TOWARDS THE DECIDABILITY OF THE THEORY OF MODULES OVER FINITE COMMUTATIVE RINGS

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Abstract. On the basis of the Klingler–Levy classification of finitely generated modules over commutative noetherian rings we approach the old problem of classifying finite commutative rings R with decidable theory of modules. We prove that if R is (finite length) wild, then the theory of all R-modules is undecidable, and verify decidability of this theory for some classes of tame finite commutative rings.

1. Introduction

Let A be a finite universal algebra in a language consisting of finitely many functional symbols and let $V(A)$ denote the variety generated by A. As a far reaching consequence of the theory of decidable locally finite varieties the following result has been proved (see [17, p. 195, Corollary 14.2]). There is an algorithm that produces from A a finite (associative) ring R such that the theory of $V(A)$ is decidable iff the theory T_R of all R-modules is decidable. Thus it appears to be important to classify finite rings with decidable (first order) theory of modules. This open problem was included in [17, p. 194, Problem 2] and then discussed at the beginning of [19]. Also [21, Chapter 17, p. 350, Problem 1] treated it within the wider question of classifying all the sufficiently recursive rings with a decidable theory of modules.

Indeed both [21, Chapter 17] and [19] are good accounts on what had been known on decidability of modules over finite (more generally sufficiently recursive countable) rings; not so much has happened since then. We will slightly update their account in the case of finite rings and clarify how recent developments could affect certain points of view.

The main conjecture is still the same as in [21, p. 332]: the theory of all modules over a finite ring R is decidable if and only if the category of finitely generated (hence finite) R-modules is tame. Unfortunately there is no generally accepted definition of tameness for finite rings, therefore one

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should stick with an informal one. We say that R is *tame* (or rather finitely generated tame) if one could 'classify' finitely generated R-modules, and R is *wild* otherwise. On the other hand there is a precise definition for R to be of finite representation type and one can prove (see [21, p. 333]) that in this case the theory of all R-modules is decidable.

If R is a finite dimensional algebra over a (finite) field then we can borrow a standard definition of tameness from representation theory (see [31, p. 290]). However it is not completely clear whether this definition corresponds to the intuitive one, its model theoretic meaning is obscure, and it is still unknown whether every finite dimensional algebra over a finite field is tame or wild (Drozd [7] proved a tame-wild dichotomy for finite dimensional algebras over algebraically closed fields).

Despite these drawbacks there is at least one advantage of using a functorial definition of wildness. Recall that a prototypical example of a wild algebra is given by the free algebra $k\langle X, Y\rangle$ (or even by some of its finite dimensional quotients) over a field k, and it is known that the theory of $k\langle X, Y\rangle$-modules (where k is an effectively given field) is undecidable. It is quite often possible to convert the existing proof of wildness of an algebra (or ring) R into an interpretation of the theory of $k\langle X, Y\rangle$-modules within the theory of R-modules showing that the latter is undecidable. Although this approach has been successfully utilized for many classes of wild algebras, the arguments still carry on ad hoc features and no general proof of 'wildness implies undecidability' has appeared yet. The best 'uniform' result is due to Prest [22]: if R is a strictly wild finite dimensional algebra (in fact the assumptions are apparently weaker) then the theory of R-modules interprets the theory of $k\langle X, Y\rangle$-modules.

Concerning the second part of the conjecture — 'tameness implies decidability' for modules over finite rings — recent advances are quite modest. As follows from Baur [4] the theory of all quadruples of vector spaces over a finite (or sufficiently recursive countable) field is decidable. This implies that the theory of modules over the finite dimensional algebra $k\widetilde{D}_4$ (over an effectively given field k) is decidable. This algebra represents a well understood class of hereditary finite dimensional algebras over a field k. If k is effectively given and A is the path algebra of a Euclidean or Dynkin quiver, the proof of decidability of the theory of all A-modules (via Ziegler spectrum and pp-interpretations) was outlined by Prest [20] and was eventually carried through by Geisler [10]. For instance, if k is algebraically closed, then every finite dimensional hereditary algebra A over k is a path algebra of a quiver without relations, and (using known results on wild hereditary finite dimensional algebras) one concludes (see [21, Theorem 17.22]) that the theory of A-modules is decidable iff A is tame.

If k is not algebraically closed, then (see [6]) there are more cases to consider, and Geisler's long proof is difficult to generalize. For the Kronecker algebra $k\widetilde{A}_1$ below we will give a different proof of decidability based on a very clear description of the Ziegler topology in [23] and [30]. However some unexpected complications will make it quite lengthy.

Switching from algebras to general (finite) rings could also add complexity — the proof of the reduction from finite rings to finite dimensional algebras in [21, p. 345–346] appears to be incomplete. Thus our variant of the proof of decidability of modules over some classes of finite rings will also overcome this difficulty.

The situation for more complex classes of tame finite dimensional algebras is even less satisfactory. Indeed the progress on decidability of modules was limited to very few cases (for some classes of domestic string algebras described in [5], the decidability should follow from the description of the Ziegler topology).

To sum up, the conjecture that the theory of all modules over a (finite) ring R is decidable iff R is tame definitely still is consistent with what is known, but needs to be tested on a wider range of examples.

In this paper we investigate decidability of the theory of modules over finite commutative rings. One reduction is easy: every finite commutative ring R is a finite direct sum of local rings, therefore we may assume that R is local. A crucial ingredient for investigating in this case is the voluminous Klingler–Levy (KL for short) investigation of finitely generated modules over commutative noetherian rings ([15] and [16] will be enough for our purposes) where the tame-wild dichotomy acquires a very precise meaning.

Suppose that R is a local commutative noetherian ring. Then [15, 16] show that either R has an artinian triad or Drozd ring as a homomorphic image, or R itself is a homomorphic image of a local Dedekind-like ring or is isomorphic to a Klein ring (see Section 3 for unexplained terminology). Furthermore in the former case R is wild (or rather finite length wild) where the term has precise meaning similar to the one in representation theory. On the other hand, with one small exception, in the latter case R is (informally) tame — [16] provides a complete list of indecomposable finitely generated R-modules. When R is finite, even this small deficiency in the KL-classification disappears, therefore we have a clear borderline between the wild and tame cases.

Here is the plan of our paper. We will recall in Section 2 the main facts about decidability of modules, and we will summarize in Section 3 the crucial points of the KL-analysis of modules over commutative noetherian rings.

After that we will deal with our main results. They are two-fold. Firstly we prove that wildness (as it is defined in [15]) implies undecidability for modules over a finite commutative ring. As we have already mentioned there are two cases to consider here. If R is an artinian triad the undecidability of the theory of all R-modules has been known (at least in principle —see [14, Proposition 8.69 and Example 8.37]). To prove undecidability for modules over Drozd rings we employ a variant (see [15, Section 4]) of Ringel's proof of wildness of the category of finite dimensional modules over Drozd algebras. With some effort we will extract from this proof an interpretation of the theory of $k\langle X, Y\rangle$-modules (where k is the residue field of R) in the theory of modules over any Drozd ring R showing that the latter is undecidable.

Secondly we tackle in Sections 5–9 one tame case in the KL-classification. Namely we prove that the theory of all modules over a (finite commutative) Klein ring is decidable. As it is customary nowadays we will exploit an approach to decidability through the Ziegler spectrum (see [20] for a detailed description of this approach). Thus to investigate decidability we have to classify the points of the Ziegler spectrum of a Klein ring (that is, indecomposable pure-injective modules) and also the topology of this space. The description of points follows from the known classification of indecomposable pure-injective $k\tilde{A}_1$-modules by applying reduction modulo the radical (see [14, p. 211] for the latter). The basis of open sets for the Ziegler topology can be similarly extracted from general results by Prest [23] and Ringel [30]. Unfortunately their description of the topology is not of the form that is required to prove decidability. Thus we will spend a great amount of time (and space) to overcome this difficulty. In detail, Section 5 will state the main decidability result as Theorem 5.1 and give the main steps of this proof. Section 6 illustrates the reduction from Klein rings to the Kronecker algebra $k\tilde{A}_1$, and Section 7 provides the aforementioned description of the Ziegler spectrum of $k\tilde{A}_1$. Section 8 transfers this analysis to an arbitrary Klein ring R via a suitable homeomorphism from the Ziegler spectrum of

R to a large clopen subset of the spectrum of $k\tilde{A}_1$ (actually consisting of all but one points). On this basis we will eventually show in Section 9 the decidability result for modules over finite Klein rings.

Finally we will briefly discuss in Section 10 decidability for the only remaining case of the KL-classification; in our setting this concerns finite factors of (complete local) Dedekind-like rings. It can be derived from general structure theory (see [16]) that the decidability question can be reduced to the case when R is a pullback of a direct sum of (at most) two finite valuation rings and a field (see Section 10 for precise description). A typical representative of this family is given by a Gelfand–Ponomarev algebra $G_{2,3}(k) = k[X, Y : X^2 = Y^3 = XY = 0]$ (k is a finite field). According to the aforementioned conjecture, the theory of $G_{2,3}(k)$-modules has to be decidable, but we have very little to say (at least in this paper) to support it even for this particular algebra.

Because the main targets of this paper are logicians and experts in general (including commutative) ring and module theory and we can expect that their knowledge of representation theory might be sparse, we will be very scrupulous in explaining some instances of representations of hereditary finite dimensional algebras, that are certainly well known to experts. On the other hand those experts may find in this paper a new encouragement in applying representation theory to decidability of modules.

Before concluding this section let us introduce some notation that will be useful later. For any commutative ring R, mod-R denotes the class of finitely generated R-modules; L_R is the first order language of R-modules (as described, for instance, in [21, p. 2]) and T_R, as said, their first order theory.

2. Decidability. General Discussion

Informally speaking, a theory T in an effectively given countable first order language L is *decidable* iff there is an algorithm producing for each sentence σ of L (as an input) the answer whether $\sigma \in T$ or not (as an output), and *undecidable* otherwise. One can express this in a rigorous way via Church's Thesis by saying that T is decidable iff the set of Gödel numbers of its theorems is recursive, and undecidable otherwise (see [25] for a discussion of this point and precise definitions).

When T is given by an effective (that is, recursively enumerable) set of axioms, then by applying deductions we can effectively produce a list of theorems of T, and thus (see again [25], or directly [21, Section 17.3]) T is decidable iff one can effectively produce a list of non-theorems of T, that is if also the complement of T is recursively enumerable. Taking negations this is the same as for every sentence σ in L to answer (uniformly) the question whether σ is true in some model of T. Of course if T is complete this difficulty disappears (hence every recursively axiomatizable complete theory is decidable), but in this paper we will mostly deal with incomplete theories.

In fact, we are going to consider the first order theory T_R of modules over a finite ring R. Of course, we can assume that R is given by a list of its elements $0, 1, r_2, \dots, r_n$ such that the operations $+, -, \times$ can be executed effectively, more generally that R is sufficiently recursive in the sense of [21, Section 17.1] and consequently that the decision problem of T_R makes sense. Furthermore T_R is recursively (indeed finitely) axiomatizable, and so to prove that T_R is decidable we have to list effectively the sentences in L_R which are true in some R-module. There is an elaborate way to do this using the Ziegler spectrum. This method was introduced by Ziegler [33] and developed by Prest (see [20]). But first we have to give some definitions.

A *positive-primitive formula* (pp-formula) $\varphi(\bar{x})$ in the free variables $\bar{x} = (x_1, \dots, x_n)$ is an L_R-formula of the form $\exists\, \bar{y}\, (\bar{y}A = \bar{x}B)$, where $\bar{y} = (y_1, \dots, y_k)$ is a tuple of bounded variables, A is a $k \times l$ and B is an $n \times l$ matrix over R. If M is a module and $\overline{m} = (m_1, \dots, m_n)$ is a tuple of elements of M, then we write $M \models \varphi(\overline{m})$, and we say that M *satisfies* $\varphi(\overline{m})$, if there is a tuple $\bar{n} = (n_1, \dots, n_k)$ in M^k such that $\bar{n}A = \overline{m}B$. The set $\varphi(M) = \{\overline{m} \in M^n \mid M \models \varphi(\overline{m})\}$ is easily seen to be a subgroup of the abelian group M^n (and is called the *pp-subgroup* defined by φ). Moreover $\varphi(M)$ is a submodule of M over the ring $S = \mathrm{End}(M)$ of R-endomorphisms of M (via the diagonal action of S); for a commutative R, $\varphi(M)$ is a submodule of M also over R. For instance if $r \in R$, then $r \mid x \doteq \exists\, y\, (yr = x)$ is a pp-formula such that $(r \mid x)(M) = Mr$ for any R-module M.

For every integer $n > 0$, the pp-formulae $\varphi(\bar{x})$ in n free variables $\bar{x}$ (more precisely their classes with respect to the equivalence relation identifying two of them if and only if they define the same subgroup in each R-module M) form a modular lattice. The underlying order relation is defined as follows: for $\varphi(\bar{x})$ and $\psi(\bar{x})$ two (equivalence classes of) pp-formulae, $\psi(\bar{x}) \leq \varphi(\bar{x})$ if and only if $\psi(M) \subseteq \varphi(M)$ for every R-module M, and hence if and only if $\psi(\bar{x})$ implies $\varphi(\bar{x})$ in T_R; accordingly $\psi(\bar{x}) < \varphi(\bar{x})$ means that

$\psi(M) \subseteq \varphi(M)$ for every R-module M but $\psi(M) \neq \varphi(M)$ for some R-module M. The meet of two pp-formulae $\varphi(\bar{x})$ and $\psi(\bar{x})$ is (the class of) their conjunction $\varphi(\bar{x}) \wedge \psi(\bar{x})$, while their join is (the class of) a pp-formula defining in every R-module M the sum $\varphi(M) + \psi(M)$ (and is consequently denoted by $\varphi(\bar{x}) + \psi(\bar{x})$).

An inclusion $M \subseteq N$ of R-modules is said to be *pure* if for every $\overline{m} \in M$ and every pp-formula $\varphi(\bar{x})$, from $N \models \varphi(\overline{m})$ it follows that $M \models \varphi(\overline{m})$. A module M is *pure-injective* if it is injective with respect to pure embeddings. For instance every injective module is pure-injective, and so is every finite module over a finite ring.

The Ziegler spectrum Zg_R of a ring R is a topological space whose points are indecomposable pure-injective modules. The basis of open neighborhoods for Ziegler topology is given by the following sets $(\varphi/\psi) = \{M \in \mathrm{Zg}_R \mid \varphi(M)/(\varphi \wedge \psi)(M) \neq 0\}$, where φ and ψ are pp-formulae in the same number of free variables (actually pp-formulae with exactly one free variable suffices, see [21, Chapter 4]). By [33, Theorem 4.9] with respect to this topology Zg_R is a compact space.

If φ and ψ are pp-formulae and M is an R-module then $\mathrm{Inv}(M, \varphi, \psi)$ denotes the cardinality of the quotient group $\varphi(M)/(\varphi \wedge \psi)(M)$ if it is finite, and ∞ otherwise. Clearly for every positive integer n the condition $\mathrm{Inv}(M, \varphi, \psi) \geq n$ can be expressed by a $\forall\exists$-sentence $\mathrm{Inv}(-, \varphi, \psi) \geq n$ and consequently is preserved under elementary equivalence. Furthermore it is a consequence of the theorem of Baur and Monk (see [33, Cor. 1.5]) that every complete theory of modules can be axiomatized by sentences $\mathrm{Inv}(-, \varphi, \psi) \geq n$ or their negations. Accordingly these sentences and their finite Boolean combinations are called *invariant statements* and, for every R-module M, the corresponding values $\mathrm{Inv}(M, \varphi, \psi)$ are called *elementary invariants* (as they characterize the elementary equivalence class of M).

Our main interest in the Ziegler spectrum is the following remarkable result (see [33, Thm. 9.4] or [21, Thm. 17.12]) which we adapt to finite rings.

Fact 2.1. Let R be a finite ring with a countable Ziegler spectrum. Suppose further that there is an effective enumeration $N_1, N_2, \ldots$ of points of Zg_R and an effective enumeration of a basis $(\varphi_1/\psi_1), (\varphi_2/\psi_2), \ldots$ of open sets for Ziegler topology. Assume also that $\mathrm{Inv}(N_i, \varphi_j, \psi_j)$ can be effectively calculated for each i and j. Then the theory of R-modules is decidable.

Note that over a finite ring R an effective list of a basis for Ziegler topology is trivial to produce. Namely make a effective list of all pp-formulae $\varphi_1, \varphi_2, \dots$ (say in one variable, which suffices — see [21, Chapter 4]) and then rearrange it into a list of pp-pairs (φ_i/φ_j) such that φ_i does not imply φ_j (hence this pair is non-trivial), where the last condition can be decided effectively. Unfortunately this brute force approach is not very effective — the main difficulty of calculating elementary invariants will become enormous; so the main point is to choose open basis for the Ziegler topology with a great care.

It could happen that a finite ring R has an uncountable Ziegler spectrum so the above result is not applicable — this is the case for Gelfand–Ponomarev algebras $G_{n,m}(k) = k[X, Y : X^n = Y^m = XY = 0]$ (n, $m \geq 2$, $n + m \geq 5$) over a finite field k — see [24]. But to investigate decidability in this paper Fact 2.1 is all what we need.

Having fixed a means to prove decidability we turn now to undecidability. The main tool to prove undecidability of theories of modules is via interpretations. For a general definition of interpretation of classes of algebraic structures see [17, pp. 9–10], or also [25]. The main point for us will be the following. If one elementary class of algebraic structures interprets another elementary class with undecidable theory, then the the theory of the former class is also undecidable. We will use two particular instances of this construction (therefore two modeling examples of classes with an undecidable theory).

Recall that for every field k, $k\langle X, Y\rangle$ denotes the free algebra in two noncommuting variables over k. It is well known that, if k is sufficiently recursive, in particular finite, the theory of all $k\langle X, Y\rangle$-modules is undecidable (see [21, Thm. 17.13]). To interpret this theory in the theory of all modules over a ring R it suffices to find pp-formulas $\varphi(\bar{u})$, $\psi(\bar{u})$ and $\varphi_X(\bar{u}, \bar{v})$, $\varphi_Y(\bar{u}, \bar{v})$ such that the following holds. For every $k\langle X, Y\rangle$-module M there exists an R-module N such that the quotient group $N' = \varphi(N)/(\varphi \wedge \psi)(N)$ inherits a structure of k-vector space, φ_X and φ_Y define an action (of X and Y) on N' and, equipped with these actions, N' is isomorphic to M as a $k\langle X, Y\rangle$-module. Thus to prove undecidability of the theory of R-modules it suffices to interpret the theory of $k\langle X, Y\rangle$-modules in the theory of R-modules in the just described (that is, *pp-definable*) way.

Another modeling example for undecidability is given by (abelian) structures (V, W, f), where $W \subseteq V$ are vector spaces over a field k, W is a subspace of V and f is a (k-linear) endomorphism of V. Again (see [21, Corollary 17.7]) if k is sufficiently recursive, then the theory of the class of these structures is undecidable. To interpret this class in the theory of R-modules it suffices to find pp-formulae $\varphi_V(\bar{u})$, $\varphi_W(\bar{v})$ and $\varphi_f(\bar{u}, \bar{v})$ such that the following holds. For every triple (V, W, f) there is an R-module N such that $\varphi_W(N) \subseteq \varphi_V(N)$ are k-vector spaces, φ_f defines a k-linear action $\bar{f}$ on $\varphi_V(N)$ and with respect to this action the structure $(\varphi_V(N), \varphi_W(N), \bar{f})$ is isomorphic to (V, W, f). So the theory of triples (V, W, f) is interpreted in T_R, and even in a pp-definable way.

This is almost all we will need from logic (or recursion theory) to discuss decidability. As we will see the remaining part is purely algebraic. Indeed we are in a position to make the first (well known) reduction. If R is a finite commutative ring, then it is a (finite) direct sum of indecomposable, hence local rings. The following remark shows that decidability of the theory of R-modules can be decided componentwise.

Remark 2.2. *Let $R = R_1 \oplus \cdots \oplus R_n$ be a decomposition of a (finite commutative) ring R into a direct sum of local rings. Then the theory of R-modules is decidable if and only if for each i the theory of all R_i-modules is decidable.*

Proof. The direct product is a very particular case of a generalized product as it is defined in Feferman and Vaught [8, Section 2]. The result follows immediately from [8, Theorem 5.4]. □

3. Klingler–Levy classification

In this section we summarize what we need for this paper from the remarkable KL-classification (see [15], [16]) of finitely generated modules over commutative noetherian rings. By Remark 2.2 we may assume that R is a finite local commutative ring. But let us momentarily refer to the wider setting the KL-classification applies to, that is, to a local commutative noetherian ring R (for our purposes — see below — we may additionally assume that R is complete). Let J denote the Jacobson radical (that is, the unique maximal ideal) of R and let $k = R/J$ be its residue field (again for our purposes we may assume that k is finite). If M is a finitely generated R-module then $\mu_R(M)$ will denote the minimal number of generators for M, hence (by Nakayama lemma) the dimension of M/MJ as a k-vector space.

We say that R is an *artinian triad* if $\mu_R(J) = 3$ and $J^2 = 0$. Thus every artinian triad is an artinian ring. A typical example of an artinian triad is given by a finite dimensional algebra $k[X, Y, Z : X^2 = Y^2 = Z^2 = XY = XZ = YZ = 0]$.

Further R is said to be a *Drozd ring* if $\mu_R(J) = \mu_R(J^2) = 2$, $J^3 = 0$ and $x^2 = 0$ for some $x \in J \setminus J^2$. Again any Drozd ring is artinian. A prototype of Drozd rings is the Drozd algebra $k[X, Y : X^2 = Y^3 = XY^2 = 0]$. A constructive description of Drozd rings as special subrings of principal ideal domains can be found in [15, Thm. 6.5].

As it will be explained in few lines artinian triads and Drozd rings are 'minimal wild'. In this perspective their counterpart are the so-called 'maximal tame' rings, that is, Klein rings and Dedekind-like rings.

R is a *Klein ring* if $\mu_R(J) = 2$, $\mu_R(J^2) = 1$, $J^3 = 0$ and $x^2 = 0$ for every $x \in J$. A typical example of Klein ring is given by the group algebra kG, where G is the Klein group and k is a field of characteristic 2.

Finally R is said to be a *Dedekind-like* ring if R is reduced (that is has no nilpotent elements) and, if Γ is the normalization of R (that is its integral closure in the ring of quotients), then Γ is a direct sum of (at most two) principal ideal domains, $\mu_R(\Gamma) = 2$ and J equals the Jacobson radical of Γ. If R is complete then (see [15, p. 351] this implies that either

i) $\Gamma/J \cong k$ that is, $R = \Gamma$ is a discrete valuation domain; or

ii) Γ/J is a 2-dimensional extension of k in which case R is called *unsplit*; or

iii) $\Gamma/J \cong k \times k$ and Γ is a direct sum of two noetherian valuation domains in which case R is said to be *strictly split*.

For various examples of Dedekind-like rings see [16, Section 12]. Let us mention among them the algebras $k[X, Y : XY = 0]$ (whose quotients include Gelfand–Ponomarev algebras) — they are strictly split Dedekind-like rings.

What will be essential for us is the following adaptation of Klingler–Levy dichotomy theorem (see [15, Theorem 2.10]). Note that in our case $k = R/J$ is finite, hence Γ/J as in ii) is a separable extension of k. This allows us to avoid the only unsettled case in KL-classification, the one just regarding unsplit rings and inseparable extensions (see a discussion in [15, p. 357]).

Theorem 3.1. *(Klingler–Levy dichotomy) Suppose that R is a finite local commutative ring. Then exactly one of the following holds.*

1) R projects itself onto an artinian triad or a Drozd ring, or

2) R is either a Klein ring or a (proper) homomorphic image of some strictly split or unsplit complete local Dedekind-like ring with finite residue field.

Furthermore in case 1) the category of finite R-modules is wild, but in case 2) it is possible to classify finite R-modules.

We do not need a precise definition of wildness (see [15, Definition 2.2]) in this paper, but rather some intermediate steps in the proof of that. Neither classification of finite R-modules in case 2) will play an essential role in this paper. However all this may be needed for the concluding discussion in Section 10.

Note that Theorem 3.1 explains why artinian triads and Drozd rings can be meant as minimal wild, and Klein rings and Dedekind-like rings as maximal tame.

4. Undecidability

In this section we prove that the wild case in the KL analysis leads to undecidability. Namely if R is a finite commutative local ring that projects itself onto an artinian triad or a Drozd ring then the theory of all R-modules is undecidable. This confirms the 'wild implies undecidable' conjecture for finite commutative rings. Clearly it suffices to prove undecidability of the theory of modules for artinian triads and Drozd rings.

The case of an artinian triad has been already tackled. Namely the theory of R-modules interprets the theory of $k\langle X, Y\rangle$-modules in a pp-definable way. This is implicitly shown in [14, Proposition 8.69 and Example 8.37], but see also [11, Lemma 3] which in its turn refers to [32, Theorem 1].

Proposition 4.1. *Let R be an artinian triad and let $k = R/J$ be the residue field of R. Then the theory of R-modules interprets in a pp-definable way the theory of $k\langle X, Y\rangle$-modules. Consequently, if R is finite, then the theory of R-modules is undecidable.*

Now let us treat Drozd rings. Suppose that R is a Drozd ring with Jacobson radical J and let $k = R/J$ be its residue field. By the definition of Drozd ring there is $x \in J \setminus J^2$ such that $x^2 = 0$. It follows that $J = \langle x, y\rangle$ is generated by x and y for some y, therefore $J^2 = \langle xy, y^2\rangle$. Consequently every element $a \in J$ can be (nonuniquely) written as

$$a = a(x)x + a(y)y + a(xy)xy + a(y^2)y^2\,, \tag{1}$$

where $a(x)$, $a(y)$, $a(xy)$ and $a(y^2)$ are units in R or zero.

Theorem 4.2. *Let R be a Drozd ring with the Jacobson radical J and residue field $k = R/J$. The theory of R-modules interprets in a pp-definable way the class (V, W, f), where V is a k-vector space, W is a subspace of V, and f is an endomorphism of V. Consequently if R is finite then the theory of all R-modules is undecidable.*

Proof. The latter claim is a direct consequence of the former (see a discussion in Section 2). So let us show how to interpret in R-modules the triples (V, W, f). We rearrange here the approach pursued for R-modules of finite length in [15, Section 4] (see also [26, p. 295]).

Let $J = \langle x, y\rangle$ as explained before. If $r \in R$ then by r_y we denote the class $r + Ry$ in the factor ring R/Ry; similarly r_{xy} denotes the class $r + Rxy$ in the quotient R/Rxy. The following statement and diagram are taken from [15, Lemma 4.3].

Fact 4.3. 1) All nonzero monomials $x, xy, x_{xy}, \ldots$ in diagram below are nonzero.

2) $\mathrm{soc}(R1_y) = Rx_y$, $\mathrm{soc}(R1_{xy}) = Rx_{xy} \oplus Ry^2_{xy}$ and $\mathrm{soc}(R) = Rxy \oplus Ry^2 = J^2$.

3) $Rx \cap Ry = Rxy$ and $Rx_{xy} \cap Ry_{xy} = 0$.

4) $Rx \cong Ry_{xy}$ as R-modules via the multiplication map $cx \to cy_{xy}$.

$$\begin{tikzcd}
 & R1_y \arrow[dl, no head] & & R1_{xy} \arrow[dl, no head] \arrow[dr, no head] & & & & R \arrow[dl, no head] \arrow[dr, no head] & & \\
Rx_y & & Rx_{xy} & & Ry_{xy} \arrow[dr, no head] & & Rx \arrow[ll, "\cong", "\text{mult}"'] \arrow[dr, no head] & & Ry \arrow[dl, no head] \arrow[dr, no head] & \\
 & & & & & Ry^2_{xy} & & Rxy & & Ry^2 .
\end{tikzcd}$$

Let us choose a k-basis of W and extend it to a k-basis of V. Thus

$$W = k^{(\beta)}, \qquad V = k^{(\alpha)}$$

for some cardinals $\beta \leq \alpha$. Construct an R-module M as follows. First form the direct sum

$$N = \frac{R^{(\beta)}}{R^{(\beta)}y} \oplus \frac{R^{(\alpha)}}{R^{(\alpha)}xy} \oplus R^{(\alpha)}$$

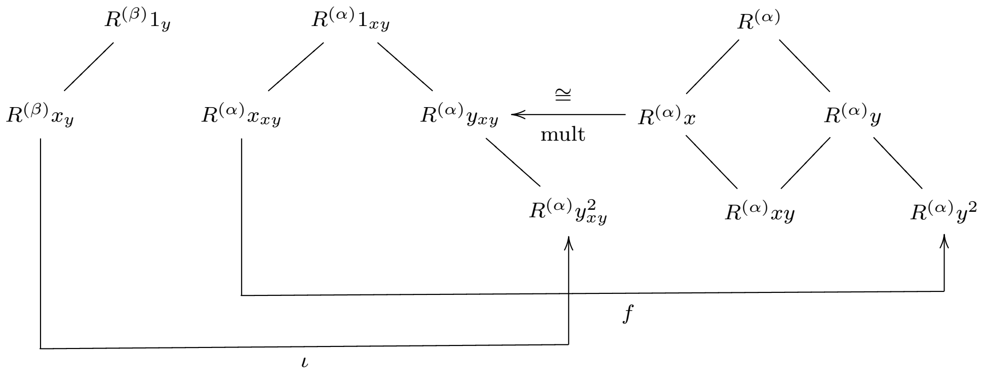


Figure 1

and put for simplicity $R_0 = R^{(\beta)}$, $R_1 = R^{(\alpha)}$. Note that R_0 embeds into R_1 in the natural way; for every $a \in R_0$ let a' denote its image via this embedding (so the juxtaposition of a and a suitably long zero sequence). Also note that R_1/R_1J is isomorphic to $k^{(\alpha)}$ so to V as an R-module, and hence as a k-vector space. Under this point of view, for $c \in R_1$, $c^* \in R_1$ is determined by the condition

$$f(c + R_1J) = c^* + R_1J\,.$$

Now we want to make some identifications in N hence factor N by some submodule $N(0)$. To see this factorization clearly let us reproduce one more diagram from [15, p. 368] (see Figure 1).

Thus we make the identifications shown by the 3 arrows on the diagram. So let $N(0)$ be a submodule of N generated by $\{ax_y - a'y_{xy}^2 \mid a \in R_0\}$, $\{by_{xy} - bx \mid b \in R_1\}$ and $\{cx_{xy} - c^*y^2 \mid c \in R_1\}$. Thus a generic element of $N(0)$ is of the form

$$(ax + R_0y, -a'y^2 + by + cx + R_1xy, -bx - c^*y^2) \tag{2}$$

with $a \in R_0$, $b, c \in R_1$.

Set $M = N/N(0)$. Looking at the representation of elements of J in the form (1) we may assume the following.

(∗) Each component a_i, $i < \beta$ of a and c_i, $i < \alpha$ of c in $N(0)$ is either zero or invertible, and each component b_i, $i < \alpha$ of b is either invertible or of the form $b_i(y)y$, where $b_i(y) \in R$ in its turn is either zero or invertible.

Now we show that V, W and f can be recovered in M by some suitable pp-formulae. More precisely, V lives in M as $R_1/R_1J \cong k^{(\alpha)}$ at the place $R^{(\alpha)}y^2_{xy}$ (see Figure 1) and W as $R_0/R_0J \cong k^{(\beta)}$ at the place $R^{(\beta)}x_y$ (hence W is identified with a subspace of V via ι).

Thus we define V by the following formula

$$\varphi_V(v) \doteq \exists\, u\, (uxy = 0 \wedge v = uy^2)\,,$$

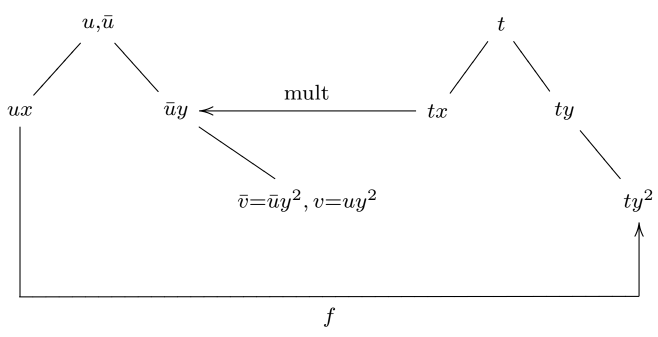
while W is given by

$$\varphi_W(v) \doteq \exists\, u\, (uy = 0 \wedge v = ux)\,.$$

The definition of f is more complex.

$$\varphi_f(v, \bar{v}) \doteq \exists\, u\, \bar{u}\, t\, (uxy = 0 \wedge v = uy^2 \wedge$$
$$\wedge \bar{u}xy = 0 \wedge \bar{v} = \bar{u}y^2 \wedge tx = \bar{u}y \wedge ty^2 = ux)\,.$$

To understand the definition of φ_f we suggest to look at the following diagram.



Thus we use t to 'twist' v via f to get $\bar{v}$.

Here are the details of this interpretation. First let us deal with $\varphi_V(v)$. Let us calculate the kernel of xy in M, so the image of this kernel in y^2 will give us $\varphi_V(M)$. Take an element of M, so the $N(0)$ class of the triple in N

$$(d + R_0y, e + R_1xy, g)$$

with $d \in R_0$, $e, g \in R_1$. Its image via xy is

$$(R_0 y, R_1 xy, gxy)\,,$$

and it is zero in M if and only if it equals in N an element (2) from $N(0)$

$$(ax + R_0 y, -a' y^2 + by + cx + R_1 xy, -bx - c^* y^2)\,,$$

where $a \in R_0$ and $b, c \in R_1$ satisfy $(*)$. Let us compare the elements of these triples componentwise. Equating the first coordinates we obtain $ax \in R_0 y$. If a_i is a unit for some $i < \beta$, then $a_i x \in Ry$ implies $x \in Ry$, whence $J = Ry$ is principal which is impossible for Drozd ring. By $(*)$ it follows that $a_i = 0$ for any i, whence a itself is 0, as well as a'.

Now comparing the second coordinates we obtain $by + cx \in R_1 xy$, hence $b_i y + c_i x \in Rxy$ for each $i < \alpha$. If b_i is a unit then $y \in Rx$, a contradiction. So by $(*)$ b_i is of the form $b_i(y) y$ where $b_i(y)$ is either 0 or invertible, therefore $b_i(y) y^2 + c_i x \in Rxy$. But then c_i has to be 0 otherwise by $(*)$ c_i invertible, hence $x \in \langle y^2, xy \rangle = J^2$, a contradiction. Because the sum of Ry^2 and Rxy is direct (see Fact 4.3), also $b_i(y)$ is zero. In conclusion $b = c = 0$ and hence we can assume $c^* = 0$ as well.

Equating the third coordinates we obtain $gxy = 0$. It follows that $g_i xy = 0$, hence $g_i \in J$ for each $i < \alpha$. To summarize, the generic element in the kernel of xy in M is the $N(0)$-class of a triple

$$(d + R_0 y, e + R_1 xy, g)\,,$$

where no component of $g \in R_1$ is a unit. Hence its image via y^2 is the class in M of

$$(R_0 y, e y^2 + R_1 xy, 0)\,.$$

Conversely, every element of this form can be seen as the y^2-image of the $N(0)$-class of the triple $(R_0 y, e + R_1 xy, 0)$ taken by xy to $(R_0 y, R_1 xy, 0)$, that is to zero.

Now observe that, for every e as before, $ey^2 \in R_1 xy$ iff $e \in R_1 \langle x, y \rangle = R_1 J$ whence the map

$$ey^2 + R_1 xy \mapsto e + R_1 J$$

yields an isomorphism between $\{ey^2+R_1xy \mid e \in R_1\}$ and $(R/J)^{(\alpha)} \cong k^{(\alpha)} \cong V$. In conclusion $\varphi_V(v)$ defines in M a k-vector space R_1/R_1J isomorphic to V.

Now let us consider W and $\varphi_W(v)$. First we calculate the kernel of y in M, so the image of this kernel in x will give us $\varphi_W(M)$. Take a generic element of M, so the $N(0)$-class of a triple $(d+R_0y, e+R_1xy, g)$ with $d \in R_0$ and $e, g \in R_1$. It image via y

$$(R_0y, ey + R_1xy, gy)\,. \tag{3}$$

is equal to 0 in M. Thus this element coincides in N with some element of the form (2)

$$(ax + R_0y, -a'y^2 + by + cx + R_1xy, -bx - c^*y^2)\,,$$

where $a \in R_0$ and $b, c \in R_1$ satisfy $(*)$. Let us compare this element with element (3) componentwise. As above equating the first coordinates one sees that a and a' must be 0. Comparing the two remaining coordinates we obtain for each $i < \alpha$,

(a) $-e_iy + b_iy + c_ix \in Rxy$ and

(b) $g_iy = -b_ix - c_i^*y^2$.

If c_i is a unit then (a) implies $x \in Ry$, which is impossible. Otherwise by $(*)$ $c_i = 0$ for every $i < \alpha$, hence $c = 0$ and then we may assume that $c^* = 0$.

In particular $c_i^* = 0$ for every i, hence (b) becomes $g_iy = -b_ix$. If b_i is invertible, this implies $x \in Ry$, a contradiction. Otherwise by $(*)$ again $b_i = b_i(y)y$, where $b_i(y)$ is either zero or invertible, hence (b) can be written as $g_iy = -b_i(y)xy$. Since $y \notin Rxy$ we conclude that g_i is not invertible. Decomposing $g_i \in J$ as in (1) we obtain $g_iy = g_i(x)xy + g_i(y)y^2$. Thus $g_i(x)xy + g_i(y)y^2 = -b_i(y)xy$ yields $g_i(y) = 0$ (since $y^2 \notin Rxy$ by Lemma 4.3). Because each g_i, $i < \alpha$, is not invertible and $g_i(y) = 0$ we conclude that $g_ix = 0$, hence $gx = 0$.

Now (a) can be written as

$$-e_iy + b_i(y)y^2 \in Rxy\,.$$

It follows that e_i is not a unit hence (see representation (1)) $e_ix \in Rxy$ for every $i < \alpha$ and then $ex \in Rxy$. Thus the image in x of our generic triple is

$$(d + R_0 y, e + R_1 xy, g) \cdot x = (dx + R_0 y, R_1 xy, 0)\,,$$

which is equivalent in M to

$$(R_0 y, -d' y^2 + R_1 xy, 0)\,.$$

Conversely every such element is an image in x of some element in the kernel of y. Thus the set of $N(0)$-classes of these triples yields the subspace W inside V isomorphic to $R_0/R_0 J \cong k^{(\beta)}$.

Finally let us deal with $\varphi_f(v, \bar{v})$. We claim that this formula defines f when $V \cong R_1/R_1 J$ is recovered in M in the way shown before.

First we prove that any two elements $v, \bar{v} \in V$ such that $f(v) = \bar{v}$ satisfy $\varphi_f(v, \bar{v})$. We can write $v = (R_0 y, ey^2 + R_1 xy, 0)$, $\bar{v} = (R_0 y, e^* y^2 + R_1 xy, 0)$, where

$$f(e + R_1 J) = e^* + R_1 J\,.$$

Choose $u = (R_0 y, e + R_1 xy, 0)$, $\bar{u} = (R_0 y, e^* + R_1 xy, 0)$, and put $t = (R_0 y, R_1 xy, -e^*)$. Clearly $uxy = 0$, $uy^2 = v$ and $\bar{u}xy = 0$, $\bar{u}y^2 = \bar{v}$ held in N and consequently in M. It remains to check that $tx = \bar{u}y$ and $ty^2 = ux$. Indeed

$$tx = (R_0 y, R_1 xy, -e^* x) \quad \text{and} \quad \bar{u}y = (R_0 y, e^* y + R_1 xy, 0)$$

represent the same class in M. Furthermore

$$ty^2 = (R_0 y, R_1 xy, -e^* y^2) \quad \text{and} \quad ux = (R_0 y, ex + R_1 xy, 0)$$

are equal in M because $f(e + R_1 J) = e^* + R_1 J$.

Conversely, take two elements $v, \bar{v} \in N$ such that their classes in M satisfy $\varphi_f(v, \bar{v})$. In particular there are $u, \bar{u} \in M$ such that $uxy = 0$, $uy^2 = v$ and $\bar{u}xy = 0$, $\bar{u}y^2 = \bar{v}$. By the first part of the proof (the description of the kernel of xy)

$$u = (d + R_0 y, e + R_1 xy, g) \quad \text{and} \quad \bar{u} = (\bar{d} + R_0 y, \bar{e} + R_1 xy, \bar{g})$$

for some $d, \bar{d} \in R_0$, $e, \bar{e} \in R_1$ and $g, \bar{g} \in R_1$ such that no component of g and $\bar{g}$ is a unit. It follows that

$$v = (R_0 y, ey^2 + R_1 xy, 0) \quad \text{and} \quad \bar{v} = (R_0 y, \bar{e} y^2 + R_1 xy, 0)\,.$$

Furthermore there exists $t \in M$ such that $ty^2 = ux$ and $tx = \bar{u}y$. Suppose that $t = (r + R_0 y, s + R_1 xy, h)$ for some $r \in R_0$ and $s, h \in R_1$. Thus

$$ty^2 = (R_0 y, sy^2 + R_1 xy, hy^2)$$

is equal in M to $ux = (dx + R_0 y, ex + R_1 xy, gx)$ which clearly coincides with

$$(dx + R_0 y, R_1 xy, gx - e^* y^2)\,.$$

Comparing ty^2 with ux we obtain that, for some $a \in R_0$ and $b, c \in R_1$ satisfying $(*)$, the following holds

(a) $dx - ax \in R_0 y$,

(b) $sy^2 + a'y^2 - by - cx \in R_1 xy$, and

(c) $hy^2 = gx - e^* y^2 - bx - c^* y^2$.

If some component of c is a unit, then (b) implies $x \in Ry$, a contradiction. Otherwise by $(*)$ $c = 0$, therefore we can assume $c^* = 0$ as well.

Recall that no component g_i of g is a unit and then observe that in (c) $g_i x$ has the form $g_i(y)xy$, where $g_i(y)$ is either 0 or invertible. Furthermore by (c) again, no component of b is a unit, whence $(*)$ yields $b_i = b_i(y)y$ where $b_i(y)$ is either 0 or invertible. So in (c) for every $i < \alpha$ we have

$$h_i y^2 = -e_i^* y^2 + (g_i(y) - b_i(y))xy\,.$$

Since $xy \notin Ry^2$ we conclude that $g_i(y) - b_i(y)$ is not a unit, therefore $(g_i(y) - b_i(y))xy = 0$ for any i. In conclusion $hy^2 = -e^* y^2$.

Now let us examine the further equality $tx = \bar{u}y$ in M. Note that

$$\bar{u}y = (R_0 y, \bar{e}y + R_1 xy, \bar{g}y)$$

and $tx = (rx + R_0 y, sx + R_1 xy, hx)$ which equals modulo $N(0)$ (hence in M)

$$(rx + R_0 y, sx - hy + R_1 xy, 0)\,.$$

Comparing these triples, we get, for a suitable choice of $a \in R_0$ and $b, c \in R_1$ satisfying $(*)$, the following.

(a′) $rx - ax \in R_0 y$,

(b′) $sx - hy - \bar{e}y + a'y^2 - by - cx \in R_1 xy$,

(c′) $\bar{g}y = -bx - c^* y^2$.

As above, (c′) implies that no component b_i of b can be a unit, hence by $(*)$ $b_i = b_i(y)y$, where $b_i(y)$ is either 0 or a unit. Similarly by (b′) no component of $s - c$ in R is a unit, whence $(s - c)x \in R_1 xy$ and (b′) becomes

$$-hy - \bar{e}y + a'y^2 - b(y)y^2 \in R_1 xy\,,$$

where $b(y) = (b_i(y))_{i<\alpha}$. Multiply by y and get $\bar{e}y^2 = -hy^2$, hence

$$\bar{e}y^2 = e^* y^2\,,$$

in other words $f(e + R_1 J) = \bar{e} + R_1 J$, as claimed. □

At this point it is straightforward to deduce (from Proposition 4.1 and Theorem 4.2) the following general result.

Corollary 4.4. *Suppose R is a finite commutative ring that is wild in the Klingler–Levy classification (that is, R projects itself onto an artinian triad or a Drozd ring). Then the theory of all R-modules is undecidable.*

5. Klein rings. The main result

The aim of the next five sections is to prove the following.

Theorem 5.1. *Let R be a finite commutative Klein ring. Then the theory of all R-modules is decidable.*

Actually this result will be shown only in Section 9, but the intermediate sections will prepare its proof and clarify several preliminaries.

First let us recall once again the definition of Klein ring (see Section 3): A finite local commutative ring R with Jacobson radical J (and residue field $k = R/J$) is said to be a *Klein ring* if J is 2-generated, J^2 is a principal ideal of R, $J^3 = 0$ and $x^2 = 0$ for every $x \in J$. In particular any Klein ring is an artinian ring of length 4.

Furthermore, it is easily shown (see [15, Lemma 2.9]) that the residue field of R has characteristic 2, and R itself has characteristic 2 or 4. As we have already mentioned, a typical example of a Klein ring is given by the group ring kG, where k is a field of characteristic 2 (finite in our case) and $G = C(2)^2$ is the Klein group. Another example, which is not an algebra over a field, is the ring $\mathbb{Z}/4\mathbb{Z}[x : x^2 = 0]$.

It follows from [16, Theorem 1.13] that every Klein ring is quasi-Frobenius with the socle J^2, therefore [14, Proposition 8.69] yields that every R-module is a direct sum of a free R-module and a module over the ring $R' = R/J^2$. Note also that, because R is artinian, the class of free R-modules is axiomatizable (see [21, Theorem 14.28]). Thus by [8, Theorem 5.4] to prove that the theory of all R-modules is decidable it suffices to show that the theory of free R-modules is decidable, and the theory of all R'-modules is decidable. The first case is easy (in fact it works for any finite ring).

Lemma 5.2. *The theory of free modules over a finite Klein ring is decidable.*

Proof. Every free R-module is of the form $R^{(I)}$ for some set I. By [8, Theorem 6.4] again the theory of this class is decidable if the theory of R_R is decidable. Because R is finite, all the elementary invariants $\mathrm{Inv}(R_R, \varphi, \psi)$ can be calculated effectively. Thus the theory T of R_R can be recursively axiomatized by invariant sentences. Since T is complete, it is decidable. $\square$

Thus to verify Theorem 5.1 it suffices to prove the following.

Proposition 5.3. *Suppose that R is a finite commutative ring whose Jacobson radical is 2-generated and satisfies $J^2 = 0$. Then the theory of all R-modules is decidable.*

Note that any further assumption on R, that is, R being a homomorphic image of a Klein ring, or having characteristic 2 or 4, is not essential for this proposition.

Incidentally it may be useful to observe that the characteristic 2 case is easy to treat. In other words the following holds.

Remark 5.4. *Let R be a finite (or even sufficiently recursive) Klein ring of characteristic 2. Then the theory of R-modules is decidable.*

In fact, by I. S. Cohen's Theorem, R, as a local commutative noetherian complete equicharacteristic ring, is an algebra over some field k of characteristic 2. Indeed R can be regarded as a quotient algebra of the algebra $k[X, Y : X^2 = Y^2 = 0]$ through a finitely generated ideal, and even equals this algebra itself or its quotient through the socle $k[X, Y : X^2 = Y^2 = XY = 0]$ [18], where k is finite (or sufficiently recursive). Hence R-modules, as a finitely axiomatizable subclass of the class of modules over $k[X, Y : X^2 = Y^2 = 0]$, have a decidable theory.

Before approaching the proof of Proposition 5.3 we should first explain quite a lot, as said. One technical tool we need is the so-called 'reduction modulo radical' functor (see [14, p. 221]). This will be the matter of the next section.

6. From Klein rings to $k\tilde{A}_1$

For the definition and basic properties of path algebras of quivers the reader can consult [2, Section III.1] or [1, Section II.1]. In this paper we need just one particular example of a path algebra. Let k be a field (with an intention to use R/J for k) and let $k\tilde{A}_1$ denote the *Kronecker algebra* over k. Thus $k\tilde{A}_1$ is the path algebra of the following quiver

$$\begin{tikzcd} 1 \circ \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"'] & \circ\ 2 \end{tikzcd}$$

In particular, $k\tilde{A}_1$ is a hereditary 4-dimensional algebra with the following basis: e_1 (the idempotent corresponding to the vertex 1), e_2 (the idempotent corresponding to the vertex 2), and α, β such that $e_1 \cdot \alpha = \alpha \cdot e_2 = \alpha$ and $e_1 \cdot \beta = \beta \cdot e_2 = \beta$ (all undefined products are set to be zero). Every $k\tilde{A}_1$-module can be seen as the direct sum of two vector spaces V and W over k with two linear maps α and β from V to W (see [1, Section III.1] for this way of defining modules over path algebras). We will sometimes denote it by (V, W).

One of the main tools in the oncoming proof of decidability will be, as already said, a functor F, called 'reduction modulo radical', from the category of R-modules to the category of $k\tilde{A}_1$-modules. In this section we gather some well known properties of F, mostly when it is restricted to the category of pure-injective modules.

First let us introduce F in detail. Choose generators x, y for J and recall that $k = R/J$ is a finite field.

Let M be a (right) R-module. Then M/MJ is an R/J-module, hence a vector space over k. From $J^2 = 0$ it follows that J annihilates MJ, hence MJ is also an R/J-module, therefore a vector space over k. We assign to M the following $k\tilde{A}_1$-module $F(M)$

$$\begin{tikzcd} M/MJ \arrow[rr, bend left, "\alpha = \times x"] \arrow[rr, bend right, "\beta = \times y"'] & & MJ\,, \end{tikzcd}$$

where $\alpha : M/MJ \to MJ$ is given by multiplication by x, and β is given by multiplication by y. Because $x, y \in J$ and $J^2 = 0$, both maps are well defined and, since R is commutative, k-linear. Thus we have defined F on objects.

To complete the definition of F, let N be another R-module and let $f : M \to N$ be a morphism (of R-modules). Thus, the $k\tilde{A}_1$-module $F(N)$ is given by the following diagram

$$\begin{tikzcd} N/NJ \arrow[r, bend left, "\alpha=\times x"] \arrow[r, bend right, "\beta=\times y"'] & NJ\,. \end{tikzcd}$$

Then (see [1, Section III.1] again) $F(f)$ has to be given by a pair of k-linear mappings $f_1 : M/MJ \to N/NJ$ and $f_2 : MJ \to NJ$ such that the two corresponding diagrams commute. One of them is

$$\begin{tikzcd} M/MJ \arrow[r, "\alpha=\times x"] \arrow[d, "f_1"'] & MJ \arrow[d, "f_2"] \\ N/NJ \arrow[r, "\alpha=\times x"'] & NJ\,, \end{tikzcd}$$

and the other is similar with α replaced by β.

Since $f : M \to N$ is a morphism, we have $f(MJ) \subseteq NJ$, hence define f_2 to be the restriction of f to MJ, and let $f_1 : M/MJ \to N/NJ$ denote the map induced by f.

It is easily verified that F is an additive covariant functor from the category of R-modules to the category of $k\tilde{A}_1$-modules. The properties of F we will list below are taken (or can be easily derived) from [14, Section at p. 211] and [2, Section X.2].

Clearly F preserves direct limits (hence direct sums) and direct products. Furthermore F is full, but not faithful. Namely, if M and N are R-modules, then the kernel of the mapping $\mathrm{Hom}(M,N) \to \mathrm{Hom}(F(M),F(N))$ is $\mathrm{Hom}(M,NJ)$, that is, consists of morphisms $f : M \to N$ whose image is contained in NJ. Since $J^2 = 0$, it follows easily that, for every R-module M, the kernel of the surjection $\mathrm{End}(M) \to \mathrm{End}(F(M))$ belongs to the Jacobson radical of $\mathrm{End}(M)$. It follows that F reflects (and preserves) the property of being isomorphic.

By [14, Proposition 8.61], F preserves and reflects pure-injectivity, that is if M is an R-module, then M is pure-injective iff $F(M)$ is a pure-injective $k\tilde{A}_1$-module. Furthermore if M is pure-injective, then M is indecomposable iff $F(M)$ is indecomposable. Note that every finite R-module is pure-injective so as every finite $k\tilde{A}_1$-module, and F preserves and reflects the property of being finitely generated (hence finite). Thus the above applies to finite modules.

Note that F is not a dense functor. Indeed, for every R-module M, its image $F(M) = (V, W)$ clearly has the property $VJ = V\alpha + V\beta = W$. For instance, the simple (projective) $k\tilde{A}_1$-module $S_2 = (0, k)$ is not isomorphic to $F(N)$, for any R-module N. However using projective covers, it can be shown that an $k\tilde{A}_1$-module $K = (V, W)$ is isomorphic to $F(M)$ for some R-module M iff $VJ = W$; and every $k\tilde{A}_1$-module L is of the form $L = F(N) \oplus L'$, where N is an R-module and L' is a direct sum of copies of S_2.

Note that $k\tilde{A}_1$ is isomorphic to the k-algebra $\left(\begin{smallmatrix} k & k\oplus k \\ 0 & k \end{smallmatrix}\right)$, where $k \oplus k$ is considered as a k-k-bimodule via the diagonal action of k. Thus we can apply the results of [2, Chapter X]. For instance (by [2, Theorem X.2.4]) the category mod-R of finite R-modules is stably equivalent to the category mod-$k\tilde{A}_1$ of finite $k\tilde{A}_1$-modules. Although it is beyond the scope of this paper, note that by [9, Corollary 3.9] this fact implies that the Krull–Gabriel dimension of R is 2.

Furthermore using a well known description of irreducible morphisms and almost split sequences in the category of finite dimensional $k\tilde{A}_1$-modules (see some examples below), we can derive a similar description of irreducible morphisms and almost split sequences in the category of finite R-modules (see [2, Chapter V] for definitions). Namely by [2, Proposition 2.5] a short exact sequence of finite R-modules $0 \to A \to B \to C \to 0$, where A and C are indecomposable and A is not simple, is almost split in mod-R iff $0 \to F(A) \to F(B) \to F(C) \to 0$ is an almost split sequence in mod-$k\tilde{A}_1$. Thus evaluating by F we can find 'almost all' almost split sequences in mod-R. Furthermore by [2, Lemma X.1.2] we have the following property of irreducible maps: if M and N are indecomposable finite non-projective R-modules, then the mapping $\operatorname{Hom}(M, N) \to \operatorname{Hom}(F(M), F(N))$ induces an isomorphism of k-vector spaces of irreducible maps $\operatorname{Irr}(M, N) \to \operatorname{Irr}(F(M), F(N))$.

Suppose that S is a simple R-module, that is, $S \cong R/J$. Then $F(S) = (R/J, 0) \cong S_1 = (k, 0) = e_1 k\tilde{A}_1/\operatorname{Jac}(e_1 k\tilde{A}_1)$ is a simple injective $k\tilde{A}_1$-module, in particular S_1 is not a source of an almost split sequence in mod-$k\tilde{A}_1$. However, since S is not injective, by [2, Proposition V.3.5] it is a source of an almost split sequence $0 \to S \to B \to C \to 0$ in mod-R, where B is a projective cover of C. It easily follows that, if $f_x : S \to R$ is given by multiplication by x, $f_y : S \to R$ is defined similarly, and $f = (f_x, f_y) : S \to R^2$, then $0 \to S \xrightarrow{f} R^2 \to R^2/\operatorname{im}(f) \to 0$ is an almost split sequence in mod-R. Therefore $f_x : S \to R$ is an irreducible morphism whose image $F(f_x) : S_1 \to e_1 k\tilde{A}_1$ is zero, hence not irreducible (but left almost split). This is because we cannot apply the aforementioned Auslander's result on irreducible maps when one of modules is projective.

Note also that $F(R) = (R/J, J) = (k, k \oplus k) \cong e_1 k\tilde{A}_1$ and every projective R-module is free, hence (see also [2, Lemma X.2.2]) F preserves (and reflects) projectives.

7. The Kronecker algebra $k\tilde{A}_1$

7.1. **Finite dimensional modules over $k\tilde{A}_1$.** In this section we recall the classification of indecomposable finite dimensional (hence finite, in our case) modules over the Kronecker algebra $k\tilde{A}_1$. Since $k\tilde{A}_1$ is a path algebra of the quiver without relations, it is hereditary, therefore the structure of finite dimensional $k\tilde{A}_1$-modules is well understood (see [2, Chapter VIII] or [27]). In particular indecomposable finite dimensional $k\tilde{A}_1$-modules can be divided in 3 subclasses

- the preprojectives,
- the regulars,
- the preinjectives.

One way to distinguish among modules in these classes is to look at the so called dimension vectors. If $M = (V, W)$ is a finite dimensional $k\tilde{A}_1$-module, then the *dimension* of M, $\dim(M)$, is defined as the ordered pair $(\dim(V), \dim(W))$.

Then every indecomposable preprojective $k\tilde{A}_1$-module has dimension $(n, n+1)$ for some integer $n \geq 0$, and there is one isomorphism type for each dimension. Here is the shape of an indecomposable preprojective module of dimension $(2, 3)$

$$\begin{tikzcd}[column sep=small] & \circ \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & & \circ \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ \circ & & \circ & & \circ \end{tikzcd}$$

which is a typical string module. The morphisms between preprojective modules are also well understood (so as, more generally, the morphisms between string modules). The following diagram of the category of preprojective $k\tilde{A}_1$-modules is taken from [28, p. 124].

$$\begin{tikzcd} & \mathbf{P}_2 & & \mathbf{P}_4 & \\ & (1,2) \arrow[dr, shift left] \arrow[dr, shift right] & & (3,4) \arrow[ll, dashed, "\tau"'] & \cdots \\ (0,1) \arrow[ur, shift left] \arrow[ur, shift right] & & (2,3) \arrow[ll, dashed, "\tau"'] \arrow[ur, shift left] \arrow[ur, shift right] & & \cdots \\ \mathbf{P}_1 & & \mathbf{P}_3 & & \end{tikzcd}$$

where solid arrows denote irreducible morphisms (so the space $\mathrm{Irr}(P_1, P_2)$ has dimension 2) and the dashed arrows show Auslander–Reiten (AR-) translates. For instance all nonzero morphisms between indecomposable preprojectives go from the left to the right, and each such morphism is mono.

Dually, every indecomposable preinjective $k\tilde{A}_1$-module has dimension $(m+1, m)$, $m \geq 0$, one isomorphism type for each dimension, and they are also string modules. Here is a diagram for an indecomposable preinjective module of dimension $(3, 2)$

$$\begin{tikzcd}[column sep=small]
\circ \arrow[dr, "\alpha"'] & & \circ \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \circ \arrow[dl, "\beta"] \\
& \circ & & \circ &
\end{tikzcd}$$

Furthermore the following diagram represents the category of preinjective $k\tilde{A}_1$-modules.

$$\begin{tikzcd}[column sep=small]
\cdots & \overset{\mathbf{I}_4}{(4,3)} \arrow[dr, shift left] \arrow[dr, shift right] & & \overset{\mathbf{I}_2}{(2,1)} \arrow[ll, dashed, "\tau"'] \arrow[dr, shift left] \arrow[dr, shift right] & \\
& \cdots & \underset{\mathbf{I}_3}{(3,2)} \arrow[ur, shift left] \arrow[ur, shift right] & & \underset{\mathbf{I}_1}{(1,0)} \arrow[ll, dashed, "\tau"]
\end{tikzcd},$$

where solid arrows stand for irreducible morphisms and dashed arrows show AR-translates. For instance two linearly independent irreducible epimorphisms from I_3 to I_2 are given by either factoring I_3 by utmost left part shown by bullets, or by utmost right part shown by diamonds.

$$\begin{tikzcd}[column sep=small]
\bullet \arrow[dr, "\alpha"'] & & \circ \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \diamond \arrow[dl, "\beta"] \\
& \bullet & & \diamond &
\end{tikzcd}$$

Note that the kernels of these two epimorphisms are nonisomorphic (simple regular) $k\tilde{A}_1$-modules.

Again, nonzero morphisms between indecomposable preinjective modules go from the left to the right in the above diagram, and each such morphism is epi. If k is finite, then indecomposable preprojective and preinjective $k\tilde{A}_1$-modules can be effectively given by generators and relations, so as irreducible maps between them.

The remaining indecomposable finite dimensional $k\tilde{A}_1$-modules are called *regular*. Each regular module has dimension (n, n), $n > 0$, but with many non-isomorphic modules of the same dimension. The easiest way to comprehend regular modules is to use a special functor G from the category of $k[X]$-modules to the category of $k\tilde{A}_1$-modules. Namely every $k[X]$-module M can be considered as a k-vector space V, where X acts as an endomorphism. Define $G(M)$ to be the following $k\tilde{A}_1$-module:

$$\begin{tikzcd} V \arrow[r, bend left, "\alpha=1"] \arrow[r, bend right, "\beta=X"'] & V \end{tikzcd},$$

where α acts as identity, and β acts as multiplication by X. The definition of $G(f)$, where f is a morphism of $k[X]$-modules, is obvious. Then G is a full and faithful additive covariant functor commuting with direct limits and (arbitrary) products. In particular (by [14, Proposition 8.61]) G preserves and reflects pure-injectivity and the property of being indecomposable (within the class of pure-injective modules). However G is not dense — its image consists of $k\tilde{A}_1$-modules (V, W) such that $\alpha : V \to W$ is an isomorphism.

Suppose that H is an indecomposable finitely generated torsion $k[X]$-module, therefore $H \cong k[X]/p^n(X)k[X]$, where $p(X)$ is an irreducible polynomial over k and $n \geq 0$. Then $G(H)$, the image of H, will be a regular indecomposable finite dimensional $k\tilde{A}_1$-module. For instance, if $p = X$ and $n = 1$, then $G(H)$ is the following string module

$$\begin{tikzcd} & \circ \arrow[dl, "\alpha"'] \\ \circ & \end{tikzcd}$$

and, if $p = X - 1$ and $n = 1$, we obtain the following band module (see [3] for more on string and band terminology)

$$\begin{tikzcd} \circ \arrow[d, bend right, "\alpha=1"'] \arrow[d, bend left, "\beta=1"] \\ \circ \end{tikzcd}$$

where α and β act as identity maps.

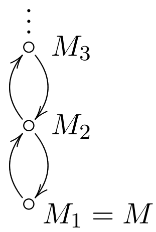
The functor G covers 'almost all' regular $k\tilde{A}_1$-modules, except a few. In fact the only regular $k\tilde{A}_1$-modules that are not covered are string modules like

$$\begin{tikzcd} & \circ \arrow[dl, "\beta"'] \arrow[dr, "\alpha"] & & \circ \arrow[dl, "\beta"] \\ \circ & & \circ & \end{tikzcd}$$

on which α does not act as an isomorphism (from V to W). One possibility to cure this situation is to consider the second functor G_1 similar to G, but where β acts as identity, and α acts as multiplication by X.

If k is finite, then we can produce a list of irreducible polynomials over k, hence (applying G or G_1) an effective list of regular $k\tilde{A}_1$-modules given by generators and relations. When drawing a diagram for the category of finite dimensional $k\tilde{A}_1$-modules, one should put regular modules on the right of preprojectives and on the left of preinjectives — then any nonzero morphism goes from the left to the right. For instance (see [28, p. 331]) there is no nonzero morphism from regular to preprojective modules.

Recall (see [2, Chapter VII]) that using irreducible maps we can arrange finite dimensional $k\tilde{A}_1$-modules into an Auslander–Reiten (AR-) quiver. We have already seen the connected components of this quiver consisting of preinjectives and preprojectives modules. The remaining components consist of regular modules and are homogeneous tubes.



The regular $k\tilde{A}_1$-module M on the mouth of the tube is called a *quasi-simple*, or *simple regular* module. Note (see [27]) that regular $k\tilde{A}_1$-modules form an abelian category, and simple regular modules are exactly simple objects in this category. Furthermore each quasi-simple module is isomorphic to the module $G(H)$ or $G_1(H)$, where H is a simple $k[X]$-module, therefore $H \cong k[X]/p(X)k[X]$ for an irreducible polynomial $p(X)$ of degree n. It follows that $\mathrm{End}(H) \cong \mathrm{End}(G(H))$ is a field K which is a simple extension of k of degree n, hence $|K| = |k|^n$. Clearly (since k is finite) this endomorphism ring, and hence the cardinality of $|K|$ can be calculated effectively.

The irreducible mappings in each tube are the images of irreducible morphisms between finitely generated torsion $k[X]$-modules, so they also can be calculated effectively. For example the multiplication by X defines an irreducible monomorphism $f : k[X]/Xk[X] \to k[X]/X^2k[X]$, hence its image $G(f)$ will be an irreducible monomorphisms $M_1 \to M_2$ in the corresponding tube of the AR-quiver.

Another important tool in the representation theory of $k\tilde{A}_1$ (more generally, any hereditary finite dimensional algebra) is a (non-symmetric) bilinear form $\langle -, - \rangle : K_0(k\tilde{A}_1) \to \mathbb{Z}$ defined by $\langle M, N\rangle = \dim \operatorname{Hom}(M, N) - \dim \operatorname{Ext}(M, N)$ with the corresponding quadratic form $q(M) = \langle M, M\rangle$ (see [2, Section VIII.3]). The value of this bilinear form depends only on dimensions of M and N. Namely if $\dim(M) = \bar{x} = (x_1, x_2)$ and $\dim(N) = \bar{y} = (y_1, y_2)$, then $\langle M, N\rangle = \langle \bar{x}, \bar{y}\rangle = x_1y_1 - 2x_1y_2 + x_2y_2$, therefore $q(M) = x_1^2 - 2x_1x_2 + x_2^2$. For instance $q(M) = 1$ for any preprojective or preinjective module M, and $q(N) = 0$ if N is a regular module.

Using this bilinear form it is not difficult to calculate Hom's and Ext's between finite dimensional $k\tilde{A}_1$-modules. Another possibility to do that is to use the AR-translate τ. Because $k\tilde{A}_1$ is hereditary, by [2, p. 258], τ provides an equivalence from the full subcategory of mod-$k\tilde{A}_1$ consisting of modules without projective direct summands onto the full subcategory of mod-$k\tilde{A}_1$ consisting of modules without injective direct summands. For instance, if M and N are indecomposable non-projective and non-injective $k\tilde{A}_1$-modules, then $\operatorname{Hom}(M, N) \cong \operatorname{Hom}(\tau M, \tau N)$ and $\operatorname{Ext}(M, N) \cong \operatorname{Ext}(\tau M, \tau N)$ (k-vector spaces isomorphisms), and the same is true with τ replaced by τ^{-1}.

Remark 7.1. *(cp.* [30, p. 114]*) If M is an indecomposable regular $k\tilde{A}_1$-module, then* $\operatorname{Hom}(M, I) \neq 0$ *for every preinjective $k\tilde{A}_1$-module I.*

Proof. Since I is preinjective, $I = \tau^k I'$ for some injective module I' and some $k \geq 0$. Because every tube over $k\tilde{A}_1$ is homogeneous, we conclude $M = \tau^k M$. Then $\operatorname{Ext}(M, I) = \operatorname{Ext}(\tau^k M, \tau^k I') \cong \operatorname{Ext}(M, I') = 0$, since I' is an injective module. It follows that $\langle M, I\rangle = \dim \operatorname{Hom}(M, I)$. From $\dim(M) = (n, n)$ and $\dim(I) = (m, m+1)$ for some $n > 0$, $m \geq 0$ we obtain $\langle M, I\rangle = \langle (n, n), (m+1, m)\rangle = n(m+1) - 2nm + nm = n > 0$, as desired. $\square$

7.2. **Pure-injective modules over $k\tilde{A}_1$.** Recall (from Section 2) that the *Ziegler spectrum* of a ring R, Zg_R, is a topological space whose points are indecomposable pure-injective R-modules, and whose topology is given by the basic open sets $(\varphi/\psi) = \{N \in \mathrm{Zg}_R \mid \varphi(N)/(\varphi \wedge \psi)(N) \neq 0\}$, where φ and ψ are pp-formulae (and restricting to formulae with only one free variable is sufficient). In this section we recall a description of the Ziegler spectrum of $k\tilde{A}_1$. All the results are taken from Ringel [30] and Prest [23], we only add some explanations.

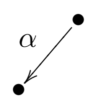
First of all each indecomposable finite dimensional $k\tilde{A}_1$-module is pure injective, therefore such modules are points of $\mathrm{Zg}_{k\tilde{A}_1}$. It follows from [23, Theorem 2.5] that every infinite dimensional point of $\mathrm{Zg}_{k\tilde{A}_1}$ is the image under G or G_1 of a non-finitely generated point of $\mathrm{Zg}_{k[X]}$. On the other hand recall (see [20]) that infinitely generated indecomposable pure-injective modules over $k[X]$ are classified as follows.

For every irreducible polynomial $p(X)$ there is a unique p-Prüfer module P_p and a unique p-adic module A_p. Furthermore there exists a unique generic module $Q = k(X)$. We will call the images of these points under G or G_1 the *Prüfer, adic and generic points* of $\mathrm{Zg}_{k\tilde{A}_1}$ (note that $G(Q) \cong G_1(Q)$, hence we have just one generic point Q). Let us say more precisely what those images are.

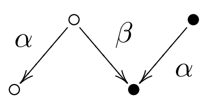
Let $p(X)$ be an irreducible polynomial over k. Then we have the following ray of irreducible monomorphisms in the category mod-$k[X]$ of finitely generated $k[X]$-modules:

$$k[X]/p(X)k[X] \xrightarrow{\times p} k[X]/p^2(X)k[X] \xrightarrow{\times p} \ldots.$$

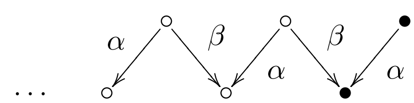
The direct limit along this ray is the p-Prüfer module P_p. Applying G or G_1 to this ray we obtain a ray of irreducible monomorphisms $M_1 \to M_2 \to \ldots$, where $M = M_1$ is the simple regular $k\tilde{A}_1$-module corresponding to $k[X]/p(X)k[X]$. The direct limit of this directed system will give us an indecomposable pure-injective (M-Prüfer) $k\tilde{A}_1$-module P_M. To grasp the shape of this module let us consider one example. Suppose that M is the following simple regular string module



Then M_2 has the following diagram



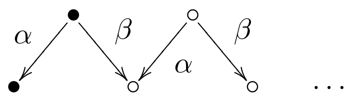
where the irreducible monomorphism $M_1 \to M_2$ is shown by identifying the bullets. Taking the direct limit along this ray, we obtain that P_M has the following diagram



Then (see [29, Section 4]) P_M is the so-called direct sum module — the underlying vector space for M is a direct sum of 1-dimensional spaces corresponding to vertices. Furthermore (see [29] again) this module is contracting — the shift by 1 to the right is a morphism of P_M which is epi but not mono (its kernel consists of the two utmost right vertices, therefore isomorphic to M). The construction of the M-adic module A_M is just dual. Namely we should start with a coray of irreducible epimorphisms

$$k[X]/p(X)k[X] \leftarrow k[X]/p^2(X)k[X] \leftarrow \dots$$

given by factoring out the socle, and apply G (or G_1) to get a coray of irreducible epimorphisms $M = M_1 \leftarrow M_2 \leftarrow \dots$ in mod-$k\tilde{A}_1$. The inverse limit along this coray will give us the M-adic module $A_M \in \mathrm{Zg}_{k\tilde{A}_1}$. For instance, if M is the simple regular string $k\tilde{A}_1$-module as above, then A_M has the following diagram:



Now A_M is a direct product module — its underlying vector space is the direct product of 1-dimensional spaces corresponding to the vertices. Furthermore this module is expanding — the shift by 1 to the right is a monomorphism f of A_M that is not epi, and whose cokernel (shown by bullets) is isomorphic to M.

Note that most Prüfer and adic modules will be counted twice (when applying G or G_1), but this does not disturb our proof of decidability.

Finally the generic $k\tilde{A}_1$-module Q has the following presentation

$$\begin{tikzcd} k(X) \arrow[r, bend left, "\alpha=1"] \arrow[r, bend right, "\beta=\times X"'] & k(X) \end{tikzcd},$$

where $k(X)$ is the field of quotients of $k[X]$. By [30, Proposition 4] if $f : A_M \to A_M$ is the aforementioned monomorphism, then the direct limit of the chain $A_M \xrightarrow{f} A_M \xrightarrow{f} \dots$ is isomorphic to a direct sum of copies of the generic module Q.

As we have already noticed in Remark 7.1, $\mathrm{Hom}(M, I) \neq 0$ for any regular $k\tilde{A}_1$-module M and any preinjective module I. Therefore by [30, Proposition 1] the Prüfer module P_M is a direct summand of a direct product of any infinite set of non-isomorphic preinjective modules.

Now we proceed describing the topology on $\mathrm{Zg}_{k\tilde{A}_1}$. It follows from [21, Proposition 13.4] that the isolated points of this space are exactly the finite dimensional ones. Furthermore because every finite dimensional $k\tilde{A}_1$-module has finite endolength, every such point is closed.

The next level of isolation is represented by Prüfer and adic points: those are exactly the points of Cantor–Bendixson (CB for short) rank 1. In fact Ringel [30] gives a nice basis of open neighborhoods of Prüfer and adic points.

Let M be a simple regular $k\tilde{A}_1$-modules and let O_M consist of P_M, all preinjective points, and all regular points on the tube with M on the mouth (often called *M-regular points*). Then O_M is a Ziegler open set containing P_M. In fact O_M consists of the points $N \in \mathrm{Zg}_{k\tilde{A}_1}$ such that $\mathrm{Hom}(M, N) \neq 0$. Furthermore every open subset of O_M containing P_M is cofinite, that is, of the form $O_M \setminus \{N_1, \ldots, N_k\}$, where the N_i are finite dimensional points. Moreover, because every finite dimensional point is closed, every such set is open.

Similarly, let J_M consist of A_M, all preprojective points, and all M-regular points. Then J_M is a Ziegler open set containing A_M. Moreover, every open subset of J_M containing A_M is of the form $J_M \setminus \{N_1, \ldots, N_k\}$, where the N_i are finite dimensional points. Note also that J_M consists of the $N \in \mathrm{Zg}_{k\tilde{A}_1}$ such that $\mathrm{Ext}(M, N) \neq 0$.

Finally the only remaining point of $\mathrm{Zg}_{k\tilde{A}_1}$ is the generic point Q. It has CB-rank 2, and the basis of open neighborhoods of Q is given by the collection of open sets $\mathrm{Zg}_{k\tilde{A}_1} \setminus \{N_1, \ldots, N_k\}$, where N_i are finite dimensional points. What follows is that $\mathrm{Zg}_{k\tilde{A}_1}$ has CB-rank 2, in particular it is a T_0-space. Then the elementary duality (see [12]) can be defined pointwise, hence provides a homeomorphism between the left and the right Ziegler spectra of $k\tilde{A}_1$. For instance the image of the right Prüfer point P_M is the left adic point $A_{M'}$, where M' is a left quasi-simple $k\tilde{A}_1$-module such that both M and M' are the images of the same simple $k[X]$-module. Note also (see [20]) that the functors G and G_1 induce homeomorphisms onto closed subsets of $\mathrm{Zg}_{k\tilde{A}_1}$.

7.3. $k\tilde{A}_1$. **More on the Ziegler spectrum.** In this section by refining the information on $\mathrm{Zg}_{k\tilde{A}_1}$ we prove the following.

Theorem 7.2. *The theory of all $k\tilde{A}_1$-modules (over a finite field k) is decidable.*

The result is definitely not new and our proof is hardly shorter. Indeed it is not difficult (see [18] for similar arguments) to interpret the theory of $k\tilde{A}_1$-modules in the theory of quadruples of k-vector spaces, hence decidability (over an effectively given field) follows from Baur's Theorem in [4]. But our main goal is to make a pattern which will be used in the next section to prove decidability in an essentially less friendly environment.

In the previous section we described the topology of $\mathrm{Zg}_{k\tilde{A}_1}$. However this description is not of the form that is required to prove decidability. For this purpose (see Fact 2.1), for each point $N \in \mathrm{Zg}_{k\tilde{A}_1}$, we should effectively produce a basis of open neighborhoods of N of the form (φ/ψ), where φ and ψ are pp-formulae, and calculate elementary invariants $\mathrm{Inv}(N, \varphi, \psi)$. Clearly this is the same as to produce a recursive set of axioms for the theory of N proving that this theory is decidable. The aim of this section is to obtain such a basis for every point of $\mathrm{Zg}_{k\tilde{A}_1}$.

Note that the basis for the Ziegler topology for N that can be extracted from Prest and Ringel (see previous section) is consisting of open sets of the form $\cap_{i=1}^{n}(\varphi_i/\psi_i)$, where φ_i and ψ_i are pp-formulae. Indeed, if P_M is a Prüfer point, then we can take $(\mathrm{Hom}(M, -)/x = 0)$ as (φ_1/ψ_1), and use the remaining (φ_i/ψ_i) to isolate, hence throw away any finite number of finite dimensional points. By [33, Theorem 4.9] there are pp-formulae φ and ψ such that $N \in (\varphi/\psi)$ and $(\varphi/\psi) \subseteq (\varphi_i/\psi_i)$ for every i. However it is not clear how to find such a pp-pair effectively and how to calculate the corresponding elementary invariant. Thus our proof will overcome this difficulty.

First we introduce some general notation. Suppose that $M = \langle \bar{x} \mid \bar{x}A = 0\rangle$ is a finitely presented module over a ring R given by generators and relations (here $\bar{x} = (x_1, \ldots, x_n)$ is a finite tuple of generators of M and A is a rectangular R-matrix). We will assign to this representation the pp-formula $\varphi \doteq \bar{x}A = 0$ (depending on the choice of representation). If N is an arbitrary R-module, then $\varphi(N) = \{\bar{n} \in N^n \mid \bar{n}A = 0\}$ can be identified with $\mathrm{Hom}(M, N)$ (both viewed as abelian groups, or even as modules over $S = \mathrm{End}(N)$). Although this identification depends on the representation of M, the condition $\mathrm{Hom}(M, N) \neq 0$ does not, and this is true exactly when $\varphi(N) \neq 0$. A more rigorous explanation is that the functors $\mathrm{Hom}(M, -)$ and φ from the category of R-modules to the category of abelian groups are isomorphic. Thus $(\varphi/x = 0)$ is an open set in the Ziegler spectrum of R

consisting of all $N \in \mathrm{Zg}_R$ such that $\mathrm{Hom}(M, N) \neq 0$. In fact this is a part of a more general construction.

Suppose that $N = \langle \bar{y} \mid \bar{y}B = 0 \rangle$ is another finitely presented R-module and let $f : M \to N$ be a morphism such that $f(\bar{x}) = \bar{y}C$ (hence $CA = BD$ for some matrix D with entries in R). Let ψ denote the pp-formula $\exists\, \bar{y}\, (\bar{y}B = 0 \wedge \bar{y}C = \bar{x})$, thus ψ generates the pp-type of $f(\bar{x})$ in N (see [21] for unexplained terminology). Then for any R-module L, we have that $\varphi(L)/(\varphi \wedge \psi)(L) \neq 0$ (that is, $L \in (\varphi/\psi)$) if and only if there is a morphism $g : M \to L$ that cannot be factored through f. Again, both the pp-formulae φ and ψ depend on the choice of representations, but the condition $L \in (\varphi/\psi)$ does not, because (φ/ψ) is isomorphic to the functor $\mathrm{Hom}(M, -)/I$, where I consists of the morphisms from M that can be factored through f. Considering the zero morphism $M \to 0$, we see that this construction includes the previous one.

If $f : M \to N$ is a left almost split morphism in the category of finitely presented R-modules, then (φ/ψ) is a minimal pair in the theory of R-modules (again see [21, Section 9.2] for a definition and properties of minimal pairs). If M has a local endomorphism ring, then (see [13, Proposition 5.3]) the pure-injective envelope of M is an indecomposable pure-injective module, hence a point of Zg_R, and this point is isolated by (φ/ψ).

That being said, let us come back to the points M in $\mathrm{Zg}_{k\tilde{A}_1}$. Our aim is to equip each of them with a basis as described before.

First suppose that M is a finite dimensional, hence an isolated point of $\mathrm{Zg}_{k\tilde{A}_1}$. As we have already mentioned, M can be effectively given by generators and relations. Assume first that M is regular, hence $M = M_n$ is the image under G (or G_1) of the $k[X]$-module $k[X]/p^n(X)k[X]$, where $p(X)$ is an irreducible polynomial. This module is the source of the left almost split monomorphism $k[X]/p^n(X)k[X] \to k[X]/p^{n+1}(X)k[X] \oplus k[X]/p^{n-1}(X)k[X]$, where the first coordinate mapping is given by multiplication by p, and the second coordinate one is factoring by the socle. The image of this morphism (under G or G_1) is an almost split monomorphism $f : M_n \to M_{n+1} \oplus M_{n-1}$ that can be calculated effectively. Then the minimal pair (φ/ψ) corresponding to f can be calculated effectively, and this pair isolates M_n. What remains to do is to calculate $\mathrm{Inv}(M_n, \varphi, \psi)$ effectively.

Since (φ/ψ) is a minimal pair in the theory of $k\tilde{A}_1$-modules and $M_n \in (\varphi/\psi)$, it is a minimal pair in the theory of M_n. By [21, Proposition 9.6], $\mathrm{Inv}(M_n, \varphi, \psi)$ is the cardinality of $\mathrm{End}(M_n)/\operatorname{Jac}\mathrm{End}(M_n)$. Since M_n has regular length n and is uniserial in the category of regular modules, it is easily seen that $\mathrm{End}(M_n)/\operatorname{Jac}\mathrm{End}(M_n)$ is isomorphic to the field $\mathrm{End}(M_1) = K$ (see Section 7.1), and (since k is finite) the cardinality of K can be calculated effectively.

Similarly if M is a finite dimensional indecomposable preprojective or preinjective $k\tilde{A}_1$-module, then from the description of morphisms between such modules (see Section 7.1) we can effectively construct a left almost split morphisms $f : M \to N$ in mod-$k\tilde{A}_1$, and this mapping defines a minimal pp-pair (φ/ψ) that isolates M. It is easily seen that $\mathrm{End}(M)/\operatorname{Jac}\mathrm{End}(M)$ is isomorphic to k, hence $\mathrm{Inv}(M, \varphi, \psi)$ is the cardinality of k.

Now consider the case of an M-Prüfer point $P_M \in \mathrm{Zg}_{k\tilde{A}_1}$, where M is a simple regular $k\tilde{A}_1$-module. Recall that O_M is a subset of $\mathrm{Zg}_{k\tilde{A}_1}$ consisting of P_M, all M-regular points, and all preinjective points. As we have already mentioned, $\mathrm{Hom}(M, N) \neq 0$ for $N \in \mathrm{Zg}_{k\tilde{A}_1}$ iff $N \in O_M$, hence O_M is an open subset of $\mathrm{Zg}_{k\tilde{A}_1}$ defined by an effectively given pair $(\varphi/x = 0)$. We will refine this pair to obtain a basis of open neighborhoods for P_M. The problem is to throw away any finite number of M-regular and preinjective points.

But first we address the problem of elementary invariants.

Proposition 7.3. $(\mathrm{Hom}(M, -)/x = 0)$ *is a minimal pair in the theory of* P_M. *If k is finite, then the corresponding elementary invariant of P_M is equal to* $|K|$.

Proof. Choose a representation $M = \langle \bar{x} \mid \bar{x}A = 0\rangle$, therefore a pp-formula $\varphi \doteq \bar{x}A = 0$ that represents $\mathrm{Hom}(M, -)$. Thus $\varphi(P_M)$ can be identified with $\{\bar{n} \in P_M \mid \bar{n}A = 0\}$.

Note (see [27]) that P_M is uniserial in the category of regular modules, and its regular socle is isomorphic to M. It follows that for every morphism $f : M \to P_M$, the image $f(\bar{x})$ is in the regular socle of P_M. Since $\mathrm{End}(M) = K$ is a (finite) field, every two such nonzero images, say $f(\bar{x})$ and $f'(\bar{x})$, can be identified by an automorphism of the socle. Thus $\varphi(P_M)$ has cardinality $|K|$.

It remains to prove that $(\varphi/x = 0)$ is a minimal pair in the theory of P_M. By [21, Proposition 9.6], it suffices to show that $\mathrm{End}(P_M)/\operatorname{Jac}\mathrm{End}(P_M) \cong K$. Since M is a regular socle of P_M, there is a natural mapping (given by restriction) $\pi : \mathrm{End}(P_M) \to \mathrm{End}(M) = K$. By [27, Proposition 4.7], P_M is injective in the category of $k\tilde{A}_1$-modules without preinjective direct summands. It follows that π is onto. Since $\mathrm{End}(P_M)$ is a local ring and K is a field, π is an isomorphism. $\square$

Now for any $n, r \geq 1$ we define a refinement (φ_n/ψ_r) of $(\varphi/x = 0)$ (in this case refinement means just that each φ_n implies φ). Let g_n be the composition of irreducible monomorphisms $M \xrightarrow{f_1} M_2 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} M_n$, and let φ_n generate the pp-type of the image $g_n(\bar{x})$ in M_n. Clearly φ_n implies φ for every n. In fact, the 'top' of the interval $(\varphi/x = 0)$ in the lattice of pp-formulae over $k\tilde{A}_1$ is a chain $\varphi > \varphi_1 > \varphi_2 > \dots$, that is, if any pp-formula ψ satisfies $\psi < \varphi$, then either ψ is equivalent to φ_s for some s, or $\psi \leq \varphi_t$ for every t. Indeed, if we consider the image of $g_n(\bar{x})$ under the almost split monomorphism $h : M_n \to M_{n+1} \oplus M_{n-1}$, only the first coordinate of $h(g_n(\bar{x}))$ is nonzero.

Now we define ψ_r. It is easily calculated that if $\dim(M) = (l, l)$, then $\dim \mathrm{Hom}(M, I_r) = l$. Let $h_1, \dots, h_l$ be a basis of $\mathrm{Hom}(M, I_r)$, and for every $i = 1, \dots, l$ let θ_i generate the pp-type of $h_i(\bar{x})$ in I_r. Then we set $\psi_r = \sum_{i=1}^{l} \theta_i$. Clearly all this can be calculated effectively.

We claim that (φ_n/ψ_r) is an open subset of O_M containing P_M but neither $M_1, \dots, M_{n-1}$ nor $I_{r-1}, \dots, I_1$. It would clearly follow that the (φ_n/ψ_r) form a basis of open neighborhoods for P_M.

First we show that $\psi_r \leq \varphi_n$ for every n and r, therefore every neighborhood (φ_n/ψ_r) is nontrivial. Because $\psi_r = \sum_{i=1}^{l} \theta_i$, it suffices to show that θ_i implies φ_n for every n (and i). By the definition of φ_n, it is enough to prove that the following diagram can be completed

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r, "g_n"] \arrow[d, "h_i"'] & M_n \arrow[dl, dashed, "h"] \\ & I_r & \end{tikzcd}$$

The cokernel of g_n is a regular module isomorphic to M_{n-1} and $\mathrm{Ext}(M_{n-1}, I_r) = 0$ by Lemma 7.1, hence h exists.

Now we check that $M_1, \dots, M_{n-1}$ do not belong to (φ_n/ψ_r). Indeed, otherwise a nonzero $\overline{m} \in M_s$, $s < n$ satisfies φ_n. In particular there exists a morphism $f : M_1 \to M_s$ such that $f(\bar{x}) = \overline{m}$. Since φ_n generates the pp-type of $g_n(\bar{x})$ in M_n, there is a morphism $h : M_n \to M_s$ such that the following diagram commutes

$$\begin{tikzcd}
 & M_s \\
M \arrow[ur, "f"] \arrow[dr, "g_n"'] & \\
 & M_n \arrow[uu, dashed, "h"']
\end{tikzcd}$$

Since M_n has regular length $n > s$ and is uniserial in the category of regular modules, h kills its regular socle M, hence $f(\bar{x}) = 0$, a contradiction.

Now we claim that $M_s \in (\varphi_n/\psi_r)$ for any $s \geq n$. Indeed by the construction, g_s factors through g_n, hence $g_s(\bar{x})$ satisfies φ_n in M_s. Suppose that this tuple satisfies ψ_r, hence $g_s(\bar{x}) = \overline{m}_1 + \cdots + \overline{m}_l$ for some $\overline{m}_j \in M_l$ such that $M_s \models \theta_j(\overline{m}_j)$ $(j = 1, \ldots, l)$. Since $h_j(\bar{x})$ is a free realization of $\theta_j(\bar{x})$ in I_r, there is a morphism $g : I_r \to M_s$ sending $h_j(\bar{x})$ to $\overline{m}_j$. But (see Section 7.1) every morphism from a preinjective to a regular module is zero, therefore $\overline{m}_j = 0$, and then $g_s(\bar{x}) = 0$, a contradiction.

Since P_M is a union of the M_s, $s \geq n$, it follows easily that $g_n(\bar{x}) \in M_n \subseteq P_M$ satisfies φ_n but not ψ_r in P_M, hence $P_M \in (\varphi_n/\psi_r)$.

It remains to prove that $I_s \notin (\varphi_n/\psi_r)$ for every $s < r$. Suppose that $0 \neq \bar{n} \in I_s$ satisfies φ_n. In particular, there exists a map $f : M \to I_s$ such that $f(\bar{x}) = \bar{n}$. Since $s < r$, there is an epimorphism $g : I_r \to I_s$ which is a composition of irreducible epimorphisms whose kernels are not isomorphic to M. This is possible because (see Section 7.1) at each step there are two irreducible morphisms with non-isomorphic regular kernels. We claim that the following diagram can be completed

$$\begin{tikzcd}
 & M \arrow[dl, dashed, "h"'] \arrow[d, "f"] & \\
I_r \arrow[r, "g"'] & I_s \arrow[r] & 0
\end{tikzcd}$$

Arguing by induction we may assume that $r = s + 1$. Then the kernel of g is a simple regular string module S of dimension $(1, 1)$. Since M is not isomorphic to S, therefore $\mathrm{Ext}(M, S) = 0$, which yields an extension as desired.

If $\overline{m} = h(\overline{x}) \in I_r$, then $I_r \models \psi_r(\overline{m})$ by the definition of that. Applying g we obtain $I_s \models \psi_r(\overline{n})$, hence $I_s \notin (\varphi_n/\psi_r)$.

Note that there is no problem in calculating the elementary invariants $\mathrm{Inv}(P_M, \varphi_n, \psi_r)$. Indeed, by Proposition 7.3, $(\varphi/x = 0)$ is a minimal pair in the theory of P_M, therefore all these invariants are equal to $|K|$.

A similar construction is possible for adic points. But to avoid technicalities we will use the following trick. Construct a basis of open neighborhoods for a left Prüfer point P_M as above, and then apply the elementary duality. Since A_M is dual to P_M, we obtain an effective basis of open neighborhoods for A_M. Moreover, by construction (or by the general theory — see [12]) the elementary invariants will be the same.

What remains is to construct an effective basis of open neighborhoods for the generic point Q of $\mathrm{Zg}_{k\tilde{A}_1}$ and calculate elementary invariants. Recall that Q has the following diagram

$$\begin{tikzcd} k(X) \arrow[r, shift left, "\alpha=1"] \arrow[r, shift right, "\beta=\times X"'] & k(X) \end{tikzcd},$$

Since Q is the image (via G) of the $k[X]$-module $k(X)$, it follows that $\mathrm{End}(Q) \cong k(X)$ is an infinite field. Hence every elementary invariant in Q is either 0 or ∞. Clearly $Q = Qe_1 \oplus Qe_2$ is a decomposition of Q as an $\mathrm{End}(Q)$-module, and both Qe_1 and Qe_2 are 1-dimensional vector spaces over $\mathrm{End}(Q)$ (recall the e_1 and e_2 denote the idempotents in the basis of $k\tilde{A}_1$ introduced at the beginning of Section 6). In particular (by [21, Proposition 9.6]) $(e_1 \mid x/x = 0)$ is a minimal pair in the theory of Q. Thus for every pp-formula φ implying $e_1 \mid x$ we have either $\varphi(Q) = 0$ or $\varphi(Q) = Qe_1$. It follows from [33, Theorem 4.9] that a basis of open neighborhoods for Q can be chosen among the pairs (φ/ψ) such that $\psi < \varphi \leq e_1 \mid x$. Thus the collection $\{(\varphi/\psi) \mid \psi < \varphi \leq e_1 \mid x \text{ and } \psi(Q) = 0, \varphi(Q) \neq 0\}$ forms a basis of open neighborhoods for Q.

As $T_{k\tilde{A}_1}$ is recursively axiomatizable, we can effectively list the pairs (φ/ψ) such that $\psi < \varphi \leq e_1 \mid x$. So the only thing to check is whether $\varphi(Q) = 0$ or not (and the same for ψ). We will reduce this question to a similar question about an adic module A_M. As we have already proved that the theory of A_M is decidable, this question can be answered effectively.

Choose any regular simple module M. Let A_M be the corresponding adic module, and let f be a monomorphism of A_M which is not epi, and whose cokernel is isomorphic to M (see an example in Section 7.2). Then, by [30, Proposition 4], the direct limit $A_M \xrightarrow{f} A_M \xrightarrow{f} \dots$ is isomorphic to a direct sum of copies of Q. Now it is easily seen that $\varphi(Q) = 0$ iff $\varphi(A_M) = 0$.

Thus we have constructed an effective basis of open neighborhoods for each point of $\mathrm{Zg}_{k\tilde{A}_1}$ and calculated the corresponding elementary invariants. By Fact 2.1 it follows that the theory of all $k\tilde{A}_1$-modules is decidable.

8. Klein rings again. The Ziegler spectrum

We deal here again with Klein rings. More precisely, we consider a finite (or countable effectively given noetherian) local commutative ring R whose Jacobson radical J is 2-generated and $J^2 = 0$. For instance every quotient of a Klein ring modulo its socle is such (see the beginning of Section 5). In this section we describe the Ziegler spectrum of R. Recall that in Section 6 we introduced a functor F, the reduction modulo radical, from the category of R-modules to the category of $k\tilde{A}_1$-modules, where $k\tilde{A}_1$ is the Kronecker algebra over the residue field $k = R/J$. In particular, this functor preserves and reflects pure-injectivity and the property of being isomorphic. Furthermore when restricted to the category of pure-injective modules F preserves and reflects the property of being indecomposable. Also F sends finite R-modules to finite $k\tilde{A}_1$-modules and vice-versa.

Recall (see Section 7.2) that every indecomposable pure-injective $k\tilde{A}_1$-module is preprojective, regular, preinjective, or Prüfer, adic, or generic. We say that an indecomposable pure-injective R-module is *preprojective (regular, ...)* if its image $F(M)$ is such. Of course, there are some drawbacks in this terminology. For instance if $S = R/J$ is the unique simple R-module, then $F(S) = I_1$ is a simple injective $k\tilde{A}_1$-module, although S is not injective.

One more caution should be taken. Recall that the image of F consists of $k\tilde{A}_1$-modules (V, W) such that $VJ = W$, and every $k\tilde{A}_1$-module is of the form $F(M) \oplus P_2^{(\alpha)}$, where P_2 is a simple projective $k\tilde{A}_1$-module. Thus every indecomposable pure-injective $k\tilde{A}_1$-module but P_2 is the image via F of the unique indecomposable pure-injective R-module. Since $\{P_2\}$ is clopen in $\mathrm{Zg}_{k\tilde{A}_1}$, therefore F induces a bijection from Zg_R onto the clopen set $\mathrm{Zg}_{k\tilde{A}_1} \setminus \{P_2\}$. In fact this map respects topology.

Proposition 8.1. *F induces a homeomorphism from Zg_R onto $\mathrm{Zg}_{k\tilde{A}_1} \setminus \{P_2\}$.*

Proof. Because R is an Artin algebra, by [2, Theorem V.1.15], the category of finite R-modules has almost split sequences. From [21, Proposition 13.1] it follows that every finite point of Zg_R (that is, every indecomposable finite R-module) is isolated, and those are the only isolated points of Zg_R. Thus F behaves well on isolated points.

Let P_M be an M-Prüfer point of Zg_R associated with a simple regular module M — this rather means that $F(P_M)$ is an $F(M)$-Prüfer point of $\mathrm{Zg}_{k\tilde{A}_1}$ associated with the simple regular module $F(M)$. Let O_M be a subset of Zg_R consisting of P_M, all M-regular points, and all preinjective points.

Recall (see Section 7.2) that $F(O_M)$ is open in $\mathrm{Zg}_{k\tilde{A}_1}$ and every open subset of $F(O_M)$ containing $F(P_M)$ is of the form $F(O_M) \setminus \{N_1, \dots, N_k\}$, where the N_i are finite points. Furthermore $F(O_M)$ is defined by a pair $(\varphi/x = 0)$, where φ corresponds to $\mathrm{Hom}(F(M), -)$ (see Section 7.3).

We will show that O_M is an open set in Zg_R, and every its open subset containing P_M is of the form $O_M \setminus \{M_1, \dots, M_k\}$, where the M_i are finite points. Consider the functor $\mathrm{Hom}(M, -)/\mathrm{Hom}(M, -)J$ whose value at an R-module N is (the abelian group) $\mathrm{Hom}(M, N)/\mathrm{Hom}(M, NJ) \cong \mathrm{Hom}(F(M), F(N))$. With respect to a given representation $M = \langle \bar{x} \mid \bar{x}A = 0\rangle$ of M this functor is isomorphic to the pp-pair (φ/ψ), where $\varphi \doteq \bar{x}A = 0$ and $\psi \doteq J \mid \bar{x}$, viewed as a functor. Therefore $O_M = (\varphi/\psi)$ is an open set which contains P_M.

Since every finite point in Zg_R is closed (being of finite endolength), each set $O_M \setminus \{M_1, \dots, M_k\}$ is open. Thus a restriction of F to O_M is continuous. To prove that this restriction is a homeomorphism it remains to check the following. If $T = \{M_1, M_2, \dots\}$ is an infinitely set of (pairwise non-isomorphic) finite points in O_M, then P_M belongs to $\overline{T}$, the closure of T. Clearly we may assume that either each M_i is M-regular, or each M_i is preinjective.

Suppose that each M_i is M-regular. Then lifting irreducible mappings from $k\tilde{A}_1$-modules, we can arrange the M_i into a ray $M_{n_1} \to M_{n_2} \to \dots$ by compositions of irreducible morphisms. Because F preserves direct limits, the direct limit along this ray is isomorphic to P_M, hence P_M is in the closure of T.

Assume now that every M_i is preinjective. By Ringel [30, Proposition 1], $F(P_M)$ is a direct summand of a direct product of the $F(M_i)$. Then P_M is a direct summand of a direct product of the M_i, hence $P_M \in \overline{T}$.

Thus F is a homeomorphism when restricted to the open neighborhood O_M of P_M. By dual arguments (or elementary duality) F preserves and reflects the basis of open neighborhoods of any adic point A_M.

It remains to look at the generic point Q. Recall that a basis of open neighborhoods for $F(Q)$ is given by the sets $\mathrm{Zg}_{k\tilde{A}_1} \setminus \{N_1, \dots, N_k\}$, where N_i are finite points. Since every finite point in Zg_R is closed, the preimage of this set in Zg_R is open.

Thus it suffices to prove that for every set $T = \{M_1, M_2, \dots\}$ of (different) points of $\mathrm{Zg}_R \setminus \{Q\}$ which either is infinite or contains an infinite point, Q is in the closure of T.

First we prove that $\overline{T}$ contains an infinite point. Indeed if T contains infinitely many preinjective points, then, as we have already seen, $\overline{T}$ contains (any) Prüfer point. By (elementary) duality, if T contains infinitely many preprojective points, its closure will contain all adic points.

If T contains infinitely many M-regular points for a given simple regular module M, then (as above, by using direct limits) it is easily shown that $P_M \in \overline{T}$. Thus it remains to consider the case when T contains infinitely many regular points $M_1, M_2, \dots$ with non-isomorphic regular socles. But in this case by [30, Proposition 5], Q is a direct summand of the module $\prod_i M_i / \oplus_i M_i$, hence $Q \in \overline{T}$.

Thus we may assume that either $P_M \in T$ or $A_M \in T$. Again by elementary duality it suffices to consider the latter case. As we have already mentioned, Q is a direct summand of a direct limit of copies of A_M, hence $Q \in \overline{T}$. □

Note that the functor F clearly defines a pp-interpretation (see [10, Definiton 1.1]) of the theory of $k\tilde{A}_1$-modules without P_2 as a direct summand in the theory of R-modules.

Question 8.2. *Is the theory of R-modules interpretable in the theory of $k\tilde{A}_1$-modules?*

If this were the case then most results of this section (on Zg_R) and of the next section (on decidability) would become trivial (modulo the corresponding results for $k\tilde{A}_1$-modules). However we believe that this not true, because too much 'individuality' of R is lost when applying F.

9. Klein rings. Decidability

In this section we prove Proposition 5.3, therefore Theorem 5.1. Recall that R is a finite commutative ring with Jacobson radical J such $J^2 = 0$ and J is 2-generated. By Fact 2.1 to prove decidability of the theory of all R-modules it suffices to equip each point of Zg_R with an effective basis of open neighborhoods and calculate the corresponding elementary invariants.

Recall (from the previous section) that each point of Zg_R is preprojective, regular, preinjective, or Prüfer, adic, or generic.

Suppose first that M is a finite, hence isolated point of Zg_R. If M is not injective, then it is a source of an almost split sequence that can be calculated effectively (say, using [2, Proposition V.2.2]). Let $M \to N$ be the corresponding left almost split morphism, and let (φ/ψ) be a pp-pair associated to this morphism as in Section 7.3. Clearly this pair can be found effectively and isolates M (see [21, Proposition 13.1]).

Similarly if M is an injective finite R-module, then the pp-pair (φ/ψ) corresponding to the (left almost split) mapping $M \to M/\operatorname{soc}(M)$ isolates M. Since M is finite, the elementary invariant $\operatorname{Inv}(M, \varphi, \psi)$ can be also calculated effectively.

Now we should do the same for an M-Prüfer point P_M (again this rather means that $F(P_M)$ is an $F(M)$-Prüfer point of $\mathrm{Zg}_{k\tilde{A}_1}$). As in Section 8 we can choose a representation $\langle \bar{x} \mid \bar{x}A = 0\rangle$ of M, and put $\varphi \doteq \bar{x}A = 0$ and $\psi \doteq J \mid \bar{x}$. Then for every R-module N we obtain

$$(**)\, \varphi(N)/(\varphi \wedge \psi)(N) = \operatorname{Hom}(M, N)/\operatorname{Hom}(M, NJ) \cong \operatorname{Hom}(F(M), F(N))\,.$$

Since $F(M)$ is a simple regular $k\tilde{A}_1$-module it follows (taking into account the description of $\mathrm{Zg}_{k\tilde{A}_1}$ in Section 7.2) that $(\varphi/\psi) = O_M$ is an open set of Zg_R consisting of P_M, all M-regular points, and all preinjective points. Using $(**)$ again we obtain $\varphi(P_M)/(\varphi \wedge \psi)(P_M) \cong \operatorname{Hom}(F(M), F(P_M))$, therefore, by Proposition 7.3, $\operatorname{Inv}(P_M, \varphi, \psi) = |K|$ which can be calculated effectively. Furthermore $\operatorname{End}(P_M)/\operatorname{Jac}\operatorname{End}(P_M)$ is clearly isomorphic to the ring $\operatorname{End}(F(P_M))/\operatorname{Jac}\operatorname{End}(F(P_M)) \cong K$ (see the proof of Proposition 7.3). It follows from [21, Proposition 9.6] that (φ/ψ) is a minimal pair in the theory of P_M.

Thus (arguing as in Section 7.3) it suffices to refine this pp-pair to a pp-pair (φ_n/ψ_r) throwing away any finite number of M-regular points $M_1, \ldots,$ M_{n-1} and any finite number of preinjective points $I_{r-1}, \ldots, I_1$.

We employ the same idea as in Section 7.3. Recall (see Section 6) that F restricted to the category of regular R-modules preserves and reflects irreducible morphisms. Let $g_n : M \to M_n$ be a composition of irreducible morphisms corresponding (via F) to irreducible monomorphisms in mod-$k\tilde{A}_1$, and let φ_n generate the pp-type of $g_n(\bar{x})$ in M_n. Clearly φ_n implies φ for every n. Before even defining ψ_r let us notice that already (φ_n/ψ) does not contain $M_1, \ldots, M_{n-1}$. Indeed, otherwise there exists a tuple $\overline{m} \in M_s$, $s < n$ such that $M_s \models \varphi_n(\overline{m}) \wedge \neg\,\psi(\overline{m})$, in particular $\overline{m} \notin M_sJ$. It readily follows that there are morphisms $f : M \to M_s$ and $h : M_n \to M_s$ such that $f(\bar{x}) = \overline{m}$ and the following diagram commutes

$$\begin{tikzcd}
 & M_s \\
M \arrow[ur, "f"] \arrow[dr, "g_n"'] & \\
 & M_n \arrow[uu, dashed, "h"']
\end{tikzcd}$$

Applying F to $M \xrightarrow{g_n} M_n \xrightarrow{h} M_s$, as in Section 7.3 we conclude that $F(f) = F(hg_n) = 0$, hence $f(M) = hg_n(M) \subseteq M_sJ$, a contradiction.

Note also that each M_s, $s \geq n$ satisfies φ on $g_s(\bar{x})$, therefore P_M satisfies φ on the direct image of this element (as we will see below $g_s(\bar{x}) \neq 0$ in P_M).

Now we are ready to define ψ_r. Let $\mathrm{Hom}(M, I_r) = \{h_1, \ldots, h_l\}$ and for every $i = 1, \ldots, l$ let θ_i generate the pp-type of $h_i(\bar{x})$ in I_r. Set $\psi_r = \sum_{i=1}^{l} \theta_i$. We claim that the interval $(\varphi_n/\psi_r + J \mid x)$ contains P_M but not $I_{r-1}, \ldots, I_1$ (neither $M_1, \ldots, M_{n-1}$ as we have already proved).

First let us check that each M_s, $s \geq n$ opens this pair on $g_s(\bar{x})$. We have already seen that $g_s(\bar{x})$ satisfies φ_n in M_s. Suppose that it satisfies $\psi_k + J \mid x$. Then $g_s(\bar{x}) = \overline{m}_1 + \cdots + \overline{m}_l + \overline{m}$ where $M_s \models \theta_j(\overline{m}_j)$ for every $j = 1, \ldots, l$ and $\overline{m} \in M_sJ$. Since $h_j(\bar{x})$ is a free realization of θ_j, there is a morphism $g : I_r \to M_s$ such that $g(h_j(\bar{x})) = \overline{m}_j$. Since $F(I_r)$ is a preinjective and $F(M_s)$ is a regular $k\tilde{A}_1$-module, it follows that $\mathrm{Hom}(F(I_r), F(M_s)) = 0$, in particular $F(g) = 0$. We conclude that $\overline{m}_j \in M_sJ$ for each j, hence $g_s(\bar{x}) \in M_sJ$. Since $\bar{x}$ generates M, it follows that $g_s(M) \subseteq M_sJ$. Applying F we obtain $F(g_s)(F(M)) = 0$, a contradiction. As in Section 7.3 it follows that $P_M \in (\varphi_n/\psi_r + J \mid \bar{x})$ for all n and r.

Before proving that $I_s \notin (\varphi_n/\psi_r + J \mid \bar{x})$ for each $s < r$, let us make a short remark. Suppose that M and N are regular R-modules. It follows from [2, Proposition IV.4.5 and Lemma X.2.3] that the modules $\mathrm{Ext}(N, \tau M)$ and $\mathrm{Hom}(M, NJ)$ have the same length. If M and N are simple regular R-modules from different tubes then using F we conclude that $f(M) \subseteq NJ$ for every morphism $f : M \to N$. However it is quite possible that $f \neq 0$, hence $\mathrm{Ext}(N, M) \neq 0$ (because $\tau(M) = M$). Thus at this stage we cannot directly repeat the arguments from Section 7.3, but we can proceed as follows.

To check that $I_s \notin (\varphi_n/\psi_r + J \mid \bar{x})$, it suffices to prove that if $\bar{n} \in I_s$, with $s < r$, satisfies φ_n, then $\bar{n}$ satisfies $\psi_r + J \mid \bar{x}$. Indeed, clearly there exists $f : M \to I_s$ such that $f(\bar{x}) = \bar{n}$; choose $g : I_r \to I_s$ as in Section 7.3. Thus we have the following diagram.

$$\begin{tikzcd} & M \arrow[d, "f"] \\ I_r \arrow[r, "g"'] & I_s \end{tikzcd}$$

Applying F we can complete the image of this diagram in mod-$k\tilde{A}_1$ by a map from $F(M)$ to $F(I_r)$ (see Section 7.3 again). It follows that there exists a morphism $h : M \to I_r$ such that $\bar{n} - gh(\bar{x}) \in I_s J$. But by the definition of that, $gh(\bar{x})$ satisfies $\psi_r = \sum_{i=1}^{l} \theta_i$, hence $\bar{n}$ satisfies $\psi_r + J \mid \bar{x}$.

Thus we have constructed an effective basis of open neighborhoods for P_M. Since each $(\varphi_n/\psi_r + J \mid x)$ is a refinement of the minimal pair (φ/ψ), it follows that $\mathrm{Inv}(P_M, \varphi_n, \psi_r + J \mid x) = |K|$. A similar basis of open neighborhoods for any adic point of Zg_R can be constructed using elementary duality (note that elementary duality preserves elementary invariants).

It remains to consider the generic point Q of Zg_R. First we will find a minimal pair in the theory of Q.

Lemma 9.1. *Let $\varphi \doteq x = x$ and let $\psi \doteq J \mid x$. Then (φ/ψ) is a minimal pair in the theory of Q and* $\mathrm{Inv}(Q, \varphi, \psi) = \infty$.

Proof. The functor (φ/ψ) is isomorphic to the functor $\mathrm{Hom}(R, -) \,/\, \mathrm{Hom}(R, -)J$. Since R goes to P_1 via F, for every R-module N we have $\varphi(N)/\psi(N) \cong \mathrm{Hom}_R(R, N)/\mathrm{Hom}(R, NJ) \cong \mathrm{Hom}_{k\tilde{A}_1}(F(R), F(N)) = \mathrm{Hom}(e_1 k\tilde{A}_1, F(N)) \cong F(N)e_1$.

For instance, $\varphi(Q)/\psi(Q) \cong F(Q)e_1 \cong k(x)$ is 1-dimensional over $\mathrm{End}(F(Q)) = k(X)$. Because $\mathrm{End}(Q)/\mathrm{Jac}\,\mathrm{End}(Q)$ is isomorphic to $\mathrm{End}\,F(Q)$, it follows that $\varphi(Q)/\psi(Q)$ is 1-dimensional over $\mathrm{End}(Q)/\mathrm{Jac}\,\mathrm{End}(Q)$. By [21, Proposition 9.6], (φ/ψ) is a minimal pair in the theory of Q. $\square$

By Ziegler [33, Theorem 4.9] the pp-pair (φ/ψ) can be refined to a basis of open neighborhoods of Q. Arguing as in Section 7.3 it suffices to decide effectively, for every pp-formula θ such that $J \mid x$ implies θ, whether $\theta(Q) = QJ$ or not. Applying F and [30, Proposition 4] we see that the direct limit $A_M \xrightarrow{f} A_M \xrightarrow{f} \dots$ of M-adic R-modules is isomorphic to a direct sum $Q^{(I)}$ of copies of Q. We claim that $QJ \subset \theta(Q)$ if and only if $A_M \in (\theta/J \mid x)$, and the last question can be answered effectively (as we have already proved the theory of A_M is recursively axiomatized, hence decidable).

Indeed, suppose that $A_M \models \theta(m)$ for some $m \notin A_M J$. Then the image of m in the above direct limit satisfies θ. If $m \notin Q^{(I)}J$, we are done. Otherwise

$m \in A_M J$ for some copy of A_M with a large enough index l. Using F we obtain the following diagram.

$$\begin{tikzcd} A_M/A_M J \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "F(f^l)"'] & A_M J \arrow[d, "F(f^l)"] \\ A_M/A_M J \arrow[r, shift left] \arrow[r, shift right] & A_M J \end{tikzcd}$$

But then $m \neq 0$ in the top copy of $A_M/A_M J$ while $F(f^l)(m) = 0$ in the bottom copy of $A_M/A_M J$, a contradiction, since $F(f^l)$ is mono.

Conversely, if $Q \models \theta(m)$ for some $m \in Q \setminus QJ$, then we obtain $m \in A_M \setminus A_M J$ and $A_M \models \theta(m)$ for some copy of A_M with a large enough index.

10. Conclusions

Recall that our ultimate goal is to classify finite commutative rings with decidable theories of modules. Thus let R be a finite commutative ring, we would like to know whether the theory T_R of all R-modules is decidable (or undecidable). We decompose R into a direct sum of local rings (this clearly can be done effectively) and then (see Remark 2.2) reduce the decidability problem to the local case. Thus we may assume that R is a finite local commutative ring with Jacobson radical J and residue field $k = R/J$.

First we control whether R has an artinian triad or a Drozd ring as a factor (clearly this can be checked effectively). If this is the case then by Corollary 4.4 the theory of all R-modules is undecidable.

Otherwise by the KL-dichotomy (see Theorem 3.1) R is a Klein ring or a homomorphic image of a complete local Dedekind-like ring. If R is a (finite) Klein ring then (by Theorem 5.1)the theory of all R-modules is decidable.

Otherwise R is a homomorphic image of a complete local (infinite) Dedekind-like ring with a finite residue field (the same as R). Since R is finite, R is a proper quotient of such a ring. Then it follows from [16, Section 11] that one of the following holds.

1) R is a finite valuation ring (hence R is of finite representation type and the theory of all R-modules is decidable);

2) there is a finite commutative valuation ring V whose residue field $K = V/\operatorname{Jac}(V)$ is a 2-dimensional extension of k, and R is isomorphic to the pullback

$$
\begin{tikzcd}
R \arrow[r, dashed] \arrow[d, dashed] & V \arrow[d, "\pi"] \\
k \arrow[r, phantom, "\subset"] & K
\end{tikzcd}
$$

3) there are two finite commutative valuation rings V_1 and V_2 with common residue field k such that R is isomorphic to the pullback

$$
\begin{tikzcd}
R \arrow[r, dashed] \arrow[d, dashed] & V_1 \oplus V_2 \arrow[d, "(\pi_1, \pi_2)"] \\
k \arrow[r] & k \oplus k\,,
\end{tikzcd}
$$

where $k \to k \oplus k$ is the diagonal embedding.

4) R has a simple socle (hence is quasi-Frobenius) and $R/\operatorname{soc}(R)$ is as in 1), 2) or 3).

Since all the properties of finite rings 1)–4) can be recognized effectively, we will eventually find a representation of R of the form 1)–3), or such a representation for $R/\operatorname{soc}(R)$. As we have already noticed in case 4) R is quasi-Frobenius, hence every R-module is a direct sum of a free module and $R/\operatorname{soc}(R)$-module. Thus arguing as after Theorem 5.1 (and case 1) being trivial) we may assume that R is of the form 2) or 3).

For instance a Gelfand–Ponomarev algebras $G_{2,3}(k)$ is in the class 3) with $V_1 = k[X : X^2 = 0]$ and $V_2 = k[Y : Y^3 = 0]$. Note that $G_{2,3}(k)$ is a non-domestic string algebra. Although some information on the Ziegler spectrum of this algebra has been obtained in [24], it is still far from being complete. Therefore the decidability of modules over this particular algebra is still a very much open problem.

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