 2 and Morley degree 1, and since f is a definable bijection, $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)$ has Morley rank 2 and Morley degree 1. □

Proposition 5.5. – *If X and Y are two distinct planes, then $\Lambda_X \cap \Lambda_Y$ has at most one element.*

Proof – Suppose toward a contradiction that λ_1 and λ_2 are two distinct elements of $\Lambda_X \cap \Lambda_Y$. By Lemma 5.3, the sets $\mathscr{L}(\lambda_1) \cap \Lambda_X$ and $\mathscr{L}(\lambda_2) \cap \Lambda_X$ have Morley rank 2. But Λ_X has Morley rank 2 and Morley degree 1 by Proposition 4.9 and Corollary 4.8, hence $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2) \cap \Lambda_X$ has Morley rank 2. By the same way, $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2) \cap \Lambda_Y$ has Morley rank 2. Thus, since $\mathscr{L}(\lambda_1) \cap \mathscr{L}(\lambda_2)$ has Morley rank 2 and Morley degree 1 (Lemma 5.4), the set $\Lambda_X \cap \Lambda_Y$ has Morley rank 2.

Since $\Lambda_X \cap \Lambda_Y$ is infinite, the set $U = \cup(\Lambda_X \cap \Lambda_Y)$ has Morley rank at least 2 by Lemma 3.10, and since U is contained in $\cup\Lambda_X$, its Morley rank is exactly 2 (Lemma 4.4). Now the set $Z = \{g \in G \mid \operatorname{rk} (\mathscr{L}(g) \cap \Lambda_X \cap \Lambda_Y) = 1\}$ has Morley rank 2 by

Lemma 4.5. But Proposition 4.6 says that Z is contained in $X \cap Y$, hence $X \cap Y$ has Morley rank 2 and Lemma 4.10 gives $X = Y$, a contradiction. □

From now on, we try to show that the set $\Lambda_X \cap \Lambda_Y$ has exactly one element (cf. Proposition 6.1). However, the final contradiction will appear earlier.

Corollary 5.6. – *Let X be a plane and $(a, b) \in G \times G$. Then the following assertions are equivalent:*

- $aXb = X$
- $a\Lambda_X b = \Lambda_X$
- $aXb \cap X$ *has Morley rank 2.*

Proof – We note that aXb is a plane, and that $a\Lambda_X b = \Lambda_{aXb}$. If $aXb \cap X$ has Morley rank 2, then $aXb = X$ by Lemma 4.10, and if $aXb = X$, then we have $a\Lambda_X b = \Lambda_{aXb} = \Lambda_X$. Moreover, if $a\Lambda_X b = \Lambda_X$, then we have $\Lambda_{aXb} = \Lambda_X$ and $aXb = X$ by Proposition 5.5, so $aXb \cap X = X$ has Morley rank 2. □

By Fact 2.4, if A is a Borel subgroup distinct from B, then $\mathrm{rk}\,(ABA) = 3$. The following result is slightly more general, and its proof is different.

We recall that, if a group H of finite Morley rank acts definably on a set E, then the *stabilizer* of any definable subset F of E is defined to be

$$\mathrm{Stab}F = \{h \in H \mid \mathrm{rk}\,((h \cdot F)\Delta F) < \mathrm{rk}\,(F)\}$$

where Δ stands for the symmetric difference. It is a definable subgroup of H by [5, Lemma 5.11].

Lemma 5.7. – *Let A and C be two Borel subgroups distinct from B. Then* $\mathrm{rk}\,(ABC) = 3$.

Proof – We consider the action of G on itself by left multiplication. Then we have $b \cdot BC = BC$ for each $b \in B$, so B is contained in $\mathrm{Stab}(BC)$.

We assume toward a contradiction that C is contained in $\mathrm{Stab}(BC)$. Since BC has Morley rank 2 and Morley degree 1 (Fact 2.4), we have $\mathrm{rk}\,(cBC \setminus BC) \leq 1$ for each $c \in C$, and since $\mathrm{rk}\,C = 1$, we obtain $\mathrm{rk}\,(CBC \setminus BC) \leq 2$ and $\mathrm{rk}\,(CBC) = 2$, contradicting Fact 2.4. Consequently, C is not contained in $\mathrm{Stab}(BC)$, and since $\mathrm{Stab}(BC)$ contains B, Fact 2.3 implies that $\mathrm{Stab}(BC) = B$.

We assume toward a contradiction that $\mathrm{rk}\,(ABC) \neq 3$. Since $\mathrm{rk}\,(BC) = 2$, we have $\mathrm{rk}\,(ABC) = 2$ and ABC is a disjoint union of finitely many definable subsets $E_1, \ldots, E_k$ of Morley rank 2 and Morley degree 1. For each $a \in A$, the set aBC has Morley rank $\mathrm{rk}\,(BC) = 2$ and Morley degree 1, so there exists a unique $i \in \{1, \ldots, k\}$ such that $\mathrm{rk}\,(aBC \cap E_i) = 2$. Since A is infinite, there are $i \in \{1, \ldots, k\}$ and two distinct elements a and a' of A such that $\mathrm{rk}\,(aBC \cap E_i) = \mathrm{rk}\,(a'BC \cap E_i) = 2$. Since E_i has Morley degree 1, the Morley rank of $aBC \cap a'BC$ is 2, and we obtain $\mathrm{rk}\,(a'^{-1}aBC \cap BC) = 2$. But BC has Morley degree 1, hence $a'^{-1}a$ belongs to $\mathrm{Stab}(BC) = B$. Thus $a'^{-1}a$ belongs to $A \cap B = \{1\}$ (Fact 2.3 (4)), contradicting that a and a' are distinct. So we have $\mathrm{rk}\,(ABC) = 3$, as desired. □

Corollary 5.8. – *Let A and C be two distinct Borel subgroups. Then* $\mathrm{rk}\,(BA \cap BC) = 1$.

Proof – We may assume $A \neq B$ and $C \neq B$. By Fact 2.4, we have

$$1 = \mathrm{rk}\,B \leq \mathrm{rk}\,(BA \cap BC) \leq \mathrm{rk}\,(BA) = 2$$

We assume toward a contradiction that $\mathrm{rk}(BA \cap BC) = 2$. Since BC has Morley rank 2 and Morley degree 1 (Fact 2.4), the set $E = BC \setminus BA$ has Morley rank at most 1. Consequently, EA has Morley rank at most $\mathrm{rk}\, E + \mathrm{rk}\, A = 2$, and since $(BA \cap BC)A \subseteq BA$ has Morley rank 2, we obtain $\mathrm{rk}(BCA) = \mathrm{rk}(EA \cup (BA \cap BC)A) = 2$, contradicting that BCA has Morley rank 3 (Lemma 5.7). □

Lemma 5.9. – *For any plane X, we have $BX \neq X$ and $XB \neq X$.*

Proof – We assume toward a contradiction that $BX = X$ for a plane X. Let $x \in X$. Since X is a plane, Proposition 4.6 gives $\mathrm{rk}(\mathscr{L}(x,X) \cap \Lambda_X) = 1$, so $\mathscr{L}(x,X) \cap \Lambda_X$ is infinite. But each line containing x has the form $B^u x$ for $u \in G$, hence there exist $u \notin B$ and $v \notin B$ such that $B^u \neq B^v$, and such that $B^u x$ and $B^v x$ belong to $\mathscr{L}(x,X) \cap \Lambda_X$. In particular, there is a co-finite subset S of B such that $S^u x$ and $S^v x$ are contained in X.

Now, since $BX = X$, the sets $BS^u x$ and $BS^v x$ are contained in X. By Fact 2.4, the set $BB^{u^{-1}}$, and so $Bu^{-1}B$, has Morley rank 2, and since $Bu^{-1}(B \setminus S)$ is a finite union of lines, the set $Bu^{-1}(B \setminus S)$ has Morley rank 1 (Lemma 3.9), so $Bu^{-1}S$ has Morley rank 2. Thus, the sets $BS^u x = Bu^{-1}Sux$ and $BS^v x = Bv^{-1}Svx$ are subsets of X of Morley rank 2, and since the Morley degree of X is 1 (Corollary 4.8), the set $BS^u x \cap BS^v x$ has Morley rank 2. This implies that $\mathrm{rk}(BB^u \cap BB^v) = 2$, contradicting Corollary 5.8. Now we have $BX \neq X$ and by the same way, we show that $XB \neq X$. □

Corollary 5.10. – *For any plane X, the stabilizer of X for the action of G on itself by left multiplication is finite.*

Proof – By Corollary 5.6, we have $\mathrm{Stab} X = \{a \in G \mid aX = X\}$. If $\mathrm{Stab} X$ is infinite, then it contains a Borel subgroup, that is impossible (Lemma 5.9). □

Proposition 5.11. – *Let uBv be a line, and X and Y be two planes such that Λ_X and Λ_Y contain uBv. Then there exists $b \in B$ such that $Y = b^{u^{-1}} X$.*

Proof – We note that the set

$$\bigcup_{a \in B} \Lambda_{a^{u^{-1}}X} = \{\lambda \in \Lambda \mid (a^{u^{-1}})^{-1}\lambda \in \Lambda_X\}$$

is a definable subset of Λ, and that $uBv = a^{u^{-1}}(uBv) \in a^{u^{-1}}\Lambda_X = \Lambda_{a^{u^{-1}}X}$ for each $a \in B$.

We consider the definable map

$$f : B \times ((\mathscr{L}(uBv) \cap \Lambda_X) \setminus \{uBv\}) \to (\mathscr{L}(uBv) \cap (\bigcup_{a \in B} \Lambda_{a^{u^{-1}}X})) \setminus \{uBv\}$$

defined by $f(a,\lambda) = a^{u^{-1}}\lambda$.

We show that f is surjective and that its fibers are finite. For each $\lambda' \in (\mathscr{L}(uBv) \cap (\bigcup_{a \in B} \Lambda_{a^{u^{-1}}X})) \setminus \{uBv\}$, there is $a \in B$ such that λ' belongs to $\Lambda_{a^{u^{-1}}X}$, so we have $(a^{u^{-1}})^{-1}\lambda' \in \Lambda_X$ and $f(a, (a^{u^{-1}})^{-1}\lambda') = \lambda'$. Thus f is surjective.

Moreover, if $f(b,\lambda) = \lambda'$ for $b \in B$ and $\lambda \in \Lambda_X$, then we have $b^{u^{-1}}\lambda = \lambda'$, so the planes $a^{u^{-1}}X$ and $b^{u^{-1}}X$ contain an infinite subset of λ' and of uBv, and by Proposition 5.5, they are equal. But by Corollary 5.10, there are finitely many elements a' of B such that $a'^{u^{-1}}X = a^{u^{-1}}X$, and for such any element a', there is a unique line $\lambda'' = (a'^{-1})^{u^{-1}}\lambda'$ such that $f(a',\lambda'') = \lambda'$. Hence the preimage of λ' is finite.

Consequently, by using Lemma 5.3, we have

$$\mathrm{rk}\,(\mathscr{L}(uBv) \cap (\bigcup_{a\in B} \Lambda_{a^{u^{-1}}X})) = \mathrm{rk}\,(B \times (\mathscr{L}(uBv) \cap \Lambda_X)) = 1 + 2 = 3$$

By the same way, we have $\mathrm{rk}\,(\mathscr{L}(uBv) \cap (\bigcup_{a\in B} \Lambda_{a^{u^{-1}}Y})) = 3$. Now it follows from Lemma 5.2 that

$$\mathrm{rk}\,(\mathscr{L}(uBv) \cap (\bigcup_{a\in B} \Lambda_{a^{u^{-1}}X}) \cap (\bigcup_{a\in B} \Lambda_{a^{u^{-1}}Y})) = 3$$

Thus we find $a \in B$ and $a' \in B$ such that $\Lambda_{a^{u^{-1}}X} \cap \Lambda_{a'^{u^{-1}}Y}$ contains a line $rBs \neq uBv$, and since we have $uBv \in \Lambda_{a^{u^{-1}}X} \cap \Lambda_{a'^{u^{-1}}Y}$, Proposition 5.5 says that $a^{u^{-1}}X = a'^{u^{-1}}Y$. Hence we have $Y = b^{u^{-1}}X$ for $b = a'^{-1}a$. □

Lemma 5.12. – *Let X be a plane. Then the set $\{v \in G \mid \exists u \in G,\ uBv \in \Lambda_X\}$ is a generic definable subset of G.*

Proof – Let $V = \{v \in G \mid \exists u \in G,\ uBv \in \Lambda_X\}$. We remember that Λ identifies with $(G/B)_l \times (G/B)_r$ (Lemma 3.2). Let $f : \Lambda_X \to (G/B)_r$ be the definable map defined by $f(uBv) = Bv$. Then we have $V = h^{-1}(f(\Lambda_X))$ where $h : G \to (G/B)_r$ is defined by $h(x) = Bx$. Thus V is a definable set, and since the preimage $h^{-1}(Bx)$ of each $Bx \in (G/B)_r$ has Morley rank $\mathrm{rk}\,B = 1$, the Morley rank of V is $1 + \mathrm{rk}\,(f(\Lambda_X))$.

We assume toward a contradiction that f is not injective. Then there exist $Bv \in (G/B)_r$ and two distinct lines in Λ_X of the form uBv and rBv. For each $b \in B$, we have $(uBv)b^v = uBv$ and $(rBv)b^v = rBv$, so $uBv \in \Lambda_X b^v = \Lambda_{Xb^v}$ and $rBv \in \Lambda_{Xb^v}$. By Proposition 5.5, we obtain $Xb^v = X$, so $XB^v = X$, contradicting Lemma 5.9. Thus f is injective, and since $\mathrm{rk}\,\Lambda_X = 2$ (Proposition 4.9), we obtain $\mathrm{rk}\,(f(\Lambda_X)) = 2$ and $\mathrm{rk}\,V = 3$. □

Proposition 5.13. – *Let X be a plane. Then for each plane Y, there exists a unique $a \in G$ and a unique $b \in G$ such that $Y = aX = Xb$.*

Proof – We show that there exists $a \in G$ and $b \in G$ such that $Y = aX = Xb$. By Lemma 5.12, there exists $v \in G$, and $u \in G$ and $u' \in G$, such that $uBv \in \Lambda_X$ and $u'Bv \in \Lambda_Y$. Then we have $Bv \in \Lambda_{u^{-1}X}$ and $Bv \in \Lambda_{u'^{-1}Y}$, and Proposition 5.11 provides $b \in B$ such that $u'^{-1}Y = bu^{-1}X$. Hence we have $Y = aX$ for $a = u'bu^{-1}$. Thus, each plane has the form aX for $a \in G$, and by the same way, each plane has the form Xb for $b \in G$.

We show the uniqueness of a and b. Let $S = \{g \in G \mid gX = X\}$. It is a finite subgroup of G by Corollary 5.10. For each $\alpha \in G$, the previous paragraph gives $\beta \in G$ such that $\alpha X = X\beta$. Then, for each $s \in S$, we have $s(\alpha X) = s(X\beta) = X\beta = \alpha X$, and we obtain $s^\alpha X = X$ and $s^\alpha \in S$. Thus any element $\alpha \in G$ normalizes the finite subgroup S, and since G is a simple group, S is trivial. This proves the uniqueness of a, and by the same way we obtain the uniqueness of b. □

By the previous result, the set of planes is $\mathscr{P} = \{aX \mid a \in G\}$, and it identifies with G, so it is definable.

Corollary 5.14. – *The set $\mathscr{P}$ of planes is uniformly definable.*

Proof – The set $\Delta = \{(u, a) \in G \times G \mid a^{-1}u \in X\}$ is definable, and it is the graph of the membership relation of an element of G to a plane, where $\mathscr{P}$ identified with G. Thus, $\mathscr{P}$ is uniformly definable. □

From now on, we fix a plane X, and we consider the action of $G \times G$ on G defined by $(u,v)\cdot g = ugv^{-1}$. This action induces an action of $G \times G$ on $\mathscr{P}$ defined by $(u,v)\cdot aX = uaXv^{-1}$. Then for each plane aX, we denote by $\mathrm{Stab}(aX)$ the stabilizer of aX for this action of $G \times G$ on $\mathscr{P}$.

Remark 5.15. –

(1) It follows from Lemma 3.2 that the stabilizer of each line $uBv \in \Lambda$ is $\mathrm{Stab}(uBv) = B^{u^{-1}} \times B^v$.
(2) Furthermore, for each pair (x,y) of nontrivial elements of G, by Fact 2.3 there are a unique pair of Borel subgroups $(B^{u^{-1}}, B^v)$ such that $x \in B^{u^{-1}}$ and $y \in B^v$. Then uBv is the unique line stabilized by (x,y).

Lemma 5.16. – *There exists $a \in G$ such that $a^{-1}Xa \neq X$.*

Proof – We assume toward a contradiction that $a^{-1}Xa \neq X$ for each $a \in G$. Then for each $uBv \in \Lambda_X$ and each $a \in G$, we have

$$(uBv)^a \in \Lambda_X^a = \Lambda_{X^a} = \Lambda_X$$

Since $\mathrm{rk}\,\Lambda_X = 2$ (Proposition 4.9), the line uBv is a Borel subgroup (Lemma 3.6), and by conjugacy of Borel subgroups, we obtain $\Lambda_X = \mathscr{B}$. Now we have $\cup\Lambda_X = G$, so $\mathrm{rk} \cup \Lambda_X = 3$, contradicting Lemma 4.4. □

Lemma 5.17. – *The stabilizer* $\mathrm{Stab}X$ *is the graph of a definable automorphism μ of $G \times G$. In particular,* $\mathrm{Stab}X$ *and G are definably isomorphic.*

Moreover, for each $b \in G$, if ν is the definable automorphism of G whose graph is $\mathrm{Stab}(Xb)$*, then $\nu \circ \mu^{-1}(x) = x^b$ for each $x \in G$.*

Proof – Since the set $\mathscr{P}$ has Morley rank $\mathrm{rk}\,G = 3$, and $G \times G$ acts transitively on $\mathscr{P}$ (Proposition 5.13), the stabilizer $\mathrm{Stab}X$ has Morley rank $6-3=3$. By Proposition 5.13 that $\mathrm{Stab}X \cap (G \times \{1\})$ and $\mathrm{Stab}X \cap (\{1\} \times G)$ are trivial. Hence $\mathrm{Stab}X$ is the graph of a definable automorphism μ of G.

By the same way, $\mathrm{Stab}(Xb)$ is the graph of a definable automorphism ν of G. For each $(x,y) \in G \times G$, we have $(x,y) \in \mathrm{Stab}X$ if and only if $xXy^{-1} = X$, that is $xXb(b^{-1}y^{-1}b) = Xb$. In other words, (x,y) belongs to $\mathrm{Stab}X$ if and only if (x,y^b) belongs to $\mathrm{Stab}(Xb)$. This implies that, for each $x \in G$, we have $\nu(x) = \mu(x)^b$ so $(\nu \circ \mu^{-1})(x) = x^b$. □

Lemma 5.18. – *The Borel subgroups of* $\mathrm{Stab}X$ *are the stabilizers of the lines $\lambda \in \Lambda_X$, for the action of* $\mathrm{Stab}X$ *on X.*

Proof – Indeed, $\mathrm{Stab}X$ has Morley rank 3 (Lemma 5.17), and it acts on the definable set Λ_X of Morley rank 2 (Lemma 4.4 and Proposition 4.9). Hence for each line $\lambda \in \Lambda_X$, the stabilizer $\mathrm{Stab}X \cap \mathrm{Stab}(\lambda)$ is either a Borel subgroup of $\mathrm{Stab}X$, or $\mathrm{Stab}X$. Since $\mathrm{Stab}(\lambda)$ has Morley rank 2 (Remark 5.15 (1)), the stabilizer $\mathrm{Stab}X \cap \mathrm{Stab}(\lambda)$ is a Borel subgroup of $\mathrm{Stab}X$. Conversely, by conjugacy of the Borel subgroups of $\mathrm{Stab}X \simeq G$, each Borel subgroup of $\mathrm{Stab}X$ stabilizes a line of Λ_X. □

Now we are ready for the final contradiction of our paper.

Proof – Let $\mathscr{U} = \{Xx^{-1} \in \mathscr{P} \mid x \in X\}$ be the set of the planes containing 1. Since X has Morley rank 2, the set $\mathscr{U}$ has Morley rank 2.

Let $\Delta = \{(x,y) \in G \times G \mid x = y\}$. We note that Δ is definably isomorphic to G, and it has Morley rank 3. Moreover, the action of Δ on G is equivalent to the action by conjugation of G on itself. In particular, Δ stabilizes $\mathscr{U}$, and no element of $\mathscr{U}$ is stabilized by Δ (Lemma 5.16). Thus, since $\mathrm{rk}\,\mathscr{U} = 2$, the stabilizer $\Delta \cap \mathrm{Stab}Y$ of each $Y \in \mathscr{U}$ is a Borel subgroup of Δ.

We fix $Y \in \mathscr{U}$. Since $\Delta \cap \mathrm{Stab}Y$ is a Borel subgroup of $\Delta \simeq G$, we may assume that $\Delta \cap \mathrm{Stab}Y = \{(b, b') \in B \times B \mid b' = b\}$. Then, since $\mathrm{Stab}Y$ is definably isomorphic to G (Lemma 5.17), the set $\{(b, b') \in B \times B \mid b' = b\}$ is a Borel subgroup of $\mathrm{Stab}Y$, so $\{(b, b') \in B \times B \mid b' = b\}$ is the stabilizer of a line $\lambda \in \Lambda_Y$ for the action of $\mathrm{Stab}Y$ on Y (Lemma 5.18), and this line λ is B (Remark 5.15 (2)). By Proposition 4.6, we have

$$\mathrm{rk}\,(\mathscr{B} \cap \Lambda_Y) = \mathrm{rk}\,(\mathscr{L}(1) \cap \Lambda_Y) = 1$$

so there is $u \in G \setminus B$ such that B^u belongs to Λ_Y. By Lemma 5.18, the stabilizer $\mathrm{Stab}(B^u) \cap \mathrm{Stab}Y$ is a Borel subgroup of $\mathrm{Stab}Y$, and it is contained in $B^u \times B^u$ (Remark 5.15 (1)).

Moreover, since the previous paragraph shows that $\mathrm{Stab}B \cap \mathrm{Stab}Y = \{(b, b') \in B \times B \mid b' = b\}$, we have

$$\mathrm{Stab}(B^u) \cap \mathrm{Stab}(Y^u) = \{(c, c') \in B^u \times B^u \mid c' = c\}$$

By Lemma 5.17, the sets $\mathrm{Stab}Y$ and $\mathrm{Stab}(Y^u)$ are the graphs of two automorphisms μ and ν respectively. Then, by Proposition 5.13 and Lemma 5.17 again, there is $g \in G$ such that $\nu \circ \mu^{-1}(x) = x^g$ for each $x \in G$. But, by the previous paragraph, $\mu(x)$ belongs to B^u for each $x \in B^u$, and since $\mathrm{Stab}(B^u) \cap \mathrm{Stab}(Y^u) = \{(c, c') \in B^u \times B^u \mid c' = c\}$, we have $\nu(x) = x$ for each $x \in B^u$. Hence, for each $x \in B^u$, we have $\mu(x) = x^{g^{-1}}$, and since $\mu(x) \in B^u$, we obtain $\mu(x) = x$ by Lemma 4.12. Thus $\mathrm{Stab}Y$ contains $\{(c, c') \in B^u \times B^u \mid c' = c\} \subseteq \Delta$, contradicting $\Delta \cap \mathrm{Stab}Y = \{(b, b') \in B \times B \mid b' = b\}$. □

6. Annex

Actually, after Lemma 5.18 we were ready for a new step to provide a structure of projective space over G, which was the initial goal of our section.

Proposition 6.1. – *If $Y \neq X$ is a plane, then $\Lambda_X \cap \Lambda_Y$ has a unique element.*

Proof – By Lemma 5.17, the subgroup $\mathrm{Stab}X$ (resp. $\mathrm{Stab}Y$) of $G \times G$ is the graph of a definable automorphism μ (resp. ν) of G. By Proposition 5.13, there is $b \in G \setminus \{1\}$ such that $Y = Xb$. We may assume that b belongs to B. Then Lemma 5.17 shows that $(\nu \circ \mu^{-1})(x) = x^b$ for each $x \in G$, so μ and ν are equal on B.

Since μ is a definable automorphism of G, the set $H = \{(b, \mu(b)) \in G \times G \mid b \in B\}$ is a Borel subgroup of $\mathrm{Stab}X \simeq G$, and Fact 2.3 provides $v \in G$ such that $\mu(B) = B^v$. Then Bv is the unique line stabilized by H (Remark 5.15 (2)), and by Lemma 5.18, we have $Bv \in \Lambda_X$, But $\mu = \nu$ on B, hence Bv belongs to Λ_Y too. Now Bv is the unique element of $\Lambda_X \cap \Lambda_Y$ (Proposition 5.5). □

Acknowledgements

Je remercie Bruno Poizat pour sa lecture attentive de cet article, et sa simplification de la démonstration de la proposition 4.13.

References

[1] Tuna Altınel, Alexandre V. Borovik, and Gregory Cherlin. *Simple groups of finite Morley rank*, volume 145 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2008.
[2] John T. Baldwin. α_T is finite for $\aleph_1$-categorical T. *Trans. Amer. Math. Soc.*, 181:37–51, 1973.
[3] John T. Baldwin. An almost strongly minimal non-Desarguesian projective plane. *Trans. Amer. Math. Soc.*, 342(2):695–711, 1994.
[4] A. V. Borovik and B. P. Poizat. Simple groups of finite morley rank without nonnilpotent connected subgroups. preprint deposited at VINITI, 1990.
[5] Alexandre Borovik and Ali Nesin. *Groups of finite Morley rank*, volume 26 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1994. Oxford Science Publications.
[6] Gregory Cherlin. Groups of small Morley rank. *Ann. Math. Logic*, 17(1-2):1–28, 1979.
[7] Luis Jaime Corredor. Bad groups of finite Morley rank. *J. Symbolic Logic*, 54(3):768–773, 1989.
[8] Robin Hartshorne. *Foundations of projective geometry*, volume 1966/67 of *Lecture Notes, Harvard University*. W. A. Benjamin, Inc., New York, 1967.
[9] Eric Jaligot. Full Frobenius groups of finite Morley rank and the Feit-Thompson theorem. *Bull. Symbolic Logic*, 7(3):315–328, 2001.
[10] Eric Jaligot and Abderezak Ould Houcine. Existentially closed CSA-groups. *J. Algebra*, 280(2):772–796, 2004.
[11] David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.
[12] Michael Morley. Categoricity in power. *Trans. Amer. Math. Soc.*, 114:514–538, 1965.
[13] Yerulan Mustafin and Bruno Poizat. Sous-groupes superstables de $\mathrm{SL}_2(K)$ et de $\mathrm{PSL}_2(K)$. *J. Algebra*, 297(1):155–167, 2006.
[14] Ali Nesin. Nonsolvable groups of Morley rank 3. *J. Algebra*, 124(1):199–218, 1989.
[15] Ali Nesin. On bad groups, bad fields, and pseudoplanes. *J. Symbolic Logic*, 56(3):915–931, 1991.
[16] Bruno Poizat. *Stable groups*, volume 87 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2001. Translated from the 1987 French original by Moses Gabriel Klein.
[17] Simon Thomas. The classification of the simple periodic linear groups. *Arch. Math. (Basel)*, 41(2):103–116, 1983.
[18] B. I. Zil′ber. Groups and rings whose theory is categorical. *Fund. Math.*, 95(3):173–188, 1977.

Laboratoire de Mathématiques et Applications, Université de Poitiers, Téléport 2 - BP 30179, Boulevard Marie et Pierre Curie, 86962 Futuroscope Chasseneuil Cedex, France

E-mail address: olivier.frecon@math.univ-poitiers.fr