# MICHAEL'S THEOREM FOR LIPSCHITZ CELLS IN O-MINIMAL STRUCTURES

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Abstract. A version of Michael's theorem for multivalued mappings definable in o-minimal structures with $M$-Lipschitz cell values ($M$ a constant) is proven.

**1. Introduction.** Assume that $R$ is any real closed field and an expansion of $R$ to some o-minimal structure is given. Throughout the paper we will be talking about definable sets and mappings referring to this o-minimal structure. (For fundamental definitions and results on o-minimal structures the reader is referred to [vdD] or [C].) In this article we adopt the following definition of a closed cell.

A subset $S$ of $R^m$ ($m \in \mathbb{Z}$, $m > 0$) will be called a *closed (respectively, closed $M$-Lipschitz) cell* in $R^m$, where $M \in R$, $M > 0$, if

(i) $S$ is a closed interval $[\alpha, \beta]$ ($\alpha, \beta \in R$, $\alpha \leqslant \beta$), or $S = [\alpha, +\infty)$, or $S = (-\infty, \alpha]$ ($\alpha \in R$), or $S = R$, when $m = 1$ and

(ii) $S = [f_1, f_2] := \{(y', y_m) : y' \in S',\ f_1(y') \leqslant y_m \leqslant f_2(y')\}$, where $y' = (y_1, \dots, y_{m-1}), S'$ is a closed (respectively, closed $M$-Lipschitz) cell in $R^{m-1}$, $f_i : S' \longrightarrow R$ ($i = 1, 2$) are continuous (respectively, $M$-Lipschitz) definable functions such that $f_1(y') \leqslant f_2(y')$, for each $y' \in S'$, or $S = [f, +\infty) = \{(y', y_m) : y' \in S', y_m \geqslant f(y')\}$, or $S = (-\infty, f] = \{(y', y_m) : y' \in S', y_m \leqslant f(y')\}$, or $S = S' \times R$, where $S'$ is as before and $f : S' \longrightarrow R$ is continuous (respectively, $M$-Lipschitz), when $m > 1$.

Let $F : A \rightrightarrows R^m$ be a multivalued mapping defined on a subset $A$ of $R^n$; i.e. a mapping which assigns to each point $x \in A$ a nonempty subset $F(x)$ of $R^m$. $F$ can be identified with its graph; i.e. a subset of $R^n \times R^m$. If this subset is definable we will call $F$ *definable*. $F$ is called *lower semicontinuous* if for each $a \in A$ and each $u \in F(a)$ and any neighborhood $U$ of $u$, there exists a neighborhood $V$ of $a$ such that $U \cap F(x) \neq \emptyset$, for each $x \in V$.

The aim of the present article is the following theorem.

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**Theorem 1.** *Let $F : A \rightrightarrows R^m$ be a definable multivalued, lower semicontinuous mapping defined on a definable subset $A$ of $R^n$ such that every value $F(x)$ is a closed $M$-Lipschitz cell in $R^m$, where a constant $M > 0$ is independent of $x \in A$. Then $F$ admits a continuous definable selection $\varphi : A \longrightarrow R^m$.*

The following generalization of Theorem 1 is immediate.

**Corollary 1.** *Let $F : A \rightrightarrows R^m$ be a definable multivalued, lower semicontinuous mapping defined on a definable subset $A$ of $R^n$. If there is a continuous definable mapping $\Phi : A \longrightarrow Aut(R^m)$ with values in the space of linear automorphisms*[1] *of $R^m$ such that $\Phi(x)(F(x))$ is a closed $M$-Lipschitz cell in $R^m$, then $F$ admits a continuous definable selection $\varphi : A \longrightarrow R^m$.*

Applying Theorem 1 to semilinear sets (see Remark 3 below) and taking into account that every closed semilinear cell is Lipschitz and for every semilinear family of semilinear cells they are $M$-Lipschitz with common $M$ [vdD, Chapter 1, (7.4)], we obtain the following application generalizing [AT, Theorem 4.10]

**Corollary 2.** *Let $F : A \rightrightarrows R^m$ be a semilinear multivalued, lower semicontinuous mapping defined on a semilinear bounded subset $A$ of $R^n$ such that every value $F(x)$ is a closed semilinear cell in $R^m$. Then $F$ admits a continuous semilinear selection $\varphi : A \longrightarrow R^m$.*

For other results on multivalued mappings in connection with o-minimal geometry we refer the reader to [AT1], [AT2] and [DP].

## 2. Proof of Theorem 1.

The proof will be by induction on $m$. Consider first the case $m = 1$. Then $F(x) = \{t \in R : f(x) \leqslant t \leqslant g(x)\}$, for each $x \in A$, where $f : A \longrightarrow R \cup \{-\infty\}$ and $g : A \longrightarrow R \cup \{+\infty\}$ are definable functions.[2] It is easy to check that $F$ is lower semicontinuous if and only if $g$ is lower semicontinuous and $f$ is upper semicontinuous. Therefore, the problem reduces to the following.

**Proposition 1.** *Let $f : A \longrightarrow R \cup \{-\infty\}$ and $g : A \longrightarrow R \cup \{+\infty\}$ be two definable functions such that $f(x) \leqslant g(x)$, for each $x \in A$, and $f$ is upper semicontinuous while $g$ is lower semicontinuous. Then there exists a definable continuous function $\varphi : A \longrightarrow R$ such that $f \leqslant \varphi \leqslant g$.*

To prove Proposition 1, which is a definable version of the Katětov-Tong Insertion Theorem, we need the following definable version of the Tietze Theorem.

**Theorem 2** (Definable Tietze's Theorem)**.** *Let $X$ and $Y$ be two definable subsets of $R^n$ such that $Y$ is closed in $X$. Then every definable continuous function $\psi : Y \longrightarrow R$ has a continuous definable extension $\Psi : X \longrightarrow R$.*

For a proof of Theorem 2 see [vdD, Chapter 8, (3.10)] (compare also [AF, Lemma 6.6]).

**Remark 1.** *According to [AT2, Theorem 3.3] Theorem 2 holds true in the semilinear o-minimal structure, provided that $Y$ is bounded.*

[1] The space $Aut(R^m)$ is naturally identified with a subset of $R^{m^2}$.

[2] This means that $f|f^{-1}(R)$ and $g|g^{-1}(R)$ are definable.

*Proof of Proposition 1.* We use induction on $d := \dim A$. The case $d = 0$ is trivial. Assume that $d > 0$. Let

$$B := \{a \in A : f \,\text{and}\, g \,\text{are both continuous in a neighborhood of}\, a \,\text{in}\, A\}.$$

Then $B$ is definable, open and dense subset of $A$. Hence $A \setminus B$ is definable closed in $A$ and $\dim(A \setminus B) < d$. By induction hypothesis there exists a definable continuous function $\psi : A \setminus B \longrightarrow R$ such that for each $x \in A \setminus B$, $f(x) \leqslant \psi(x) \leqslant g(x)$. By the Definable Tietze Theorem there exists a definable continuous extension $\Psi : A \longrightarrow R$ of $\psi$. Now put $\varphi(x) := \min(\max(\Psi(x), f(x)), g(x))$, for each $x \in A$. It is clear that $f \leqslant \varphi \leqslant g$. Continuity of $\varphi$ on $B$ is obvious, since $\Psi, f$ and $g$ are continuous on $B$. We are checking continuity at any $a \in A \setminus B$. Then $\varphi(a) = \psi(a) \in [f(a), g(a)]$. Fix any $\varepsilon > 0$. There exists a neighborhood $V$ of $a$ in $A$ such that $\psi(a) + \varepsilon > f(x), \psi(a) - \varepsilon < g(x)$, $\psi(a) + \varepsilon > \Psi(x)$ and $\psi(a) - \varepsilon < \Psi(x)$ for each $x \in V$. Then $\varphi(x) - \varepsilon = \psi(a) - \varepsilon < \Psi(x) \leqslant \max(\Psi(x), f(x)) < \psi(a) + \varepsilon = \varphi(a) + \varepsilon$ and $\varphi(a) - \varepsilon < g(x)$. Hence $\varphi(a) - \varepsilon < \varphi(x) = \min(\max(\Psi(x), f(x)), g(x)) < \varphi(a) + \varepsilon$.

**Remark 2.** *Since Theorem 2 holds true for the o-minimal structure of semilinear sets under the assumption that $X$ semilinear is bounded (see Remark 1), Proposition 1 holds true in this case too.*

Assume now that $m > 1$ and our theorem is true for $m - 1$. To make the induction hypothesis work we prove the following.

**Proposition 2.** *Under the assumptions of Theorem 1, let*

$$\pi : R^m \ni y = (y_1, \ldots, y_m) \longmapsto y' = (y_1, \ldots, y_{m-1}) \in R^{m-1}$$

*be the natural projection. Let $\pi \circ F : A \rightrightarrows R^{m-1}$ denote the composition defined by the formula $(\pi \circ F)(x) = \pi(F(x))$.*

*Then $F$ treated as a multi-valued mapping $F : \pi \circ F \rightrightarrows R$ is lower semi-continuous.*

*Proof of Proposition 2.* Put for each $x \in A$

$$F(x) = \{(y', y_m) : y' \in \pi(F(x)), y_m \in R, f_x(y') \leqslant y_m \leqslant g_x(y')\}.$$

Fix any $(a, b') \in \pi \circ F$, $u \in F(a, b') = \{y_m \in R : f_a(b') \leqslant y_m \leqslant g_a(b')\}$ and any open interval $U_\varepsilon := (u - \varepsilon, u + \varepsilon)$. Let $W$ be the open ball $\{y' \in R^{m-1} : |y' - b'| < \frac{\varepsilon}{4M}\}$, where $|\,.|$ is defined by $|\,y'| = |(y_1, \ldots, y_{m-1})| = \max_j |\,y_j|$. By lower semi-continuity of $F$ there exists a neighborhood $V$ of $a$ in $A$ such that $F(x) \cap (W \times U_{\frac{\varepsilon}{2}}) \neq \emptyset$, whenever $x \in V$.

Let now $(x, y') \in (\pi \circ F) \cap (V \times W)$. There exists $(z', v) \in F(x) \cap (W \times U_{\frac{\varepsilon}{2}})$. Then $y' \in \pi(F(x))$, $z' \in \pi(F(x))$; hence $|y' - z'| < \frac{\varepsilon}{2M}$ and $f_x(z') \leqslant v \leqslant g_x(z')$. Thus, if $f_x \not\equiv -\infty$, $|f_x(y') - f_x(z')| \leqslant M|y' - z'| < \frac{1}{2}\varepsilon$. Hence $f_x(y') \leqslant f_x(z') + \frac{1}{2}\varepsilon \leqslant v + \frac{1}{2}\varepsilon < u + \varepsilon$, also in the case when $f_x \equiv -\infty$. Similarly, if $g_x \not\equiv +\infty$, $|g_x(y') - g_x(z')| < \frac{1}{2}\varepsilon$ and consequently $g_x(y') \geqslant g_x(z') - \frac{1}{2}\varepsilon \geqslant v > u - \varepsilon$. Finally, $[f_x(y'), g_x(y')] \cap U_\varepsilon \neq \emptyset$, which ends the proof.

To finish the proof of Theorem 1, observe that the mapping $\pi \circ F$ is lower semi-continuous as a composition of a lower semicontinuous mapping with a continuous one, so by the induction hypothesis there exists a continuous definable selection $\varphi'$ for $\pi \circ F$. By Proposition 2, $F|\varphi' : \varphi' \rightrightarrows R$ is lower semi-continuous; hence, by Proposition 1, it admits a continuous definable selection $\sigma : \varphi' \longrightarrow R$, which gives a required selection $\varphi = (\varphi', \sigma \circ (id_A, \varphi'))$.

**Remark 3.** *Proof of Proposition 2 holds true for the o-minimal structure of semilinear sets, so in view of Remark 2, the Theorem 1 holds true for the semilinear structure under the assumption that $X$ semilinear is bounded.*

## 3. A counterexample.

We are going to present an example of a semialgebraic mapping $G : A \rightrightarrows R^2$, with $A \subset R^2$, which is not only lower semicontinuous, but even continuous with respect to the Hausdorff distance in the space of definable, closed, bounded and nonempty subsets, and which does not admit a continuous selection, although its values $G(x_1, x_2)$ are $M$-Lipschitz cells but not with a constant $M$ independent of $(x_1, x_2)$. Let $A = T_1 \cup T_2$, where

$$T_1 = \{(x_1, x_2) : x_1 \in [0, 1], -x_1 \leqslant x_2 \leqslant x_1\}$$

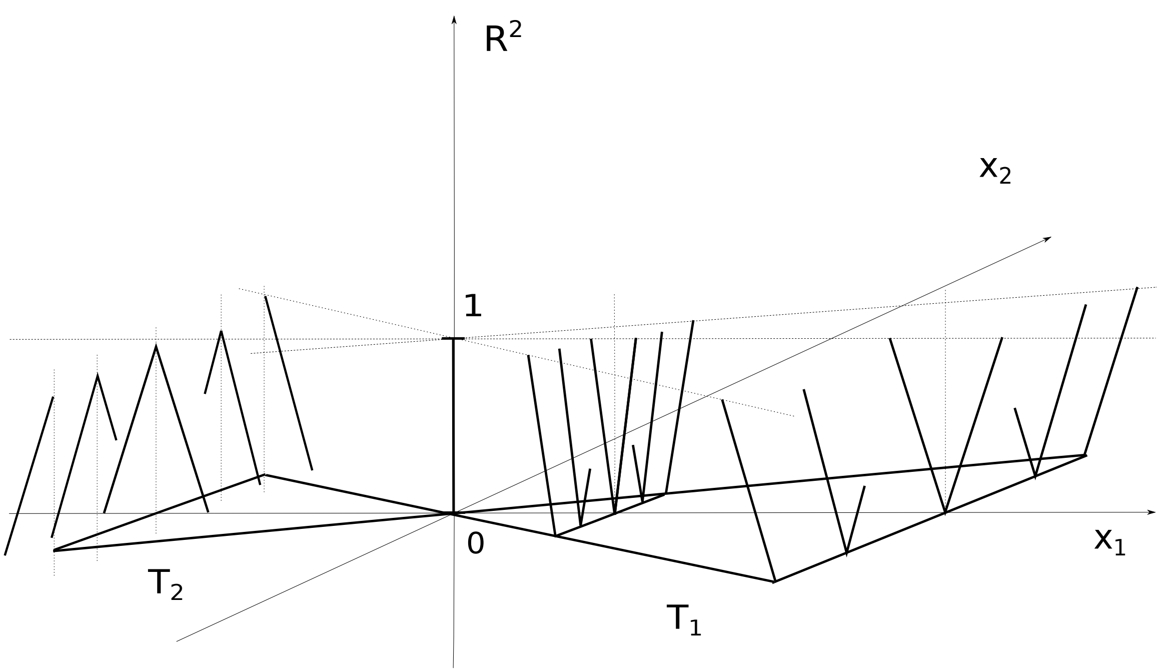
and

$$T_2 = \{(x_1, x_2) : x_1 \in [-1, 0], x_1 \leqslant x_2 \leqslant -x_1\}.$$

We define $G$ by the following

$$G(x_1, x_2) = \begin{cases} \{0\} \times [0, 1]\,, & (x_1, x_2) = (0, 0), \\ \left\{(y, \dfrac{|y|}{|x_1|}) : -x_1 + x_2 \leqslant y \leqslant x_1\right\}, & x_1 > 0, x_2 \geqslant 0, \\ \left\{(y, \dfrac{|y|}{|x_1|}) : -x_1 \leqslant y \leqslant x_1 + x_2\right\}, & x_1 > 0, x_2 \leqslant 0, \\ \left\{(y, 1 - \dfrac{|y|}{|x_1|}) : x_1 + x_2 \leqslant y \leqslant -x_1\right\}, & x_1 < 0, x_2 \geqslant 0, \\ \left\{(y, 1 - \dfrac{|y|}{|x_1|}) : x_1 \leqslant y \leqslant -x_1 + x_2\right\}, & x_1 < 0, x_2 \leqslant 0. \end{cases}$$

The graph of $G$ is imagined by the following picture.



Suppose that the mapping $G$ admits a continuous semialgebraic selection $\varphi = (\sigma, \rho) : A \longrightarrow R^2$. Then, for $x_1 > 0$, $\sigma(x_1, x_1) \geqslant 0$ and $\sigma(x_1, -x_1) \leqslant 0$; hence, there exists $\xi \in [-x_1, x_1]$ such that $\sigma(x_1, \xi) = 0$, so $\rho(x_1, \xi) = \frac{|\sigma(x_1,\xi)|}{|x_1|} = 0$ and $\varphi(x_1, \xi) = (0, 0)$. Consequently, by continuity, $\varphi(0, 0) = (0, 0)$. Similarly, for any $x_1 < 0$, there exists $\xi \in [x_1, -x_1]$, such that $\varphi(x_1, \xi) = (0, 1)$; hence $\varphi(0, 0) = (0, 1)$, a contradiction.

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