PILLAY'S CONJECTURE AND ITS SOLUTION-A SURVEY

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1. Introduction

These notes were originally written for a tutorial I gave in a Modnet Summer meeting which took place in Oxford 2006. I later gave a similar tutorial in the Wroclaw Logic colloquium 2007. The goal was to survey recent work in model theory of o-minimal structures, centered around the solution to beautiful conjecture of Pillay on definable groups in o-minimal structures. The conjecture (which is now a theorem in most interesting cases) suggested a connection between arbitrary definable groups in o-minimal structures and compact real Lie groups.

All the results discussed here have already appeared in print (mainly [28], [5], [24], [16]). The goal of the notes is to put the results together and to provide a direct path through the proof of the conjecture, avoiding side-tracks and generalizations which are not needed for the proof. This is especially true for the last paper in the list [16] which was often written with an eye towards generalizations far beyond o-minimality.

The last section of the paper has gone through substantial changes in the final stages of the writing. Originally, it contained several open questions and conjectures which arose during the work on Pillay's Conjecture. However, most of these questions were recently answered in a paper of Hrushovski and Pillay, [15], in which the so-called Compact Domination Conjecture has been solved. In another paper, [25], the assumptions for Pillay's Conjecture were weakened from o-minimal expansions of real closed fields to o-minimal expansions of groups. These recent results are now briefly discussed here. I also list some related work which appeared since the original conjecture was formulated.

The paper is aimed for readers who are familiar with the basic model theoretic language and with the introductory definitions of o-minimality (for more on o-minimality, see v.d. Dries' book, [6]).

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2. A motivating example and the conjecture

Before stating Pillay's conjecture, with all its technical terminology, let's consider the main motivating example.

Consider the group:

$$G = SO(2,\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL(2,\mathbb{R}) : a^2 + b^2 = 1 \right\}$$

G is isomorphic, as a Lie group, to the circle group. Namely,

$$G \simeq \mathbb{T}^1 = \{z \in \mathbb{C} : |z| = 1\},$$

with its complex field-multiplication. Both groups, together with their group operations and the isomorphism between them, are definable in the real field $\overline{\mathbb{R}} = \langle \mathbb{R}, <, +, \cdot, 0, 1\rangle$, so from a model theoretic view-point they are equivalent to each other.

Consider now a κ-saturated real closed field $\mathcal{R} \succ \overline{\mathbb{R}}$ (κ large). We write $G(\mathcal{R})$ for the realization of G in $\mathcal{R}$. Namely, $G(\mathcal{R}) = SO(2,\mathcal{R})$.

Because $SO(2,\mathbb{R})$ is a compact group the *standard-part mapping*, which sends every element of $\mathcal{R}$ of "finite" size to its nearest real element, induces a group-homomorphism $st : SO(2,\mathcal{R}) \to SO(2,\mathbb{R})$, defined by:

$$st \begin{pmatrix} a & b \\ -b & a \end{pmatrix} = \begin{pmatrix} st(a) & st(b) \\ st(-b) & st(a) \end{pmatrix}.$$

We have

$$ker(st) = \mu(I) = \bigcap_{n \in \mathbb{N}} \{A : |A - I| < 1/n\},$$

the intersection of countably many definable sets in $\mathcal{R}$.

One says in this case that $ker(st)$ is *type-definable*, i.e., it can be written as the intersection of less than κ-many definable sets.

The map $st(g)$ is a-priori just an abstract group homomorphism. The first observation of Pillay, [28], establishes a connection between definability in $G(\mathcal{R})$ and the Euclidean topology on G:

Two topologies on $G(\mathcal{R})/\mu(I)$

We identify $G(\mathcal{R})/\mu(I)$ with $SO(2,\mathbb{R})$ and denote by the *E-topology* its standard Euclidean topology. We define another topology on this quotient, called the *Logic topology* (L-topology), by: $F \subseteq SO(2,\mathbb{R})$ is *L-closed* iff $st^{-1}(F) \subseteq G(\mathcal{R})$ is *type-definable* in the ordered field structure on $\mathcal{R}$.

Logical compactness, together with the saturation of $\mathcal{R}$ relative to the size of $SO(2,\mathcal{R})/\mu(I)$ imply (see [28]) that the L-topology is compact and Hausdorff.

Fact 2.1. *A set $F \subseteq SO(2,\mathbb{R})$ is E-closed if and only if it is L-closed.*

Proof. Assume $F \subseteq SO(2,\mathbb{R})$ is L-closed. It follows that $st^{-1}(F) = p(\mathcal{R})$ for some type $p(x) = \{\phi_i(x) : i \in I\}$, with $|I| < \kappa$. We take g in the Euclidean closure of F and show that it belongs to F.

For all $n \in \mathbb{N}$, there exists $g' \in F$ such that $|g' - g| < 1/n$ (with $|\cdot|$ being the Euclidean distance). If we now take any $h \in st^{-1}(g')$ then, since h is infinitesimally close to g', we have $|h - g| < 1/n$ and moreover, $h \models p(x)$.

Because we can do the above for every n, we can replace the order of quantifiers (with the help of saturation) and obtain an element $h_0 \models p(x)$ such that for all $n \in \mathbb{N}$, $|h_0 - g| < 1/n$. This implies that $st(h_0) = g$ and therefore

$$p(\mathcal{R}) \cap st^{-1}(g) = st^{-1}(F) \cap st^{-1}(g) \neq \emptyset.$$

Clearly, this implies that $g \in F$.

For the converse, assume that $F \subseteq SO(2,\mathbb{R})$ is closed in the Euclidean topology. We will show that $st^{-1}(F)$ is type-definable.

Because $F \subseteq SO(2,\mathbb{R})$ is compact, for every $g \in SO(2,\mathbb{R}) \setminus F$ there is $n_g \in \mathbb{N}$ such that the distance between g and F is $> 1/n_g$.

Claim: $st^{-1}(F) = p(\mathcal{R})$ for the type:

$$p(x) = \{x \in SO(2,\mathcal{R}) \,\&\, |x - g| \geqslant 1/n_g \;:\; g \in SO(2,\mathbb{R}) \setminus F\}.$$

Indeed, assume that $st(h) = g' \in F$. Then, for every $g \in SO(2,\mathbb{R}) \setminus F$, we have $|g' - g| > 1/n_g$. Because h is infinitesimally close to g' we have $|h - g| > 1/n_g$. Hence, $h \models p(x)$.

For the opposite inclusion, assume that $h \notin st^{-1}(F)$. It follows that $st(h) = g \in SO(2,\mathbb{R}) \setminus F$, and therefore $|h - g| < 1/n_g$, and $h \notin p(\mathcal{R})$. □

Remarks

1. The type $p(x)$ defining $st^{-1}(F)$ is parameterized by a subset of $SO(2,\mathbb{R})$ hence uses at most $2^{\aleph_0}$-many formulas. Moreover, the type is given *uniformly*, namely there is a fixed formula $\phi(x,y)$ such that all formulas in p are of the form $\phi(x,b)$ for varying b's. As we will later see, this extra feature is still lacking in the general theory.

2. The quotient group $G(\mathcal{R})/\mu(I)$ ($\simeq SO(2,\mathbb{R})$) is independent of $\mathcal{R}$. I.e. every coset, even in elementary extensions, is already represented in $\mathcal{R}$. In such a case $\mu(I)$ is said to have *of bounded index* in G. An equivalent condition is that the cardinality of $G(\mathcal{R})/\mu(I)$ is smaller than κ (recall that $\mathcal{R}$ is κ-saturated). Note that if H is *a definable* subgroup of G of bounded index then the quotient is necessarily finite.

An example of a type-definable subgroup which is *not* of bounded index is the infinitesimal subgroup $\mu(0)$ of $\langle \mathcal{R}, <, + \rangle$. The quotient in this case is *not* $\langle \mathbb{R}, + \rangle$ because as the elementary extension extends one can realize more and more elements which are not infinitesimally close to each other.

3. The Logic topology on $G(\mathcal{R})/\mu(I)$ is *not* the quotient topology with respect to the topology of the real closed field, because $\mu(I)$ is open in this topology (so the quotient topology is discrete).

4. One can carry out the above process starting with any compact Hausdorff topological space X, instead of $SO(2,\mathbb{R})$, as long as a definable basis for the topology is uniformly definable. In this case, if we consider an elementary extension X^* of X then $\pi : X^* \to X$ is defined by: $\pi(x) =$ the unique $y \in X$ such that every X-definable open set containing y also contains x.

Generalizing the example

Assume now that we move in the opposite direction. Namely, we start with an arbitrary group G definable in an arbitrary (sufficiently saturated) o-minimal structure. The goal is to associate to G a real Lie group H and a surjective homomorphism $\pi : G \to H$ whose kernel is type-definable, such that the logic topology agrees with the Euclidean topology on H. Ideally, H should capture certain properties of G, such as dimension, the structure of torsion points, cohomological structure and elementary theory. This is the idea behind Pillay's Conjecture.

Before stating the conjecture in full we need to review some topological concepts in the theory of definable groups in o-minimal structures:

Assume that $\mathcal{M} = \langle M, <, \cdots \rangle$ is an o-minimal structure. $\mathcal{M}$ is an ordered structure and as such it is a topological space. The cartesian products of M admit the product topology. Now, if G is a definable group in $\mathcal{M}$ whose universe is a subset of M^n then the set G inherits the subspace topology from M^n but this might not be compatible with the group operation on G. (Consider for example, the the interval $[0, 1)$ in $\mathbb{R}$, with addition $mod\,1$. This is a definable group in the real field but the group operation is cont continuous with respect to the real topology).

A fundamental theorem of Pillay, [29], says: Let $\langle G, \cdot\rangle$ be definable group whose underlying set G is a subset of M^n. Then there exists a topology τ on G with the properties:
(1) For all $g \in G$ outside a definable set of small small dimension, if $\{U_s : s \in S\}$ is a basis for the open neighborhoods of g in M^n then $\{h \cdot U_s : s \in S, h \in G\}$ is a basis for τ.
(2) G, together with τ, is a topological group. Namely, the group operation, and the group-inverse map are continuous with respect to τ.

Actually, Pillay proves a much stronger result, as he shows that G can be covered by finitely many τ-open sets, each definably homeomorphic to an open subset of M^k for some fixed k (the o-minimal dimension of G). This implies for example, that just like definable sets in the o-minimal topology, every definable subset of G has finitely many definably τ-connected components (a set is called *definably τ-connected* if it not contained the disjoint union of two non-empty definable τ open sets).

It turns out, [6], that if $\mathcal{M}$ expands a real closed field then every definable group G is definably homeomorphic (with its τ-topology) to a a definable group $H \subseteq M^r$, for some r, such that the topology on H is the subspace topology. We call such an H an *affinely embedded* group.

Definable compactness

If one works in a sufficiently saturated o-minimal structure $\mathcal{M}$ then the underlying topology on M^n is very far from being locally compact. In fact, it is not difficult to see that *no* infinite definable subset of M is compact. Also, sequences are quite useless in this setting since the only converging sequences are those which are eventually constant.

What should be then the correct analogue of compactness? The first attempt is to restrict oneself to definable covers of open sets. However, this fails as the following example shows:

Consider the interval $[0, 1]$ in a nonstandard real closed field $\mathcal{R}$, and take $\alpha \in \mathcal{R}$ to be a positive infinitesimal. The family

$$\mathcal{U} = \{(x - \alpha, x + \alpha) : x \in [0, 1]\}$$

is a *definable* open cover but it has no finite subcover.

So, instead of using either open covers or converging sequences, we use "converging" definable curves (see [27]):

Definition 2.2. *A definable group G is* definably compact *if every definable continuous $f : (a, b) \to G$ has a limit point in G (with respect to the τ-topology), as t tends to either a or b in $\mathcal{M}$.*

Examples of definably compact groups

1. If $G \subseteq M^n$ is an affinely embedded group then G is definably compact if and only if it is closed and bounded. In particular, if we work over $\mathbb{R}$, the notions of definable compactness and compactness are the same for definable groups.
2. Compact real Lie groups are definably compact in any o-minimal structure in which they are definable.
3. If A is an abelian variety over a real closed field R then $A(R) =$ the set R-points of A, is definably compact.
4. The interval $[0, a)$, in any ordered divisible abelian, with addition $mod\, a$ is definably compact.

As a result of the work on Pillay's conjecture, and mainly as a result of the work of Dolich, [7], one obtains an equivalent definition for definable compactness, in terms of open covers:

Fact 2.3. [24] *G is definably compact if and only if every uniformly definable open cover of G which is parameterized by a complete type, has a finite sub-cover.*

Pillay's Conjecture

We are now ready to state Pillay's conjecture in full:

Pillay's Conjecture PC [28]*Let G be a definable group in a κ-saturated o-minimal structure $\mathcal{M}$ (large κ). Then:*

(1) *G has a minimal (minimum) type-definable normal subgroup of bounded index, call it G^{00}.*
(2) *G/G^{00}, equipped with the Logic topology, is isomorphic, as a topological group, to a compact Real Lie group.*
(3) *If G is definably compact then*

$$\dim_{Lie}(G/G^{00}) = \dim_{\mathcal{M}}(G).$$

The beauty of this conjecture is that it offers a surprising connection between the pure lattice of definable sets in definable groups in o-minimal structures and Real Lie groups. It implies that every definably compact group in an o-minimal (large) structure has a homomorphism onto a canonical Real Lie group that is associated to it. The pull-back under this homomorphism of every Euclidean closed set is type-definable and vice-versa. Such quotients are called in Model Theory "hyper-imaginaries" (in contrast to standard imaginaries, which are quotients of definable sets by definable equivalence relations).

Some examples

(1) If G an elementary extension of a compact Lie group H then, just as in the case of $SO(2,\mathbb{R})$, the group G^{00} is just $\mu(e) \cap G$ and $G/G^{00} \simeq H$. If G is definably isomorphic to such a group we say that G has *very good reduction*.

In these examples the choice of G^{00} is determined by the infinitesimals of the associated saturated real closed field $\mathcal{R}$, i.e by the valuation ring of $\mathcal{R}$. This is not the case in the next example.

(2) Consider a sufficiently saturated real closed field $\mathcal{R}$, α a positive infinitesimal element, and let $G = \langle [0,\alpha), + \bmod \alpha\rangle$. In this case the whole of G is contained in the kernel of the standard part map, so we need to use an "internal" notion of valuation:

$$G^{00} = \{g \in [0,\alpha) : \forall n \in \mathbb{N} \;\; g < \alpha/n \vee 1 - \frac{1}{n} < g < 1\}$$

and G/G^{00}, as a Lie group is again $SO(2,\mathbb{R})$.

(3) $G = \langle R, +\rangle$ (R a real closed field). In this case $G^{00} = G$, so G/G^{00} is trivial.

(4) A non-elementary example: Take $A(R)$ to be the R-points of an abelian variety A defined over a real closed field R, $\dim A = n$, and let $G = A(R)^0$ be its semi-algebraic connected component. By **PC**, there exists a homomorphism from G onto an n-dimensional real torus $\mathbb{T}^n$, whose kernel is type-definable in R, and such that the logic toplogy agrees with the Euclidean topology on $\mathbb{T}^n$.

The current status of PC.

The existence of G^{00}, and the fact that G/G^{00} is a Lie group was proven in [5] without any restrictions. **PC** is now proven in full when $\mathcal{M}$ expands a real closed field (the last step in the proof is in [16]). **PC** was also proved in the case when $\mathcal{M}$ is an ordered vector space over an ordered division ring, [19], [12].

It is still unknown whether **PC** holds in arbitrary o-minimal structures, or even in an o-minimal expansion of an ordered group. As we will point out, the only obstacle here is the understanding of torsion points in definably compact groups in such structures.

3. The existence of G^{00} and some corollaries

The material in this section is contained in [5]*.*

In [28] Pillay shows, for a group G definable anywhere, that the existence of G^{00} and the fact that G/G^{00} with the group topology is a compact Lie group are together equivalent to the Descending Chain Condition for type-definable subgroups of bounded index.

Throughout this section G is a definable group in an arbitrary o-minimal structure.

Theorem 3.1. [5]

(1) *G satisfies DCC for type-definable subgroups of bounded index. Namely, there is no infinite descending chain of type-definable subgroups of G of bounded index.*
(2) *If G is definably connected then G/G^{00} is connected.*

About the proof By [22], every definable group in an o-minimal structure has a definable normal solvable subgroup H such that G/H is semisimple, namely has no infinite definable normal abelian subgroup. DCC for a semisimple group follows from its decomposition into an almost direct product of definably almost simple groups (see [22]) and the fact that definably simple groups have very good reduction, [23]. By analyzing each abelian step which makes up the solvable group H, we are reduced to the abelian case, so we assume that G is abelian.

An important ingredient the proof is the notion of a *definably connected* type-definable set X. By that we mean that there are no definable open sets $U_1, U_2 \subseteq G$ such that $U_1 \cap X$ and $U_2 \cap X$ are both nonempty and pairwise disjoint. As is proved in the paper, every type-definable, definably connected subgroup of G has a type-definable subgroup of bounded index which is definably connected. This latter subgroup can be written as the directed intersection of definably connected sets.

Assume now that DCC fails. Then there exists a descending chain of type-definable subgroups of bounded index $H_1 > \cdots H_n > \cdots$, which we may assume are all definably connected. Using standard model theoretic arguments one may assume that all groups are definable over a countable model M_0 using a countable langauge. Let H be the minimal type-definable subgroup of bounded index definable over M_0 (this does exist!). Most of the work now is towards proving that G/H, equipped with the Logic topology, is a compact Lie group. That is done using topological arguments, together with the fact that G has a finite number of elements of every given finite order (see [32]). Once it is established that G/H is Lie group, the sequence H_i/H is a descending chain of closed subgroups, which is impossible.

Remark
1. In [30], Shelah proves the existence of G^{00} (but not DCC!) for any group with NIP and therefore in particular for o-minimal structures.

(*The following discussion and example were suggested by Pillay*):
2. There are two other related notions for a group G (in a sufficiently saturated structure): Consider all definable subgroups of G of finite index. If the intersection of these groups has bounded index (equivalently, the intersection does not change when we move to an elementary extension) then it is called G^0. In o-minimal structures and in groups of finite Morley rank, G^0 itself is definable and has finite index in G.

Another notion is that of G^{000}: For $A \subseteq M$ a small subset, let G^{000}_A be the smallest subgroup of G of bounded index which is invariant under automorphisms fixing A point-wise. If G^{000}_A does not depend on A then we call this group G^{000}.

The existence of G^{000} implies the existence of G^{00} and this in turn implies the existence of G^0. In stable theories all exists and are equal to each other.

It was shown by Shelah, [31], that if G is abelian and has NIP (see definition below) then G^{000} exists. Later on this was generalized by Gismatulin, [14], to an arbitrary group with NIP. However, it is still unknown in the NIP context (and even in the o-minimal case), whether $G^{00} = G^{000}$.

Example Consider the group $G = \langle \mathbb{Z}^\omega, \cdot \rangle$ in the two-sorted structure $\langle G; \mathbb{N} \rangle$, with a predicate $P \subseteq \mathbb{Z}^\omega \times \mathbb{N}$ such that $(x, n) \in P$ if and only if $x_n = 0$.

The theory of the structure says that for every $0 \neq g \in G$ there exists an $n \in \mathbb{N}$ such that $P(G, n)$ is a subgroup of index 2 which avoids g. This is easily seen to imply that the group G^0 (and therefore also G^{00} and G^{000}) does not exist in elementary extensions.

We now return to the o-minimal context. Here are two important corollaries of Theorem 3.1;

Corollary 3.2. *Assume that G is abelian. Then:*

(1) *G^{00} is divisible.*
(2) *Let H be a type-definable subgroup of bounded index. If H is torsion-free then $H = G^{00}$.*

Proof. (1) We need to see that for every n, the map $\sigma_n(x) = x^n$ sends G^{00} onto itself. It is easy to see that $\sigma_n(G^{00})$ has bounded index in $\sigma_n(G)$. However, σ_n has finite kernel, [32], and therefore $\dim \sigma_n(G) = \dim(G)$, so $\sigma_n(G)$ has finite index in G. It now follows that $\sigma_n(G^{00})$ has bounded index in G, and because it is contained in G^{00} it follows from minimality that $\sigma_n(G^{00}) = G^{00}$.

(2) The group H/G^{00} is a closed subgroup of the compact Lie group G/G^{00}, therefore either $H = G^{00}$ or H/G^{00} has torsion. If the latter holds then, because G^{00} is divisible H must have torsion as well. Contradiction. □

Corollary 3.3. *If G is torsion-free then $G^{00} = G$.*

Notice that up until now we have not even established that in a definably compact group we have $G^{00} \neq G$. Indeed, the main remaining difficulty in proving **PC** is the dimension equality:

Remaining Conjecture *If G is definably compact then* $\dim_M(G) = \dim_{Lie}(G/G^{00})$.

4. Some theory of generic sets I

Most of the material in this section is taken from [24].

Here G is definable in an o-minimal structure. However, some of the results work in any model theoretic setting, or at least when there is a reasonable notion of rank.

Definition 4.1. (1) *A set $X \subseteq G$ is called* left k-generic *if $G = \bigcup_{i=1}^{k} g_i X$, for some $k \in \mathbb{N}$ and $g_i \in G$. X is* left generic *if it is left k-generic for some $k \in \mathbb{N}$. X is* generic *if it is both left and right generic.*

(2) *If $X \subseteq G$ is definable then X is called* large *if* $\dim(G \setminus X) < \dim G$.

Remark In ω-stable connected groups the notions of "generic" and "large" are the same and both are equivalent to $RM(X) = RM(G)$. In o-minimal structures generic sets are not necessarily large and $\dim(X) = \dim(G)$ does not imply that X is generic:

1. In $\langle R, + \rangle$ (R a r.c.f), a set is generic if and only if it is of the form $(-\infty, a) \cup (b, +\infty)$, for $a, b \in R$.

2. In elementary extensions of $\mathbb{T}^1$ a definable set is generic if and only if it contains a segment of standard length.

Fact 4.2. *If X is large in G and $\dim(G) = n$ then X is (both left and right) n-generic.*

Proof. Without loss of generality, X is 0-definable.

We show: If g is generic in G and $h \in G \setminus (X \cup gX)$ then

$$\dim(h/g) < \dim(h/\emptyset) < n.$$

Indeed, if the left inequality fails then $\dim(h/g) = \dim(h/\emptyset)$ and hence (by the addition formula for dimension) we have $\dim(g/h) = \dim(g/\emptyset) = n$. It follows that

$$\dim(g^{-1}h/h) = \dim(g^{-1}/h) = \dim(g/h) = n.$$

In particular, $g^{-1}h$ is generic in G and because X was large we must have $g^{-1}h \in X$ and hence $h \in gX$, contradicting the assumption on h.

The inequality $\dim(h/\emptyset) < n$ follows from the fact that $h \in G \setminus X$ and X is large.

It follows from the above dimension inequality that $\dim(G \setminus (X \cup gX)) < \dim(G \setminus X) < dim(G)$. We now replace X by $X \cup gX$ and proceed by induction. □

Our goal in this section is to discuss the following result:

Theorem 4.3. [24] *Assume that G is a definably compact affinely embedded group, $\mathcal{M}$ expands an ordered group and and $X \subseteq G$ is not left-generic. Then $G \setminus X$ is right generic.*

Fact 4.4. *(i) If $X \subseteq G$ is not left-generic then $Cl(X)$ is not left-generic.*
(ii) If $X \subseteq G$ is generic then $Int(X)$ is also generic. (Here and below we use the τ-topology of G which was described above).

Proof. (i) We use the following basic fact about a definable set X in o-minimal structures: For $Fr(X) = Cl(X) \setminus X$, $\dim Fr(X) < \dim(X)$.

If $Cl(X)$ is left-generic then

$$G = \bigcup_{i=1}^{k} g_i Cl(X) = \bigcup_{i=1}^{k} g_i X \,\cup\, \bigcup_{i=1}^{k} g_i Fr(X).$$

But $\dim(Fr(X)) < \dim(G)$, hence $\dim(\bigcup_{i=1}^{k} g_i Fr(X)) < \dim(G)$, and therefore the set $\bigcup_{i=1}^{k} g_i X$ is large in G. By Fact 4.2, this last set is generic and therefore X is generic.

(ii) Use the fact that for any $X \subseteq G$, we have $\dim(X \setminus Int(X)) < \dim(G)$, and proceed as in (i). □

The connection of generic sets to Pillay's Conjecture comes through:

Fact 4.5. *If $H \subseteq G$ is a type-definable subgroup then H has bounded index in G if and only if it is the intersection of left generic sets.*

Proof. If H has bounded index and is contained in a definable set X then G can be covered by boundedly many left translates of X (namely the number of cosets of H). By compactness, finitely many left translates of X cover G.

If $H = \bigcap_{i<\lambda} X_i$ is the intersection of left generic sets, let $A = \{g_j : j < \lambda\}$ be a set of elements such that for every X_i, we have $G = AX_i$. Let $\mathcal{M}_0$ be a small model realizing all complete types over A. We claim that every coset of H has a representative in M_0. Indeed, if $g \in G$ then for every X_i there is $g_i \in A$ such that $g_i^{-1} g \in X_i$. By compactness we can find $h \in M_0$ such that $h^{-1} g \in \bigcap_i X_i = H$. □

Fact 4.6. *If $X \subseteq G$ is not left-generic then for any small $M_0 \subseteq M$ (where "small" means $|M_0| < \kappa$) there exists $g \in G$ such that $Xg \cap M_0 = \emptyset$.*

Proof. By assumption, for every $h_1, \ldots, h_k \in G$, $k \in \mathbb{N}$, there is $g \in G$ such that

$$\bigwedge_{i=1}^{k} g \notin h_i X,$$

or equivalently

$$\bigwedge_{i=1}^{k} h_i^{-1} \notin X g^{-1}.$$

Clearly then, for every $h_1, \ldots, h_k \in G$, $k \in \mathbb{N}$, there is $g \in G$ such that

$$\bigwedge_{i=1}^{k} h_i \notin Xg.$$

It follows that if M_0 is any small subset of M then, by the saturation of $\mathcal{M}$, there is $g \in G$ such that $M_0 \cap Xg = \emptyset$. □

Digression: Dolich's work In [7], Dolich examines the notion of forking and dividing in o-minimal structures. The paper contains many interesting and highly nontrivial results about types in o-minimal structures. In [24] we extract from his work the following:

Theorem 4.7. *Let $X(a) \subseteq M^n$ be a closed and bounded a-definable set in a sufficiently saturated o-minimal structure $\mathcal{M}$ expanding an ordered group and let $\mathcal{M}_0 \subseteq \mathcal{M}$ be a small model. Assume that the set $\{X(a') : a' \equiv_{M_0} a\}$ has the finite intersection property (namely, the intersection of every finite sub-family is nonempty).*

Then $X(a) \cap M_0 \neq \emptyset$.

The proof of this result is too long to discuss here. We only make few observations:
1. Consider the simplest example for 4.7, where $X(a_1a_2)$ is the closed interval $[a_1, a_2] \subseteq M$, and M_0 a small submodel of M. If $[a_1, a_2] \cap M_0 = \emptyset$ then $a_1 \equiv_{M_0} a_2$ (otherwise, by o-minimality, there is an interval J over M_0, containing one of the a_i's but not the other. It follows that one of the endpoints of this interval, which must be in M_0, also belongs to $[a_1, a_2]$. Contradiction). Moreover, because $a_1 \notin M_0$ there exists $a_2' < a_1$ with $a_2' \equiv_{M_0} a_1 \equiv_{M_0} a_2$. By homogeneity, there is $a_1' < a_2'$ such that $a_1'a_2' \equiv_{M_0} a_1a_2$, and hence $X(a_1'a_2') \cap X(a_1a_2) = \emptyset$, so the family

$$\{X(a_1', a_2') : a_1'a_2' \equiv_{M_0} a_1a_2\}$$

does not have the finite intersection property.

2. Theorem 4.7 is false when $X(a)$ is not closed and bounded. Consider for example the open interval $(0, \alpha)$ in a nonstandard real closed field, for an infinitesimal $\alpha > 0$, and take M_0 to be the real algebraic numbers. The family

$$\{(0, \alpha') : \alpha' \equiv_{M_0} \alpha\}$$

is finitely satisfiable but $(0, \alpha) \cap M_0$ is empty.

3. In the stable case, the analogous theorem to 4.7 is true for any definable set because the assumption is equivalent to the forking of $X(a)$ over $\mathcal{M}_0$.

4. The description of a definably compact set using a type-definable open covering (see Fact 2.3) easily follows from Fact 4.7.

End of Digression

Proof. of Theorem 4.3

Because G is affinely embedded it is closed and bounded in M^k. Assume $X \subseteq G$ is not left generic. By 4.4 we may assume that X is closed. Fix $\mathcal{M}_0$ such that X is definable over M_0. By 4.6, there exists $g \in G$ such that $M_0 \cap Xg = \emptyset$. By 4.7, there are $g_1, \ldots, g_k$ (each realizing the same type as g over M_0) such that

$$\bigcap_{i=1}^{k} Xg_i = \emptyset.$$

By taking complement we get

$$\bigcup_{i=1}^{k}(G \setminus X)g_i = G.$$

Hence, $G \setminus X$ is right-generic. $\square$

Remarks

1. Theorem 4.3 fails without the definable compactness assumption: The set $(a, +\infty)$ and its complement are both not generic in $\langle R, + \rangle$ (here left and right-generic are the same).

2. The analogue of Theorem 4.3 in the stable setting is true for any definable subset of the group G.

3. Recently, Eleftheriou has pointed out that the assumption that G is affinely embedded can be omitted Theorem 4.7 by working in the manifold charts of G. Also, the assumption that $\mathcal{M}$ expands an ordered group seems to be unnecessary.

Here are two easy consequences:

Fact 4.8. *Assume that G is definably compact and abelian.*

(1) *The non-generic sets form an ideal.*
(2) *Every formula defining a generic set in G belongs to a complete "generic" type p (over $\mathcal{M}$). Namely, every formula in p defines a generic set in G.*

5. Some Theory of generic sets II: Measure and the NIP

The content of Sections 4-8 is mostly taken from [16]. *The connection between the Non Independence Property and measure is due to Keisler* [17] *and the proof of 5.4 below is modeled after a proof from Keisler's paper.*

The next notion is due to Shelah. The definition we use is equivalent to the original one.

Definition 5.1. *A theory T is said to be* dependent *or to have the* Non Independence Property *(NIP) if for every indiscernible sequence $\langle a_i : i < \omega \rangle$ over A and $\phi(x, y)$ a formula over A the type $\{\phi(x, b_{2j}) \triangle \phi(x, b_{2j+1}) : j < \omega\}$ is inconsistent. (We take $\phi \triangle \psi$ to mean $(\phi \wedge \neg\psi) \vee (\neg\phi \wedge \psi)$)*

Stable theories, o-minimal theories, the theory of p-adically closed fields all have the NIP, while the theory of pseudo-finite fields fails to have it.

Definition 5.2. *We say that G admits a left invariant* Keisler measure *if there exists a real valued finitely additive measure $\mu : Def(G) \to \mathbb{R}$ on the definable subsets of G, such that $\mu(G) = 1$ and for every definable $X \subseteq G$ and $g \in G$, we have $\mu(gX) = \mu(X)$.*

In the rest of this section we make the following assumptions on the group G (equipped with the definable sets induced by the ambient structure):

- *G has NIP.*
- *The non left-generic sets form an ideal.*
- *G admits a left-invariant Keisler measure.*

As we will eventually show, every definably compact satisfies all of the above. For now, notice that any abelian definably compact group satisfies the above assumptions. Indeed, o-minimality implies NIP, and by Fact 4.8 the non-generics form and ideal. Because every abelian group is amenable, it admits a left-invariant real valued finitely additive probability measure on all subsets.

Definition 5.3. *For $X, Y \subseteq G$ definable, we write $X \approx_{ng} Y$ if $X \triangle Y$ is not left-generic.*

Notice that since the non left-generics form an ideal $\approx_{ng}$ is an equivalence relation. The NIp assumption is crucial for the following.

Lemma 5.4. *The equivalence relation $\approx_{ng}$ is bounded. I.e., there exists a fixed small model M_0 such that every equivalence class of $\approx_{ng}$ is already represented by a definable set over M_0.*

Proof. Let μ denote the finitely additive left-invariant measure on $Def(G)$, the family of definable subsets of G. Note that if $X \subseteq G$ is a definable n-generic set then we have $\mu(X) \geqslant 1/n$.

Assume that $\approx_{ng}$ is unbounded. Then there exists a formula $\phi(x, y)$ over the empty set, with the variable x for elements in G, and unboundedly many b_i's, such that $\phi(G, b_i) \triangle \phi(G, b_j)$ is generic.

By standard Ramsey-type arguments, there exists a fixed n and an indiscernible sequence $\langle a_i : i < \omega \rangle$ such that for every $i \neq j$, the set $\phi(G, a_i) \triangle \phi(G, a_j)$ is n-generic.

Consider the family $\mathcal{F} = \{Y_j = \phi(G, a_{2j}) \triangle \phi(G, a_{2j+1}) : j < \omega\}$. By indiscernibility, there exists a natural number k such that every k sets in $\mathcal{F}$ have empty intersection. However, for every j, $\mu(Y_j) \geqslant 1/n$, and because $\mu(G) = 1$ it is impossible that every k sets in $\mathcal{F}$ intersect trivially. Contradiction. $\square$

Definition 5.5. *For $X \subseteq G$ definable, let*

$$Stab_{ng}(X) = \{g \in G : gX \approx_{ng} X\}.$$

Under our assumptions, the set $Stab_{ng}(X)$ is a subgroup of G. It is type-definable because for every n, the set of all g such that n translates of $gX\triangle X$ *do not* cover G, is definable. The map $g \mapsto gX/\approx_{ng}$ is an injective map from $G/Stab_{ng}(X)$ into $Def(G)/\approx_{ng}$ and therefore we proved

Theorem 5.6. *For any definable $X \subseteq G$, the subgroup $Stab_{ng}(X)$ has bounded index in G.*

6. The proof of **PC** in the abelian case

We assume in this section that $\mathcal{M}$ expands a real closed field and that G is definably compact and abelian.

Our goal here is to prove:

Theorem 6.1. *If G is definably compact and abelian then $\dim_{Lie}(G/G^{00}) = \dim_{\mathcal{M}} G$.*

Proof. Because G^0 has finite index in G we may assume that G is definably connected.

The proof is based on two ingredients. The first one is a deep theorem of Edmundo and Otero on the torsion points in definably connected, definably compact abelian groups. (Presumably, this was one of the most important justifications to the original conjecture of Pillay). Its proof is based on Cohomological tools and uses extensively the triangulation theorem which is true only in o-minimal expansions of real closed fields:

Theorem 6.2. [11] *Assume that $\mathcal{M}$ expands a real closed field and that G is definably compact, definably connected abelian group of dimension n. Then*

$$Tor(G) \simeq Tor(\mathbb{T}^n).$$

(where $\mathbb{T}^n$ is the n-dimensional torus).

The second ingredient, which we will prove below is:

Lemma 6.3. *G^{00} is torsion-free.*

Let us see how the two results, taken together, imply **PC** in the abelian case:

Lemma 6.3, together with the divisibility of G^{00} (see 3.2 (1)) imply that

$$Tor(G/G^{00}) \simeq Tor(G).$$

If $\dim(G) = n$, it follows from

$$Tor(G/G^{00}) \simeq Tor(\mathbb{T}^n).$$

Because G/G^{00} is a connected (3.1) abelian compact Lie group, it is Lie isomorphic to a direct sum of $\mathbb{T}^1$'s. The number of these $\mathbb{T}^1$'s is determined by, say, the number of 2-torsion points, therefore $G/G^{00} \simeq (\mathbb{T}^1)^n$ and so the real dimension of G/G^{00} is n.

Proof. of Lemma 6.3. For every $n \in \mathbb{N}$, consider the map $\sigma_n : g \mapsto g^n$ from G onto G. By definable choice, there exists a definable set $X \subseteq G$ such that $\sigma_n|X$ is a bijection of X and G (we assume that G is definably connected).

By [11] (or actually by [32]), $T_n = ker(\sigma_n)$ is finite. It clearly contains all n-torsion points and, as easily checked, G equals a finite disjoint union of the gX's, for $g \in T_n$. Thus X and all the gX's are generic and pair-wise disjoint, and therefore $T_n \cap Stab_{ng}(X) = \{0\}$. Because this is true for every n, the group $Stab_{ng}(X)$ is torsion-free.

By 5.6, the type-definable subgroup $Stab_{ng}(X)$ above has bounded index in G, and therefore $G^{00} \subseteq Stab_{ng}(X)$. It follows that G^{00} is torsion-free, ending the proof of Lemma 6.3 and thus **PC** in the abelian case.

7. Proof of PC for arbitrary definably compact G

We assume in this section that G is a definably compact group in an o-minimal expansion of a real closed field.

Here are some preliminary facts about noncommutative definably compact groups:

As shown in [26], $G/Z(G)$ is "semisimple", namely has no infinite definable abelian normal subgroup. We let $N = Z(G)^0$. By [22], G/N can be written as an almost direct product of definably almost simple groups. Namely, each component is a noncommutative definable group which, modulo its finite center, has no definable normal subgroup. The word "almost" implies that up to a finite central subgroup the product of the groups is direct.

Finally, as is shown in [23], every definably simple group has "very good reduction", namely it is definably isomorphic to a semialgebraic linear group defined over the real algebraic numbers. The proof of **PC** for groups with very good reduction is partly contained in the Introduction (see [24] for more details). It easily follows that **PC** holds for definably almost simple groups and therefore also for an almost direct product of such groups. Therefore, **PC** holds for both $Z(G)$ (Theorem 6.1) and for G/N.

We thus have:

$$\dim_{\mathcal{M}}(G) = \dim_{\mathcal{M}}(G/N) + \dim_{\mathcal{M}}(N) =$$

$$\dim_{Lie}((G/N)/(G/N)^{00}) + \dim_{Lie}(N/N^{00}).$$

We also have

$$\dim_{Lie}(G/G^{00}) = \dim_{Lie}(G/G^{00}N) + \dim_{Lie}(G^{00}N/G^{00}).$$

It is easy to verify that $G^{00}N/N = (G/N)^{00}$. Hence,

$$(G/N)/(G/N)^{00} \backsimeq G/G^{00}N.$$

In order to show that $\dim_{\mathcal{M}}(G) = \dim_{Lie}(G/G^{00})$ it is therefore sufficient to prove:

$$G^{00}N/G^{00} \backsimeq N/N^{00}.$$

The group on the left is isomorphic to $N/(G^{00} \cap N)$, hence in order to prove **PC** we are left to prove:

Lemma 7.1. *If G is definably compact then $N^{00} = G^{00} \cap N$.*

The fact that $N^{00} \subseteq (G^{00} \cap N)$ follows from the fact that the group on the right has bounded index in N. However, in order to prove the opposite inclusion (which fails for arbitrary groups, even with NIP) we need to take one more de'tour, through the general theory of generic sets.

8. Some theory of generic sets III

In this section we make no assumption on the group G unless otherwise stated.

Definition 8.1. *The theory of G is said to have* finitely satisfiable generics[1] *(in short f.s.g) if there exists a complete type p over $\mathcal{M}$ such that if $\phi(x) \in p$ then:*
(i) $\phi(G)$ is left generic.
(ii) There exists a small model $M_0 \subseteq M$ such that every left translate of $\phi(G)$ intersects M_0.

Our goal is to show that every definably compact group in an o-minimal structure has f.s.g. This is useful because of the following properties:

Fact 8.2. *Assume that $T = Th(G)$ has f.s.g . Then*
(i) There exists a small $\mathcal{M}_0 \subseteq \mathcal{M}$ such that every left generic set and every right generic set intersect M_0.
(ii) Given $X \subseteq G$ definable, X is left-generic if and only if it is right generic.
(iii) The definable non generic sets in G form an ideal.
(iv) G^{00} exists and $G^{00} = \bigcap\{Stab_{ng}(X) : X \in Def(G)\}$.

[1]For simplicity, we slightly modified the definition from [16]

Proof. Assume that p and $\mathcal{M}_0$ witness f.s.g.
(i) If X is a left generic set then there are $g_1, \ldots, g_k \in G$ such that the formula $x = x$ is equivalent to the finite disjunction of the formulas $x \in g_i X$. Hence, there is g_i such that "$x \in g_i X$" is in p. By assumption on p, $X \cap M_0 \neq \emptyset$. Consider the type $p(x^{-1})$. Because $x \mapsto x^{-1}$ is a 0-definable bijection of G it easily follows that every definable set Y in $p(x^{-1})$ is right generic and every right translate of Y intersects M_0. As above, it follows that every right generic set intersects M_0.
(ii) Assume X is not a left generic set. By 4.6, there exists a right translate of X which does not intersect M_0, hence by (i), X is not right generic as well.
(iii) Since p is a complete generic type it must contain the complement of every nongeneric set.
(iv) For the existence of G^{00}, see [16], Corollary 4.3.

Now fix a small model $\mathcal{M}_0$ witnessing (i). Given $X \subseteq G$ definable, let $\mathcal{M}_1$ be a small model containing $\mathcal{M}_0$ over which X is definable. If $g \equiv_{\mathcal{M}_1} h$ then $gX \cap M_1 = hX \cap M_1$ and hence $(gX \triangle hX) \cap M_0 = \emptyset$. By (i), $gX \triangle hX$ is nongeneric. Thus, every coset of $Stab_{ng}(X)$ contains all the realizations of some complete type over $\mathcal{M}_1$. In particular, in a (still small) model where every complete type over $\mathcal{M}_1$ is realized, there is a representative for every coset of $Stab_{ng}(X)$, so $Stab_{ng}(X)$ has bounded index, and therefore it is contained in G^{00}.

For the opposite inclusion, since G^{00} has bounded index it can be obtained as the intersection of definable generic sets (Fact 4.5). If g belongs to $Stab_{ng}(X)$ for every such X then it must belong to G^{00} (otherwise, by compactness, there is X containing G^{00} such that $gX \cap X = \emptyset$, which implies that $g \notin Stab_{ng}(X)$. □

Lemma 8.3. *Assume that N is a definable normal subgroup of G. If N and G/N have fsg then G has fsg.*

Proof. See Proposition 4.5 in [16]. □

We return to the o-minimal setting.

Lemma 8.4. *If G is definably compact and abelian then G has fsg.*

Proof. Since we do have complete generic types in abelian groups, it is sufficient to show that there is a small $\mathcal{M}_0$ such that every generic set intersects $\mathcal{M}_0$.

Let $\mathcal{M}_0$ be a small model such that every $\approx_{ng}$-class in $Def(G)$ has a representative definable over $\mathcal{M}_0$ (recall, 5.4, that this equivalence relation is bounded).

Given $X \subseteq G$ generic, there exists $X_1 \subseteq X$ such that X_1 is still generic and $Cl(X_1) \subseteq X$. Indeed, the following argument for that fact was suggested by the UIUC seminar:

$$G = \bigcup_{i=1}^{k} g_i X.$$

By 4.4, $Int(X)$ is also generic. For every $\epsilon > 0$ let

$$X^\epsilon = \{g \in X : d(g, Fr(X)) > \epsilon\}.$$

We have,

$$X = \bigcup_{\epsilon > 0} X^\epsilon.$$

It is now sufficient to take ϵ in a complete type $p(x)$ right of zero. So,

$$G = \bigcup_{\epsilon \models p} \bigcup_{i=1}^{k} g_i X^\epsilon.$$

We obtained a definable open covering of G parameterized by a complete type. By the equivalent definition to definable compactness, 2.3, there is a finite subcover, which easily implies that some X^ϵ is generic. Let $X_1 = X^\epsilon$. Because $Cl(X_1) \subseteq X$ we may assume that X is closed.

Let $Y \subseteq G$ be a definable set definable over $\mathcal{M}_0$ such that $Y \triangle X$ is nongeneric. Again, by 4.4, we may assume that Y is closed, so both X and Y are definably compact. We will show that $(Y \cap X) \cap M_0 \neq \emptyset$. Notice that both X and Y are definably compact.

By 4.7, if $X \cap Y \cap M_0 = \emptyset$ then there are finitely many $\mathcal{M}_0$-conjugates of $X \cap Y$ whose intersection is empty. Because Y is $\mathcal{M}_0$-definable there are $X_1, \ldots, X_k$ all $\mathcal{M}_0$-conjugates of X such that

$$\bigcap_{i=1}^{k} X_i \cap Y = \emptyset.$$

Since Y is generic this implies that for some X_i we must have $Y \setminus X_i$ generic. Contradiction to $Y \triangle X$ being non-generic. □

Lemma 8.5. *If H is definably compact and semisimple then H has fsg.*

The proof of this lemma is based on the almost-decomposition into definably almost simple groups. The definably simple case is handled in [24] using measure theoretic arguments based on [4] and [1].

Using 8.4, 8.5, and 8.3, we can conclude:

Theorem 8.6. *Every definably compact group has fsg.*

The above theorem, together with 8.2, implies that the set of left (hence also right) generics in G form an ideal, and that for any definable set X, $Stab_{ng}(X)$ is a type-definable group of bounded index. Finally (and this is the main fact which forced us to take this de'tour through the notion of "f.s.g"), the group G^{00} is the intersection of all stabilizers of definable subsets of G.

9. Completing the proof of **PC**

We can now return to the missing ingredient in the proof of **PC**, namely the proof of 7.1. We need to show that $G^{00} \cap N = N^{00}$, where N is a definably connected normal central subgroup. By 3.2, it is sufficient to prove that $G^{00} \cap N$ is torsion-free.

Given $n \in \mathbb{N}$, let $T_n = Tor_n(N)$ and $X \subseteq N$ be a definable set such that $g \mapsto g^n$ gives a bijection of X and N. By definable choice, there is $D \subseteq G$ which intersects every coset of G/N exactly once. It is now easy to verify that G is the finite disjoint union of the translates of DX by the elements of T_n. In particular, DX is generic and

$$T_n \cap Stab_{ng}(DX) = \{e\}.$$

Because this is true for every n, we have $Stab_{ng}(DX) \cap Tor(G) \neq \emptyset$.

By the f.s.g property, $G^{00} \subseteq Stab_{ng}(DX)$, therefore $G^{00} \cap Tor(N) = \{e\}$.

We thus proved that $G^{00} \cap N = N^{00}$, completing the proof of **PC** (see the argument preceding Lemma 7.1). □

9.1. **Defining measure on G.** As a result of the work on Pillay's Conjecture, the following theorem was established in [16].

Theorem 9.1. *If G is definably compact in an o-minimal structure then it admits a left-invariant Keisler measure on the definable subsets of G. For a definable $X \subseteq G$, we have $\mu(X) = 0$ if and only if X is non-generic.*

Proof. As we already pointed out, the existence of such measure is immediate when G is abelian. In the general case, we first note that G/G^{00}, as a compact Lie group, admits a left-invariant finitely additive probability measure on a boolean algebra of sets containing all Borel measurable sets- the Haar measure $\mathbf{m}$.

We first fix a complete generic type $p(x)$ over G. Given a definable set X, we consider the set

$$\hat{X} = \{gG^{00} \in G/G^{00} : p \models \text{“}x \in gX''\}.$$

(note that X is well defined. Namely, if $gh^{-1} \in G^{00}$ then in particular, $gX \triangle hX$ is non-generic and therefore not in p. It follows that $gX \in p$

if and only if $hX \in p$). The main part of the proof is to show that $\hat{X}$ is a Borel set in G/G^{00} (see Proposition 6.2 in [16]). We then define

$$\mu_p(X) = \mathbf{m}(\hat{X}).$$

Clearly, μ_p is left invariant, and it is easy to verify that it is also finitely additive (if $X_1 \cap X_2 = \emptyset$ then $\hat{X}_1 \cap \hat{X}_2 = \emptyset$).

Finally, if X is generic then finite additivity implies that $\mu_p(X) > 0$ and if X is non-generic then $\hat{X} = \emptyset$ and therefore $\mu_p(X) = 0$. $\square$

10. Related work and some open questions

This section has gone through substantial changes in the last stages of writing. As will be explained below, most of the open questions listed here were solved in a recent paper by Hrushovski and Pillay, [15].

10.1. **Omitting the real closed field assumption.** As was pointed out early on, the only remaining obstacle for proving **PC** without the assumption that $\mathcal{M}$ expands a real closed field is the lack of an analogue to Theorem 6.2 on the number of torsion points, without the field assumption. Such a theorem was proved by Eleftheriou and Starchenko [13] when $\mathcal{M}$ was assumed to be an ordered division ring over an ordered vector space and hence **PC** holds in this case as well. Actually, a very clear description of definable groups in this setting is given in the paper, out of which the number of torsion points is easily read.

In order to prove the torsion points result under weaker assumptions it seems important to develop similar topological tools to the ones originally used, but this time without the triangulation theorem. Indeed, Sheaf Cohomology in expansions of ordered groups has been the subject of several papers of Edmundo, Jones and Peatfield (see [8] and [9]) and of Beraducci and Fornasiero (see [3]). A first application to counting torsion points is given in [8] where the correct upper bound is obtained.

In a very recent result, [25]*, the author was able to prove the question about the torsion point and hence Pillay's Conjecture, in o-minimal expansions of ordered groups*

The questions formulated below were written prior to the publication of the recent pre-print by Hrushovski and Pillay [15]*. As I will eventually point out, most of these questions are now solved by that paper, either explicitly or implicitly. I leave them here because I find that their discussion could still be of some interest*

10.2. **Uniform definability of G^{00}.** An important feature of the basic example of Pillay's conjecture (where we start with a compact real Lie group and view it in an elementary extension) is the fact that the type defining G^{00} is given by a single formula, with varying parameters. Namely,

$$G^{00} = \{g \in G : |g| < a : a \in \mathbb{R}\}.$$

Consider the structure G_{ind} whose universe is G/G^{00}, with a function symbol for the group operation and a predicate for every set of the form $\pi(X)$, for $X \subseteq G^n$ definable in the o-minimal structure $\mathcal{M}$. In [16] we showed, using a theorem of Baysalov and Poizat, that if $G = \langle [0,1)^n, +mod1\rangle$ then structure G_{ind} is definable in an o-minimal structure over the reals. Later, in [18], Marikova re-proved this result without referring to [1], and provided a much finer analysis of the definable sets in this structure. The uniformity in parameters plays an important role in both works.

Conjecture *If G is definably compact then there is a formula $\phi(x, y)$, where x varies over element of G, and a set of parameters A, such that*

$$G^{00} = \{g \in G : \bigwedge_{a \in A} \phi(g, a)\}.$$

Conjecture *The structure on the compact Lie group G_{ind} is definable in some o-minimal structure over the real numbers.*

Related to the above conjecture is the following:

Question *What is the structure which G induces $Tor(G)$? In particular, what subsets of $Tor(G)$ are of the form $X \cap Tor(G)$ for a definable subset of G?*

Note that when G is abelian its torsion group can be realized as a definable set in the o-minimal structure $\langle \mathbb{Q}, +, <\rangle$, namely it is isomorphic as a group to $\langle [0,1)^n, +mod1\rangle$, viewed inside of $\mathbb{Q}$. It was shown by Wilkie, [33], that there are nontrivial o-minimal expansions of this structure. Moreover, if G itself equals to the real points of $\langle [0,1)^n, +mod1\rangle$, in the structure of the real field, then the torsion points of G inherit the ring operations and therefore the induced structure is unstable and undecidable. However, even in this case it is interesting to ask which definable subsets of $\mathbb{Q}^n$ can be obtained as the trace of a definable set in G.

10.3. **The distribution of torsion points.** Somewhat surprisingly for those of us who have worked on this problem, the solution of Pillay's Conjecture did not yield a much better understanding of the distribution of torsion points in a definably compact abelian G. Here are some conjectures on this matter:

Conjecture A *If $X \subseteq G$ is generic then it contains a torsion point.*

Conjecture B *If $G \backslash X$ is non-generic then X contains a torsion point.*

Clearly, (A) implies (B) and both imply the following result, recently proved in [21]:

Theorem *If $\dim(G \setminus X) < \dim G$ then X contains a torsion point.*

10.4. **Other related work.** In other work generated by Pillay's conjecture the precise relationship between G and G/G^{00} was investigated. In [2], Berarducci discusses the o-minimal spectrum $\tilde{G}$ of a definably compact G and proves that G/G^{00} is a topological quotient of $\tilde{G}$. In [10], Edmundo, Jones and Peatfield examine the connections between the cohomology groups of G and of G/G^{00}.

In [20], Onshuus and Pillay study the analogous conjecture in the p-adic setting and show cases where it fails and other cases where the conjecture holds.

In an unpublished result, Hrushovski, Pillay and the present author, prove that every definably compact group G is elementarily equivalent, as a pure group, to G/G^{00}. A better understanding of the group theoretic structure of G can then be deduced, and in particular, one concludes that the commutator subgroup of G is definable and that G is the almost direct product of $Z(G)$ and $[G, G]$.

Finally, the recent paper of Hrushovski and Pillay [15] puts some of the notions which were examined in [16] in a very general context, and examines forking, stability and measure in several different abstract settings, mainly in groups under the assumptions of NIP and the existence of some measure. The machinery and results obtained there are very powerful and, as I now explain, yielded answers to most of the questions raised above.

10.5. **The Compact Domination Conjecture and its recent solution.** At first, it seems as if the most natural way to define measure on definable subsets of G would be directly through the map

$\pi : G \to G/G^{00}$. Namely, to let $\mu(X)$ equal $\mathbf{m}(\pi(X))$ (the Haar measure of $\pi(X)$) for any definable set $X \subseteq G$. However, a difficulty arises when one tries to prove finite additivity:

Take $X_1, X_2 \subseteq G$ two disjoint definable sets. Finite additivity should imply that the Haar measure of $\pi(X_1) \cap \pi(X_2)$ is zero (note that $\pi(X_1), \pi(X_2)$ need not be disjoint anymore). However, until very recently this remained an open question and, as we will soon see, it is equivalent to the Compact Domination Conjecture below.

We first introduce some notation: Given $X \subseteq G$, we let

$$B(X) = \{y \in G/G^{00} : \pi^{-1}(g) \cap X \neq \emptyset \,\&\, \pi^{-1}(g) \cap G \setminus X \neq \emptyset\}.$$

We say that G is *compactly dominated by G/G^{00}, via π, in a measure theoretic sense* if for every definable $X \subseteq G$, the Haar measure of $B(X)$ is zero. We say that G is *compactly dominated by G/G^{00}, via π, in a topological sense* if every such $B(X)$ is nowhere dense in G. (The term "compact domination" is modeled after the notion "stably dominated" referring to a situation where an unstable set is "controlled" by a stable one).

The Compact Domination Conjecture, formulated in [16] stated that every definably compact group is compactly dominated (in both senses). In an earlier version of these notes several equivalences to the above conjectures were proved, implying for example that the measure theoretic conjecture implies the topological one. Both are known to imply Conjecture (A) above about the density of torsion points. However, the recent preprint of Hrushovski and Pillay [15] proves the Compact Domination Conjecture, and at the same time the torsion point and the uniform definability conjectures formulated above.

Theorem 10.1. [15] *Every definably compact group is compactly dominated, in the measure theoretic (and hence topological) sense.*

Since the paper is very new I will only try to roughly sketch the ideas behind this solution:

As in [16], the authors make use of the theorem of Baysalov and Poizat mentioned above. Namely, they consider an elementary extension $\mathcal{M}^*$ of $\mathcal{M}$, and for every $\mathcal{M}^*$-definable set X they add a predicate to $\mathcal{M}$ for the the trace of X, on M^n. The main theorem in [1] (later generalized by Shelah to any theory with NIP), implies that this new structure eliminates quantifiers and in particular it is weakly o-minimal. We denote it by $\bar{\mathcal{M}}$.

The main difficulty is to prove that G^{00} is definable in $\bar{\mathcal{M}}$. Hrushovski and Pillay do it after showing first that it can be written as the set theoretic stabilizer of any global generic type in G. It is here that they also show prove the uniform definability of G^{00}, which was conjectured above.

G/G^{00} is now a compact Lie group, given as a quotient of two definable sets in the weakly o-minimal structure $\bar{\mathcal{M}}$. After a fine analysis of the topological situation (and using the knowledge of the fundamental group of G), they prove that G/G^{00} is *semi o-minimal* in this weakly o-minimal structure. Namely it is in the definable closure of finitely many o-minimal structures, all definable in $\bar{\mathcal{M}}$. In particular, this settles the conjecture on G_{ind} mentioned above.

Once this machinery is established, definable subsets of G/G^{00} of Haar measure zero are just sets of smaller dimension (in the o-minimal sense). It is now not difficult to prove compact domination similarly to the simple cases handled in [16].

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