STRICT C^1-TRIANGULATIONS IN O-MINIMAL STRUCTURES

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ABSTRACT. Let R be a real closed field and let an expansion of R to an o-minimal structure be given. We prove that for any closed bounded definable subset A of R^n and a finite family $B_1, \dots, B_r$ of definable subsets of A there exists a definable triangulation $h : |\mathcal{K}| \longrightarrow A$ of A compatible with $B_1, \dots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n and h extends to a definable C^1-mapping defined on a definable open neighborhood of $|\mathcal{K}|$ in R^n.

1. Introduction. Assume that R is any real closed field and an expansion of R to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [vdD] or [C].) We adopt the following definitions of a simplex and a simplicial complex. Let $k, n \in \mathbb{N}$ and $k \leqslant n$. A *simplex of dimension k in R^n* is the convex hull

$$\Delta = (a_0, \dots, a_k) = \left\{ \sum_{i=0}^{k} \alpha_i a_i : \alpha_i > 0 \ (i = 0, \dots, k), \ \sum_{i=0}^{k} \alpha_i = 1 \right\}$$

of $k + 1$ affinely independent points a_i of R^n which are called the *vertices* of Δ. An *l-dimensional face of Δ* is any of the following simplexes $\Delta' = (a_{\nu_o}, \dots, a_{\nu_l})$, where $0 \leqslant \nu_o < \dots < \nu_l \leqslant k$.

A *simplicial complex* in R^n is a finite family $\mathcal{K}$ of simplexes in R^n which satisfies the following conditions:

(1) If $\Delta_1, \Delta_2 \in \mathcal{K}$ and $\Delta_1 \neq \Delta_2$, then $\Delta_1 \cap \Delta_2 = \emptyset$.

(2) If $\Delta \in \mathcal{K}$ and Δ' is a face of Δ, then $\Delta' \in \mathcal{K}$.

The closed bounded definable subset $|\mathcal{K}| = \bigcup \mathcal{K}$ of R^n is called the *polyhedron of the symplicial complex $\mathcal{K}$*.

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Let A be any closed bounded subset of R^n. A *definable C^1-triangulation of A* is a pair $(\mathcal{K}, h)$, where $\mathcal{K}$ is a simplicial complex in some space R^m, $h : |\mathcal{K}| \longrightarrow A$ is a definable homeomorphism such that for each $\Delta \in \mathcal{K}$, $h(\Delta)$ is a definable C^1-submanifold of R^n and $h|\Delta : \Delta \longrightarrow h(\Delta)$ is a C^1-diffeomorphism. When $B_1, \dots, B_r$ are definable subsets of A, we say that a *triangulation* $(\mathcal{K}, h)$ *is compatible with the sets* $B_1, \dots, B_r$ if each of the sets $h^{-1}(B_j)$ is a union of some simplexes of $\mathcal{K}$. A *definable strict C^1-triangulation* is such a definable C^1-triangulation $(\mathcal{K}, h)$ that $h : |\mathcal{K}| \longrightarrow R^n$ is of class C^1; i.e. it has an extension to a C^1-mapping defined on an open definable neighborhood of $|\mathcal{K}|$ in R^m.

The main result of the present article is the following.

Main Theorem. *Let A be a closed bounded definable subset of R^n and let $B_1, \dots, B_r$ be a finite family of definable subsets of A. Then there exists a definable strict C^1-triangulation $(\mathcal{K}, h)$ of A compatible with $B_1, \dots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n.*

This theorem improves the results of Coste-Reguiat [CR] and Ohmoto-Shiota [OS]. The interest for proving such a theorem is in its application to integration theory on sets definable in o-minimal structures (cf. [OS]). The proof of the main theorem below is divided into two parts; in the first one it is proven that there exists a definable $\mathcal{C}^1$-triangulation $(\mathcal{K}, h)$ of A compatible with $B_1, \dots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n, $h : |\mathcal{K}| \longrightarrow R^n$ is Lipschitz and $\{h|\Delta : \Delta \in \mathcal{K}\}$ is a $\mathcal{C}^1$-stratification with the Whitney (A) condition and in the second part this triangulation will be improved to a strict $\mathcal{C}^1$-triangulation.

2. Proof of Main Theorem.

Part I. First we will prove that there exists a definable $\mathcal{C}^1$-triangulation $(\mathcal{K}, h)$ of A compatible with $B_1, \dots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n, $h : |\mathcal{K}| \longrightarrow R^n$ is Lipschitz and $\{h|\Delta : \Delta \in \mathcal{K}\}$ is a $\mathcal{C}^1$-stratification with the Whitney (A) condition.

The proof is by induction on n. Without any loss of generality we assume that A is the closure of its interior $A = \overline{\text{int}A}$. By Theorem 3.12 from [Cz2] there exists a definable C^1-triangulation $(\mathcal{K}, f)$ of A compatible with $B_1, \dots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n and $f : |\mathcal{K}| \longrightarrow A$ is a Lipschitz mapping. By the assumption about A, $|\mathcal{K}| = \bigcup\{\overline{\Delta} : \Delta \in \mathcal{K}, \dim \Delta = n\}$. After perhaps a linear change of coordinates in R^n, we can assume that there exists a finite number of affine functions $\varphi_j : R^{n-1} \longrightarrow R$ $(j = 1, \dots, s)$, such that

$$\bigcup\{\partial\Delta : \dim \Delta = n\} \subset \bigcup_{j=1}^{s} \varphi_j,$$

where φ_j stands for the graph of $\varphi_j = \{(x_1, \dots, x_n) \in R^n : x_n = \varphi_j(x_1, \dots, x_{n-1})\}$. (Throughout the article we adopt the convention to identify mappings with their graphs.) Then $\{f|\Delta : \Delta \in \mathcal{K}\}$ is a finite definable C^1-stratification of (the graph of) f. By [L2] (see also [L1] or [ŁSW], or [Ł]) it admits a finite definable C^1-refinement $\mathcal{S}$ with Whitney (A) condition such that strata from S of dimension n are exactly $\{f|\Delta : \Delta \in K, \dim \Delta = n\}$. There exists a corresponding $\mathcal{C}^1$-stratification $\mathcal{T}$ of $|\mathcal{K}|$ which is a refinement of $\mathcal{K}$ such that $\mathcal{S} = \{f|\Lambda : \Lambda \in \mathcal{T}\}$ and $\mathcal{T}$ contains all open simplexes of $\mathcal{K}$. Then for any pair $M, N \in \mathcal{T}$, such that $M \subset \overline{N}$ and for any $x_o \in M$ and any definable arc $\alpha : (0,\varepsilon) \longrightarrow N$ $(\varepsilon > 0)$ such that $\lim_{t\to 0}\alpha(t) = x_o$, we have

$$\lim_{t\to 0} d_{\alpha(t)}(f|\, N) \supset d_{x_o}(f|\, M). \tag{1}$$

Here we use the fact that the limit $\lim\limits_{t\to 0} d_{\alpha(t)}(f|\, N)$ always exists due to the o-minimality condition and common boundedness of the differentials $d_{\alpha(t)}(f|\, N)$ following from the lipschitzianity condition.

Let $\pi : R^n \ni (x_1,\dots,x_n) \longmapsto (x_1,\dots,x_{n-1}) \in R^{n-1}$ denote the natural projection. $\pi(|\mathcal{K}|)$ is a definable closed and bounded subset of R^{n-1}. Take $\rho > 0$ such that $|\varphi_j(y)| < \rho$, for each $y \in \pi(|\mathcal{K}|)$ and $j \in \{1,\dots,s\}$. By the induction hypothesis there exists a strict C^1-triangulation $(\mathcal{L}, g)$ of $\pi(|\mathcal{K}|)$ compatible with all the subsets $\pi(N)$, where $N \in \mathcal{T}$, and at the time with all the subsets $\{y \in R^{n-1} : \varphi_{j_1}(y) = \varphi_{j_2}(y)\}$ and $\{y \in R^{n-1} : \varphi_{j_1}(y) < \varphi_{j_2}(y)\}$, where $j_1 \neq j_2$. Replacing perhaps $\mathcal{L}$ by its barycentric subdivision we can assume that

(2) $\Lambda \in \mathcal{L}, \varphi_{j_1}\circ g < \varphi_{j_2}\circ g$ on $\Lambda \Rightarrow (\varphi_{j_1}\circ g)(c) < (\varphi_{j_2}\circ g)(c)$, for some vertex c of Λ.

Put $\varphi_o \equiv -\rho$ and $\varphi_{s+1} \equiv \rho$.

Similarly as in the classical proofs of triangulation (compare [vdD, Chapter 8]), we built a polyhedral complex $\mathcal{P}$ in R^n the polyhedron of which is $|\mathcal{L}| \times [-\rho,\rho]$ and such that its projection under π is $\mathcal{L}$. To this end fix any simplex $\Lambda \in \mathcal{L}$. Put

$$\{\psi_o^\Lambda, \dots, \psi_{r+1}^\Lambda\} = \{\varphi_j \circ g|\, \Lambda : j = 0,\dots,s+1\},$$

where $\psi_o^\Lambda < \dots < \psi_{r+1}^\Lambda$, $r = r_\Lambda$ depending on Λ. Let $c_o,\dots,c_k$ be all vertices of Λ. For each $i \in \{0,\dots,r+1\}$, define also $\Psi_i^\Lambda : \Lambda \longrightarrow R$ by the the formula

$$\Psi_i^\Lambda\Big(\sum_{\nu=0}^{k}\alpha_\nu c_\nu\Big) := \sum_{\nu=0}^{k}\alpha_\nu \psi_i^\Lambda(c_\nu),$$

where $\alpha_\nu > 0$, for each $\nu \in \{0,\dots,k\}$, and $\sum\limits_{\nu=0}^{k}\alpha_\nu = 1$. Now we define the polyhedral complex

$$\mathcal{P} := \{\Psi_i^\Lambda : \Lambda \in \mathcal{L}, i = 0,\dots,r_\Lambda+1\} \cup \{(\Psi_i^\Lambda, \Psi_{i+1}^\Lambda) : \Lambda \in \mathcal{L}, i = 0,\dots,r_\Lambda\}.$$

The complex is well defined because ψ_i^Λ have continuous extensions to $\overline{\Lambda}$ and because of (2) (for more detailed explanation, see Lemma 1 below). There exists a unique definable homeomorphism $H : |\mathcal{L}| \times [-\rho,\rho] \longrightarrow |\mathcal{L}| \times [-\rho,\rho]$, such that for each $\Lambda \in \mathcal{L}$ and $i \in \{0,\dots,r_\Lambda+1\}$, $H(u,\Psi_i^\Lambda(u)) = (u,\psi_i^\Lambda(u))$, for each $u \in \Lambda$, and for each $i \in \{0,\dots,r_\Lambda\}$ and $u \in \Lambda$, H is an affine isomorphism of the line segment $[(u,\Psi_i^\Lambda(u)),(u,\Psi_{i+1}^\Lambda(u))]$ onto the line segment $[(u,\psi_i^\Lambda(u)),(u,\psi_{i+1}^\Lambda(u))]$ (see Lemma 1). Since each of the functions ψ_i^Λ has a $\mathcal{C}^1$-extension to $\overline{\Lambda}$, according to Lemma 1, H is Lipschitz, $\mathcal{C}^1$ on every polyhedron $\Theta \in \mathcal{P}$ and $\{H|\,\Theta : \Theta \in \mathcal{P}\}$ is a $\mathcal{C}^1$-stratification of H with Whitney (A) condition. By Lemma 2 below, all the above properties of H hold when we replace $\mathcal{P}$ by a simplicial complex $\mathcal{P}^*$ which is a barycentric subdivision of $\mathcal{P}$, and since $g : |\mathcal{L}| \longrightarrow \pi(|\mathcal{K}|)$ is $\mathcal{C}^1$, the same properties inherits the mapping $\tilde{H} := (g \times id_R) \circ H : |\mathcal{L}| \times [-\rho, \rho] \longrightarrow \pi(|\mathcal{K}|) \times [-\rho, \rho]$. It is clear from the definitions that there exists a subcomplex $\mathcal{R}$ of $\mathcal{P}$ such that $\{\tilde{H}(\Theta) : \Theta \in \mathcal{R}\}$ is a $\mathcal{C}^1$-stratification of $|\mathcal{K}|$ which is a refinement of $\mathcal{K}$ such that $\tilde{H}$ is Lipschitz and $\{\tilde{H}|\,\Theta : \Theta \in \mathcal{R}\}$ is a $\mathcal{C}^1$-stratification of $\tilde{H}$ with Whitney (A) condition. Now the mapping $G := f \circ \tilde{H}$ is the desired Lipschitz triangulation such that $\{G|\,\Theta : \Theta \in \mathcal{R}\}$ is a $\mathcal{C}^1$-stratification of G with Whitney (A) condition (see Lemma 2).

Lemma 1(cf. [Cz; Lemma 3.10]). *Let $\Lambda = (c_o, \ldots, c_k)$ be a simplex in R^n of dimension k. Let $\mathcal{L}_\Lambda$ be the simplicial complex of all faces of Λ; so $|\mathcal{L}_\Lambda| = \overline{\Lambda}$. Let $\psi_i : \overline{\Lambda} \longrightarrow R$ $(i = 1, 2)$ be definable C^1-functions such that*

(3) $\Delta \in \mathcal{L}_\Lambda, \psi_1|\Delta \not\equiv \psi_2|\Delta \Rightarrow$ *there is a vertex c_ν of Δ such that $\psi_1(c_\nu) < \psi_2(c_\nu)$.*

Let $\Psi_i : |\overline{\Lambda}| \longrightarrow R$ $(i = 1, 2)$ be defined by the formula

$$\Psi_i\Big(\sum_{\nu=0}^{k} \alpha_\nu c_\nu\Big) = \sum_{\nu=0}^{k} \alpha_\nu \psi_i(c_\nu),$$

where $\sum_{\nu=0}^{k} \alpha_\nu = 1$, $\alpha_\nu \geqslant 0$. Consider the following polyhedral complex

$$\mathcal{P} = \{\Psi_i|\,\Delta : \Delta \in \mathcal{L}_\Lambda, i = 1, 2\} \cup \{(\Psi_1|\Delta, \Psi_2|\Delta) : \Delta \in \mathcal{L}_\Lambda, \Psi_1|\Delta < \Psi_2|\Delta\}.$$

Then there exists a unique definable homeomorphism

$$H : |\mathcal{P}| \longrightarrow \{(y, z) \in \overline{\Lambda} \times R : \psi_1(y) \leqslant z \leqslant \psi_2(y)\}$$

such that, for each $y \in \overline{\Lambda}$ and $i = 1, 2$, $H(y, \Psi_i(y)) = (y, \psi_i(y))$ and H is an affine isomorphism of the line segment $[(y, \Psi_1(y)), (y, \Psi_2(y))]$ onto the line segment $[(y, \psi_1(y)), (y, \psi_2(y))]$. Moreover, we have that

- (a) *H is Lipschitz,*
- (b) *H is $\mathcal{C}^1$-mapping on each $\Theta \in \mathcal{P}$ and*
- (c) *$\{H|\Theta : \Theta \in \mathcal{P}\}$ is a $\mathcal{C}^1$-stratification of H with Whitney (A) condition.*

Proof of Lemma 1. It is clear that, for each $\Delta \in \mathcal{L}_\Lambda$,

$$H(y, w) = \begin{cases} (y, \psi_1(y)), & \text{if } (y, w) \in \Psi_1|\Delta \\ \big(y, \frac{w - \psi_1(y)}{\psi_2(y) - \psi_1(y)} \psi_2(y) + \frac{\psi_2(y) - w}{\psi_2(y) - \psi_1(y)} \psi_1(y)\big), & \text{if } (y, w) \in (\Psi_1|\Delta, \Psi_2|\Delta) \\ (y, \psi_2(y)), & \text{if } (y, w) \in \Psi_2|\Delta. \end{cases}$$

Notice that H is well-defined bijection due to (3), which implies that $\psi_1 < \psi_2$ on Δ if and only if $\Psi_1 < \Psi_2$ on Δ, otherwise $\psi_1 \equiv \psi_2$ on Δ and $\Psi_1 \equiv \Psi_2$ on Δ. To prove (a), (b) and (c), first observe that using the following $\mathcal{C}^1$-diffeomorphism

$$\overline{\Lambda} \times R \ni (y, w) \mapsto (y, w - \psi_1(y)) \in \overline{\Lambda} \times R$$

we can assume without any loss of generality that $\psi_1 \equiv \Psi_1 \equiv 0$. Of course, we can assume that $\psi := \psi_2 > 0$ and $\Psi := \Psi_2 > 0$ on Λ. The condition (b) is clearly fulfilled. Put $\Pi = (0|\Lambda, \Psi|\Lambda)$ and $H(y,w) = (y, H^*(y,w))$. In order to prove (a) it suffices to show that all first-order partial derivatives of H^* are bounded on Π. Since

$$\frac{\partial H^*}{\partial y_j}(y,w) = \frac{z}{\Psi(y)} \cdot \frac{\partial \psi}{\partial y_j}(y) - \frac{z}{\Psi(y)} \cdot \frac{\psi(y)}{\Psi(y)} \cdot \frac{\partial \Psi}{\partial y_j}(y) \tag{4}$$

$$\text{and} \qquad \frac{\partial H^*}{\partial w}(y,w) = \frac{\psi(y)}{\Psi(y)}$$

it is enough to show that $\dfrac{\psi}{\Psi}$ is bounded on Λ. This is clear if $\psi(c_\nu) = \Psi(c_\nu) > 0$, for all ν, so assume that $\{c_o, \ldots, c_l\} = \{c_\nu : \psi(c_\nu) = 0\}$, where $0 \leqslant l < k$. By an affine change of coordinates one can assume that $c_o = 0$ and c_ν $(\nu = 1, \ldots, k)$ are vectors of the canonical basis. Let $y = (y_1, \ldots, y_k) \in \Pi$. Put $u = (y_1, \ldots, y_l, 0, \ldots, 0)$. We have

$$\Big|\frac{\psi(y)}{\Psi(y)}\Big| = \Big|\frac{\psi(y) - \psi(u)}{\Psi(y)}\Big| \leqslant \frac{M \sum_{\nu=l+1}^{k} y_\nu}{\sum_{\nu=l+1}^{k} y_\nu \psi(c_\nu)} \leqslant \frac{M}{\min\{\psi(c_\nu) : \nu = l+1, \ldots, k\}},$$

where M is the upper bound for the absolute value of the first-order partial derivatives of ψ. In order to check (c), first observe that H is a $\mathcal{C}^1$-diffeomorphism of $\{(y,w) \in |\mathcal{P}| : \Psi(y) > 0\}$ onto $\{(y,z) \in \overline{\Lambda} \times R : 0 \leqslant z \leqslant \psi(y), \psi(y) > 0\}$. Therefore, without any loss of generality, it suffices to check the Whitney (A) condition for Π and

$$\Theta \subset \{(y,w) \in \overline{\Lambda} \times R : \Psi(y) = 0 = w\} = \{(y,w) \in \overline{\Lambda} \times R : \psi(y) = 0 = w\} =$$

$$\operatorname{conv}\{c_o, \ldots, c_l\} \times \{0\}.$$

Hence, without any loss of generality, one can assume that $\Theta = (c_o, \ldots, c_p) \times \{0\}$, where $p \leqslant l$. Fix any $(a, 0) \in \Theta$. By (4), since ψ and Ψ are $\mathcal{C}^1$, we have

$$\frac{\partial H^*}{\partial y_j}(y,w) \to 0, \quad \text{for } j = 1, \ldots, p, \text{ when} \quad \Pi \ni (y,w) \to (a,0).$$

This ends the proof of (c) and of Lemma 1.

The next lemma is a particular case of the general fact that the Whitney (A) condition is preserved in a transversal intersection (see [Cz1]).

Lemma 2. *Let $H : A \longrightarrow R^m$ be a definable Lipschitz mapping defined on a closed subset $A \in R^n$. Let $\mathcal{S}$ be a definable finite $\mathcal{C}^1$-stratification of A such that $H|M$ is $\mathcal{C}^1$ for each $M \in \mathcal{S}$ and $\{H|M : M \in \mathcal{S}\}$ is a $\mathcal{C}^1$-stratification of H with Whitney (A) condition. Let $\mathcal{T}$ be a definable finite C^1-stratification of A with Whitney (A) condition which is a refinement of $\mathcal{S}$.*

Then $\{H|N : N \in \mathcal{T}\}$ is a $\mathcal{C}^1$-stratification of H with Whitney (A) condition.

Proof. It follows from the Lipschitz condition that the differentials of $H|M$ are commonly bounded. Hence the proof is immediate.

Part II. Let $(\mathcal{K}, f)$ be a definable $\mathcal{C}^1$-triangulation of A compatible with $B_1, \dots, B_r$ such that $\mathcal{K}$ is a simplicial complex in R^n such that:

$$f : |\mathcal{K}| \longrightarrow R^n \text{ is Lipschitz} \tag{5}$$

and

$$\{f|\Delta : \Delta \in \mathcal{K}\} \text{ is a } \mathcal{C}^1\text{-stratification with the Whitney (A) condition.} \tag{6}$$

Now we will improve f to get a strict $\mathcal{C}^1$-triangulation of A. To this end we will modify f in some tubular neighborhoods of simplexes.

Fix any simplex $\Gamma \in \mathcal{K}$ of dimension $p < n$. Without any loss of generality we can assume that $0 \in \Gamma$ and $\Gamma \subset R^p = \{(x_1, \dots, x_n) \in R^n : x_{p+1} = \dots = x_n = 0\}$. Let $R^{n-p} = \{(x_1, \dots, x_n) \in R^n : x_1 = \dots = x_p = 0\}$. There are affine functionals $\rho_j : R^p \longrightarrow R$ $(j = 0, \dots, p)$ such that $\Gamma = \{u \in R^p : \rho_j(u) > 0 \ (j = 0, \dots, p)\}$.

Consider the star $\mathrm{St}(\Gamma, \mathcal{K})$ of Γ in $\mathcal{K}$; i.e. $\mathrm{St}(\Gamma, \mathcal{K}) = \{\Lambda \in \mathcal{K} : \Gamma \text{ is a face of } \Lambda\}$. Then $\Omega := \bigcup\{\Lambda \in \mathrm{St}(\Gamma, \mathcal{K})\}$ is an open neighborhood of Γ in $|\mathcal{K}|$. There exists $\alpha > 0$ such that, for each $u \in \Gamma$,

$$\mathrm{dist}(u, \partial\Omega) > \alpha \min_j \rho_j(u).$$

Put $\omega(u) := \rho_o^2(u) \cdot \ldots \cdot \rho_p^2(u)$, for each $u \in \Gamma$. There exists $\varepsilon > 0$ such that, for each $u \in \Gamma$,

$$2\varepsilon\omega(u) \leqslant \alpha \min_j \rho_j(u) < \mathrm{dist}(u, \partial\Omega). \tag{7}$$

Then

$$G := \{(u, v) \in |\mathcal{K}| : u \in \Gamma, v \in R^{n-p}, |v| \leqslant \varepsilon\omega(u)\}$$

is a neighborhood of Γ in $|\mathcal{K}|$ contained in Ω due to (7).

Let $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ be a definable $\mathcal{C}^1$-function such that $\varphi(0) = \varphi'(0) = 0$, $\varphi(t) = 1$, for $t \geqslant 1$, and $\varphi'(t) > 0$, for $t \in (0, 1)$. Now we define $g : \Gamma \times R^{n-p} \longrightarrow \Gamma \times R^{n-p}$ by the formula

$$g(u, v) := \Big(u, \varphi(\frac{|v|}{\varepsilon\omega(u)})v\Big).$$

Then $g(G) = G$ and g is identity outside G. Besides, g is a $\mathcal{C}^1$-diffeomorphism of $\Gamma \times R^{n-p} \setminus \Gamma$ onto $\Gamma \times R^{n-p} \setminus \Gamma$, because its inverse on $\Gamma \times R^{n-p} \setminus \Gamma$ is

$$g^{-1}(u, w) = \Big(u, \psi^{-1}(\frac{|w|}{\varepsilon\omega(u)})\frac{w}{|w|}\Big),$$

where $\psi : (0, +\infty) \longrightarrow (0, +\infty)$ is a $\mathcal{C}^1$-diffeomorphism defined by the formula $\psi(t) := \varphi(t)t$.

Furthermore g is C^1 on $\Gamma \times R^{n-p}$, because for any $j \in \{1, \dots, n-p\}$

$$\frac{\partial g}{\partial v_j}(u,v) = \Big(0, \frac{v_j}{|v|} \cdot \frac{1}{\varepsilon\omega(u)} \cdot \varphi'\big(\frac{|v|}{\varepsilon\omega(u)}\big)v + \varphi\big(\frac{|v|}{\varepsilon\omega(u)}\big)e_j\Big), \tag{8}$$

where $e_j = (0, \dots, \underset{(j)}{1}, \dots, 0)$. It follows that $\frac{\partial g}{\partial v_j}(u,v) \to (0,0)$, when $(u,v) \to (u_o, 0) \in \Gamma$.

Now we define $h : |\mathcal{K}| \longrightarrow |\mathcal{K}|$ by putting $h(x) = g(x)$, for each $x \in G$, and $h(x) = x$ on $|\mathcal{K}| \setminus G$. It is clear that h is a homeomorphism of $|\mathcal{K}|$ onto $|\mathcal{K}|$ and a $\mathcal{C}^1$-diffeomorphism of each simplex $\Lambda \in \mathcal{K}$ onto itself. It follows from (8) and the boundedness of first-order partial derivatives of $f|\Lambda$ (due to (5)) that

$$\frac{\partial(f|\Lambda \circ h)}{\partial z}(u,v) \to (0,0), \qquad \text{when} \qquad (u,v) \to (u_o,0) \in \Gamma, \tag{9}$$

where $\Lambda \in \mathrm{St}(\Gamma, \mathcal{K}) \setminus \{\Gamma\}$ and z is any nonzero vector from the intersection of the linear subspace L generated by Λ with R^{n-p}. On the other hand we have for any $i \in \{1, \dots, p\}$ and $(u,v) \in G \cap \Lambda$

$$\frac{\partial(f|\Lambda \circ h)}{\partial u_i}\big(u,v\big) = \frac{\partial(f|\Lambda)}{\partial u_i}\Big(u, \varphi\big(\frac{|v|}{\varepsilon\omega(u)}\big)v\Big) \tag{10}$$

$$+ \sum_{\nu=1}^{q} \frac{\partial(f|\Lambda)}{\partial z_\nu}\Big(u, \varphi\big(\frac{|v|}{\varepsilon\omega(u)}\big)v\Big)(-1)\frac{\partial\omega}{\partial u_i}(u)\frac{|v|}{\varepsilon\omega^2(u)}\varphi'\big(\frac{|v|}{\varepsilon\omega(u)}\big)v_\nu,$$

where $z_1, \dots, z_q$ is an orthogonal basis of $L \cap R^{n-p}$ and v_ν are coefficients of v with respect to this basis. It follows from (5) and from flatness of ω on $\partial\Gamma$ that

$$\frac{\partial(f|\Lambda \circ h)}{\partial u_i}\big(u,v\big) \to \frac{\partial(f|\Delta)}{\partial u_i}\big(u,0\big), \tag{11}$$

when $\Lambda \ni (u,v) \to (u_o, 0) \in \Delta$, for any simplex $\Delta \in \mathcal{K}$ contained in $\overline{\Gamma}$. This has two consequences. Firstly, all first-order partial derivatives of $f|\Lambda \circ h$ have finite limits when approaching Γ (see (9) and (11)). Secondly, the new triangulation $f \circ h$ satisfies the condition (6) at faces Δ of Γ where it may fail to be $\mathcal{C}^1$-extendable. But such Δ are of dimension less then $p = \dim\Gamma$, and our procedure works by decreasing induction on $p = \dim\Gamma$.

Consequently, after finite number of steps, we obtain a definable $\mathcal{C}^1$-triangulation $f : |\mathcal{K}| \longrightarrow R^n$ of A which has on $|\mathcal{K}|$ all first-order continuous partial derivatives. Hence, by a definable version of Whitney's extension theorem [KP], f can be extended to a definable $\mathcal{C}^1$-mapping defined on the whole space R^n.

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